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ON A TRANSITIVE TRANSFORMATION GROUP OF A COMPACT GROUP MANIFOLD

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1. Introduction. Let K be a connected compact Lie group and H a closed subgroup of K. Suppose a connected Lie subgroup G of K acts simply transitively on the coset space K/H by the left translation. Then the composition mapping

$$F: G\times H\to K$$

defined by $F(g, h)=gh$ $(g\in G, h\in H)$ gives rise to a diffeomorphism of the product manifold $G\times H$ onto K. Consequently, for their Lie algebras, we have

$$\mathfrak{k} = \mathfrak{g}+\mathfrak{h} \quad \text{(direct sum of vector spaces)}.$$

We shall prove in this paper the following:

Theorem 1. *Let $\mathfrak{k}$ be a compact Lie algebra. Suppose there exist two subalgebras $\mathfrak{g}$ and $\mathfrak{h}$ of $\mathfrak{k}$ such that*

$$\mathfrak{k} = \mathfrak{g}+\mathfrak{h} \quad \textit{(direct sum of vector spaces)}.$$

Then there exist a direct sum decomposition

$$\mathfrak{k} = \mathfrak{g}_1\oplus\mathfrak{h}_1$$

of Lie algebras and Lie algebra homomorphisms

$$\varphi: \mathfrak{g}_1\to\mathfrak{h}_1 \quad \textit{and} \quad \psi:\mathfrak{h}_1\to\mathfrak{g}_1$$

with the following properties:

(i) $\mathfrak{g} = \{(X, \varphi(X))\,|\,X\in\mathfrak{g}_1\}$.

(ii) $\mathfrak{h} = \{(\psi(Y), Y)\,|\,Y\in\mathfrak{h}_1\}$.

(iii) $\psi\circ\varphi$ *has no non-zero fixed vector.*

As a result we see that the Lie algebra $\mathfrak{k}$ is isomorphic with the direct sum $\mathfrak{g}\oplus\mathfrak{h}$ of Lie algebras. This theorem gives us an infinitesimal characterization of a homogeneous space of the type mentioned in the above. Some application and remarks will be added after its proof.

Such a homogeneous space is related with a study of isometries of a compact group manifold. Let G be a connected compact Lie group and choose a left invariant Riemannian metric ds^2 on G. Denote by K the identity component of the isometry group of (G, ds^2). We identify an element g of G with its left translation L_g on G. Ochiai-Takahashi [2] proved that if G is simple then G is normal in K. Their theorem follows immediately from our Theorem 1. The conclusion of their theorem does not hold in general if G is not simple, as our example shows. However, our Theorem 3 asserts that if G is simply connected then we have a similar conclusion by a suitable change of the action of G on the space.

2. Recall that a Lie algebra $\mathfrak{k}$ is said to be *compact* if it can be represented as a Lie algebra of a compact Lie group. For a compact Lie algebra $\mathfrak{k}$, we denote by $\mathrm{c}(\mathfrak{k})$ its center and by $\mathrm{s}(\mathfrak{k})$ its maximal semi-simple ideal, so that we have $\mathrm{s}(\mathfrak{k})=[\mathfrak{k}, \mathfrak{k}]$ and

$$\mathfrak{k} = \mathrm{s}(\mathfrak{k})\oplus\mathrm{c}(\mathfrak{k})$$

(direct sum of Lie algebras). The same notation will be used for a connected Lie group K when the Lie algebra $\mathfrak{k}$ of K is compact. $\mathrm{c}(K)$ and $\mathrm{s}(k)$ are the connected Lie subgroups of K corresponding to Lie subalgebras $\mathrm{c}(\mathfrak{k})$ and $\mathrm{s}(\mathfrak{k})$ respectively.

Note that a connected Lie group K has a compact Lie algebra if and only if K has a bi-invariant Riemannian metric and also that any subalgebra of a compact Lie algebra is compact. In the sequel, for a Lie group homomorphism, the induced Lie algebra homomorphism is denoted by the same symbol.

Lemma 1. *Let K, G and H be connected Lie groups with Lie algebras $\mathfrak{k}$, $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Suppose $\mathfrak{k}$ is compact. Let $\phi: G\to K$ and $\psi: H\to K$ be Lie group homomorphisms such that the induced homomorphisms $\phi: \mathfrak{g}\to\mathfrak{k}$ and $\psi: \mathfrak{h}\to\mathfrak{k}$ are both injective and*

$$\mathfrak{k} = \phi(\mathfrak{g})+\psi(\mathfrak{h}) \quad (\textit{direct sum of vector spaces}).$$

Then the composition mapping

$$F: G\times H\to K$$

defined by $F(g, h)=\phi(g)\cdot \psi(h)$ is a covering map.

Proof. In general, we denote the left translation and the right translation of a group induced by an element x in it by L_x and R_x respectively. Then, for the mapping F, we have the following commutative diagram:

$$\begin{array}{ccc} G\times H & \xrightarrow{F} & K \\ {\scriptstyle (L_g,\, R_h)}\downarrow & & \downarrow{\scriptstyle L_{\phi(g)}\circ R_{\varphi(h)}} \\ G\times H & \xrightarrow{F} & K \end{array}$$

for $(g, h)\in G\times H$. This gives an indentity

$$F = (L_{\phi(g)}\circ R_{\varphi(h)})\circ F\circ(L_{g^{-1}}, R_{h^{-1}}) .$$

Taking the differentials, we have

$$(dF)_{(g,h)} = (d(L_{\phi(g)}\circ R_{\varphi(h)e}))_e\circ(dF)_{(e,e)}\circ(d(L_{g^{-1}h}, R_{h^{-1}}))_{(g,h)} .$$

We identify $T_{(e,e)}(G\times H)$ with $T_e(G)+T_e(H)=\mathfrak{g}+\mathfrak{h}$ (direct sum of vector spaces). Since $(dF)_{(e,e)}\,|\,T_e(G)=\phi$ and $(dF)_{(e,e)}\,|\,T_e(H)=\psi$, our assumption in the lemma implies that $(dF)_{(e,e)}$ gives an isomorphism of $T_{(e,e)}(G\times H)$ onto $T_e(K)=\mathfrak{k}$. By the above identity, we see that $(dF)_{(g,h)}$ is isomorphic at each point (g, h) of $G\times H$. Since $\mathfrak{k}$ is compact, we can choose a bi-invariant Riemennian metric ds^2 on K. Then $d\bar{s}^2=F^*(ds^2)$ gives a Riemannian metric on the manifold $G\times H$, which is locally isometric with (K, ds^2) via F. In virtue of the first commutative diagram, the Riemannian metric $d\bar{s}^2$ on $G\times H$ is $L(G)$ and $R(H)$-invariant, and hence it is complete. Thus we see that F is a locally isometric mapping of a complete Riemannian manifold $(G\times H, d\bar{s}^2)$ into (K, ds^2). This proves that F is a covering map. q.e.d.

Lemma 2. *Let $\mathfrak{k}$ be a compact Lie algebra, and let $\mathfrak{g}$ and $\mathfrak{h}$ be two subalgebras of $\mathfrak{k}$ such that*

$$\mathfrak{k} = \mathfrak{g}+\mathfrak{h} \quad (\textit{direct sum of vector spaces}).$$

Then, $\mathfrak{k}$ is isomorphic with the direct sum $\mathfrak{g}\oplus\mathfrak{h}$ of Lie algebras. consequently, we have

$$\dim \mathfrak{c}(\mathfrak{k}) = \dim \mathfrak{c}(\mathfrak{g})+\dim \mathfrak{c}(\mathfrak{h}) .$$

Proof. For $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{k}$, choose simply connected Lie groups G, H and K with the corresponding Lie algebras respectively. Let

$$\phi\colon \mathfrak{g}\to\mathfrak{k} \quad\text{and}\quad \psi\colon \mathfrak{h}\to\mathfrak{k}$$

be the inclusion mappings. They induce Lie group homomorphisms

$$\phi\colon G\to K \quad\text{and}\quad \psi\colon H\to K .$$

The composition mapping F of the product manifold $G\times H$ into K defined by

$$F(g, h) = \phi(g)\psi(h)$$

is a covering map by Lemma 1. Since K is assumed to be simply connected, we have a diffeomorphism of $G\times H$ onto K. $\mathfrak{k}$ is compact and hence $\mathfrak{g}$ and $\mathfrak{h}$ are compact. Since G, H and K are simply connected and their Lie algebras are compact, we see $G=\mathrm{s}(G)\times\mathrm{c}(G)$, $H=\mathrm{s}(H)\times\mathrm{c}(H)$ and $K=\mathrm{s}(K)\times\mathrm{c}(K)$. Since F is a diffeomorphism of the product manifold $G\times H$ onto K we see

$$\dim \mathrm{c}(K) = \dim \mathrm{c}(G)+\dim \mathrm{c}(H)$$

and hence

$$\dim \mathrm{c}(\mathfrak{k}) = \dim \mathrm{c}(\mathfrak{g})+\dim \mathrm{c}(\mathfrak{h}) .$$

Note that $\mathrm{s}(K)$ is a maximal compact subgroup of K. Also we see that F induces a homotopy equivalence between $\mathrm{s}(G)\times\mathrm{s}(H)$ and $\mathrm{s}(K)$.

A theorem in homotopy theory ([3], [4]) states that if two simply connected compact Lie groups are homotopicall equivalent then they are isomorphic as Lie groups. Thus, we see that the Lie group $\mathrm{s}(K)$ is isomorphic with the direct product $\mathrm{s}(G)\times\mathrm{s}(H)$ of Lie groups. Finally we can conclude that the Lie algebra $\mathfrak{k}$ is isomorphic with the direct sum $\mathfrak{g}\oplus\mathfrak{h}$ of Lie algebras. q.e.d.

Corollary 1. *Under the same assumption as above, we have*

$\mathrm{s}(\mathfrak{k}) = \mathrm{s}(\mathfrak{g})+\mathrm{s}(\mathfrak{h})$ (*direct sum of vector spaces*).

Proof. Since $\mathfrak{k}$ and $\mathfrak{g}\oplus\mathfrak{h}$ are isomorphic, $\mathrm{s}(\mathfrak{k})$ and $\mathrm{s}(\mathfrak{g})\oplus\mathrm{s}(\mathfrak{h})$ are isomorphic. Especially, we have

$$\dim \mathrm{s}(\mathfrak{k}) = \dim \mathrm{s}(\mathfrak{g})+\dim \mathrm{s}(\mathfrak{h}) .$$

On the other hand, we know

$$\mathrm{s}(\mathfrak{k}) = [\mathfrak{k}, \mathfrak{k}], \quad \mathrm{s}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}], \quad \mathrm{s}(\mathfrak{h}) = [\mathfrak{h}, \mathfrak{h}] .$$

Thus, we have

$$\mathrm{s}(\mathfrak{k})\supset\mathrm{s}(\mathfrak{g}) \quad\text{and}\quad \mathrm{s}(\mathfrak{k})\supset\mathrm{s}(\mathfrak{h}) .$$

The assumption $\mathfrak{k}=\mathfrak{g}+\mathfrak{h}$ (direct sum of vector spaces) shows that $\mathrm{s}(\mathfrak{g})+\mathrm{s}(\mathfrak{h})$ is a direct sum of vector spaces in $\mathrm{s}(\mathfrak{k})$. The first equality on dimension proves our corollary. q.e.d.

3. Theorem 1 will follow easily from the following:

Proposition 1. *Let $\mathfrak{k}$ be a compact Lie algebra and let $\mathfrak{g}$ and $\mathfrak{h}$ be its subalgebras such that*

$\mathfrak{k} = \mathfrak{g}+\mathfrak{h}$ (*direct sum of vector spaces*).

Then $\mathfrak{k}$ has a direct sum decomposition of Lie algebras

$$\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{l}'$$

with the following properties:

(i) *The projection* π *of* $\mathfrak{k}$ *onto* $\mathfrak{l}$ *with respect to the above decomposition induces an isomorphism of* $\mathfrak{g}$ *onto* $\mathfrak{l}$.

(ii) $\mathfrak{k}=\mathfrak{k}+\mathfrak{h}$ (*direct sum of vector spaces*).

Proof. We prove the propoosition by induction on $\dim \mathfrak{k}$. When $\dim \mathfrak{k}=1$, the proposition holds since $\mathfrak{k}=\mathfrak{g}$ or $\mathfrak{k}=\mathfrak{h}$. Now assume that the proposition holds when $\dim \mathfrak{k}<N$. Let $\dim \mathfrak{k}=N$. To simplify the argument we prepare the following:

Sublemma. *Suppose* $\mathfrak{k}$ *has a non-trivial proper ideal* $\mathfrak{k}_1$ *such that*

$$\mathfrak{k}_1=(\mathfrak{g}\cap\mathfrak{k}_1)+(\mathfrak{h}\cap\mathfrak{k}_1) \quad (\textit{direct sum of vector spaces}).$$

Then the assertion of Proposition 1 *holds for* $\mathfrak{k}$, $\mathfrak{g}$ *and* $\mathfrak{h}$.

Proof. For $\mathfrak{k}_1$, we choose a complementary ideal $\mathfrak{k}_2$ so that we have a direct sum decomposition

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2 .$$

Let π_2 be the projection of $\mathfrak{k}$ onto $\mathfrak{k}_2$. We have

$$\dim \mathfrak{g} = \dim \mathfrak{g}\cap\mathfrak{k}_1+\dim \pi_2(\mathfrak{g}) ,$$
$$\dim \mathfrak{h} = \dim \mathfrak{h}\cap\mathfrak{k}_1+\dim \pi_2(\mathfrak{g}) .$$

Thus,

$$\dim \mathfrak{k}_2 = \dim \mathfrak{k}-\dim \mathfrak{k}_1$$
$$= \dim \pi_2(\mathfrak{g})+\dim \pi_2(\mathfrak{h}) .$$

Since $\mathfrak{k}=\mathfrak{g}+\mathfrak{h}$, $\mathfrak{k}_2=\pi_2(\mathfrak{k})$ is spanned by $\pi_2(\mathfrak{g})$ and $\pi_2(\mathfrak{h})$, and hence we have

$$\mathfrak{k}_2 = \pi_2(\mathfrak{g})+\pi_2(\mathfrak{h}) \quad \text{(direct sum of vector spaces).}$$

Consider $\mathfrak{k}_1$ and its subalgebras $\mathfrak{g}\cap\mathfrak{k}_1$ and $\mathfrak{h}\cap\mathfrak{k}_1$ and also $\mathfrak{k}_2$ and its subalgebras $\pi_2(\mathfrak{g})$ and $\pi_2(\mathfrak{h})$. By the inductive hypothesis, we have direct sum decompositions

$$\mathfrak{k}_1 = \mathfrak{l}_1\oplus\mathfrak{l}_1' \quad \text{and} \quad \mathfrak{k}_2 = \mathfrak{l}_2\oplus\mathfrak{l}_2'$$

with the properties:

i. The projections $\mathfrak{g}\cap\mathfrak{k}_1\to\mathfrak{l}_1$ and $\pi_2(\mathfrak{g})\to\mathfrak{l}_2$ are isomorphisms.

ii. $\mathfrak{k}_1=\mathfrak{l}_1+\mathfrak{h}\cap\mathfrak{k}_1$, $\mathfrak{k}_2=\mathfrak{l}_2+\pi_2(\mathfrak{h})$ (direct sums of vector spaces).

Let

$$\mathfrak{l} = \mathfrak{l}_1\oplus\mathfrak{l}_2 \quad \text{and} \quad \mathfrak{l}' = \mathfrak{l}_1'\oplus\mathfrak{l}_2' .$$

We claim that the direct sum decomposition

$$\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{l}'$$

satisfies the required properties.

First suppose $X \in \mathfrak{g} \cap \mathfrak{l}'$. Then $\pi_2(X) \in \pi_2(\mathfrak{g}) \cap \mathfrak{l}_2'$. However $\pi_2(\mathfrak{g}) \cap \mathfrak{l}_2' = \{0\}$ from the assumption. Thus $\pi_2(X)=0$, and hence $X \in \mathfrak{k}_1$. Then $X \in (\mathfrak{g} \cap \mathfrak{k}_1) \cap \mathfrak{l}_1' = \{0\}$. Consequently we have $\mathfrak{g} \cap \mathfrak{l}' = \{0\}$. This shows that the projection of $\mathfrak{g}$ into $\mathfrak{l}$ with respect to $\mathfrak{l} \oplus \mathfrak{l}'$ is injective. Since they have the same dimension, we have the property (i). Next suppose $\mathfrak{l} \cap \mathfrak{h} \in X$. $\pi_2(X) \in \pi_2(\mathfrak{h}) \cap \mathfrak{l}_2 = \{0\}$, and hence $X \in \mathfrak{k}_1$. We see that $X \in \mathfrak{l}_1 \cap (\mathfrak{h} \cap \mathfrak{k}_1) = \{0\}$. Thus, we have $\mathfrak{l} \cap \mathfrak{h} = \{0\}$. Since $\dim \mathfrak{k} = \dim \mathfrak{l} + \dim \mathfrak{h}$, we see $\mathfrak{k} = \mathfrak{l} + \mathfrak{h}$ (direct sum of vector spaces). Thus we have the property (ii) also. q.e.d.

We continue our proof of Proposition 1. First consider easy cases.

(1) *Suppose* $\mathfrak{k}$ is abelian.
Then $\mathfrak{l}=\mathfrak{g}$ and $\mathfrak{l}'=\mathfrak{h}$ satisfy the required properties.

(2) *Suppose* $\mathfrak{k}$ is simple.
Then, by Lemma 2 we see $\mathfrak{k}=\mathfrak{g}$ or $\mathfrak{k}=\mathfrak{h}$. Thus our assertion holds trivially.

(3) *Suppose* $\mathfrak{g}$ contains a non-trivial proper ideal, say $\mathfrak{k}_1$, of $\mathfrak{k}$.
Then choose a complementary ideal $\mathfrak{k}_2$ of $\mathfrak{k}_1$ in $\mathfrak{k}$, so that we have

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2 .$$

Clearly, $\mathfrak{k}_1 \cap \mathfrak{g} = \mathfrak{k}_1$, $\mathfrak{k}_1 \cap \mathfrak{h} \subset \mathfrak{g} \cap \mathfrak{h} = \{0\}$. Applying the above sublemma we see that Proposition 1 holds in this case.

(4) *Suppose* $\mathfrak{h}$ contains a non-trivial proper ideal, say $\mathfrak{k}_1$, of $\mathfrak{k}$.
Then again we have $\mathfrak{k}_1 \cap \mathfrak{g} = \{0\}$, and $\mathfrak{k}_1 \cap \mathfrak{h} = \mathfrak{k}_1$. Thus we can apply the sublemma in this case also.

(5) *Suppose* $\mathfrak{k}$ is not semi-simple.
We may suppose $\mathfrak{k}$ is not abelian. Then the semi-simple part $\mathrm{s}(\mathfrak{k})$ is a non-trivial proper ideal of $\mathfrak{k}$. By Corollary 1, we have

$$\mathrm{s}(\mathfrak{k}) = \mathrm{s}(\mathfrak{g}) + \mathrm{s}(\mathfrak{h}) \quad \text{(direct sum of vector spaces)}.$$

Since $\mathrm{s}(\mathfrak{g}) \subset \mathfrak{g} \cap \mathrm{s}(\mathfrak{k})$, $\mathrm{s}(\mathfrak{h}) \subset \mathfrak{h} \cap \mathrm{s}(\mathfrak{k})$ and $(\mathfrak{g} \cap \mathrm{s}(\mathfrak{k})) \cap (\mathfrak{h} \cap \mathrm{s}(\mathfrak{k})) = \{0\}$, we have $\mathrm{s}(\mathfrak{g}) = \mathfrak{g} \cap \mathrm{s}(\mathfrak{k})$ and $\mathrm{s}(\mathfrak{h}) = \mathfrak{h} \cap \mathrm{s}(\mathfrak{k})$. Thus

$$\mathrm{s}(\mathfrak{k}) = \mathfrak{g} \cap \mathrm{s}(\mathfrak{k}) + \mathfrak{h} \cap \mathrm{s}(\mathfrak{k})$$

is a direct sum of vector spaces, and hence we can apply our sublemma.

The above argument shows that we may suppose $\mathfrak{k}$ is semi-simple and not simple.

(6) *Suppose* $\mathfrak{k}$ is semi-simple and all simple factors of $\mathfrak{k}$ are mutually isomorphic with each other.

In this case we shall show that either $\mathfrak{g}$ or $\mathfrak{h}$ contains a proper ideal of $\mathfrak{k}$, so that the proposition holds by (3) or (4). Suppose neither $\mathfrak{g}$ nor $\mathfrak{h}$ contains a non trivial proper ideal of $\mathfrak{k}$. Let

$$\mathfrak{k}=\sum_{i\in I}\mathfrak{k}_i,\ \mathfrak{g}=\sum_{j\in J}\mathfrak{g}_j \quad \text{and} \quad \mathfrak{h}=\sum_{k\in K}\mathfrak{h}_k$$

be the decompositions of $\mathfrak{k}$, $\mathfrak{g}$ and $\mathfrak{h}$ into simple factors. By the present assumption, all $\mathfrak{k}_i$'s are mutually isomorphic. By Lemma 2, we see also that all $\mathfrak{g}_j$'s, $\mathfrak{h}_k$'s and $\mathfrak{k}_i$'s are mutually isomorphic, and that

$$|I|=|J|+|K|$$

where $|\ |$ indicates the number of elements.

Denote by π_i the projection of $\mathfrak{k}$ onto $\mathfrak{k}_i$. One sees that

$$\begin{aligned}\pi_i(\mathfrak{g}_j)&=\mathfrak{k}_i \quad \text{or} \quad \{0\}\,,\\ \pi_i(\mathfrak{h}_k)&=\mathfrak{k}_i \quad \text{or} \quad \{0\}\end{aligned}$$

for all i, j, k. Put

$$\begin{aligned}A_j&=\{i\in I\,|\,\pi_i(\mathfrak{g}_j)\neq\{0\}\}\,,\\ B_k&=\{i\in I\,|\,\pi_i(\mathfrak{h}_k)\neq\{0\}\}\end{aligned}$$

for each $j\in J$ and $k\in K$. Let $j_1, j_2\in J$, and $j_1\neq i_2$. Then $[\pi_i(\mathfrak{g}_{j_1}), \pi_i(\mathfrak{g}_{j_2})]=0$. Thus $A_{j_1}\cap A_{j_2}=\phi$. Hence A_j's are mutually disjoint and so are B_k's.

Suppose A_j consists of exactly one element, say i. Then we see $\mathfrak{g}_j=\mathfrak{k}_i$ and hence $\mathfrak{g}$ contains a non trivial proper ideal. This is a contradiction. Thus each A_j contains at least two elements. Similarly we have $|B_k|\geq 2$. Thus we have

$$\sum|A_j|+\sum|B_k|\geqq 2(|J|+|K|)=2|I|$$

On the other hand,

$$\sum|A_j|\leq|I| \quad \text{and} \quad \sum|B_k|\leq|I|\,.$$

Combining together, we see

$$|A_j|=|B_k|=2$$

for every $j\in J$ and $k\in K$.

By an elementary combinatorial argument one can decompose the index set I into two disjoint subsets I_1 and I_2 such that, for every j, k, the sets $A_j\cap I_1$,

$A_j \cap I_2$, $B_k \cap I_1$ and $B_k \cap I_2$ are all non empty. Let

$$\mathfrak{a}_1 = \sum_{i \in I_1} \mathfrak{k}_i \quad \text{and} \quad \mathfrak{a}_2 = \sum_{i \in I_2} \mathfrak{k}_i ,$$

so that we have $\mathfrak{k} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$. Denote by p_i the projection of $\mathfrak{k}$ onto $\mathfrak{a}_i$ (for $i=1, 2$). It follows from our construction that the homomorphisms $p_1|\mathfrak{g}$, $p_2|\mathfrak{g}$, $p_1|\mathfrak{h}$ and $p_2|\mathfrak{h}$ are all onto isomorphisms. Using the decomposition $\mathfrak{k} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$, we can write

$$\mathfrak{g} = \{(X, \phi(X)) \mid X \in \mathfrak{a}_1\}$$

and

$$\mathfrak{h} = \{(\psi(Y), Y) \mid Y \in \mathfrak{a}_2\}$$

by suitable onto isomorphisms $\phi: \mathfrak{a}_1 \to \mathfrak{a}_2$ and $\psi: \mathfrak{a}_2 \to \mathfrak{a}_1$. Consider an automorphism $\psi \circ \phi$ of $\mathfrak{a}_1$. By a result due to Borel and Mostow [1], every automorphism of a semi-simple Lie algebra has a non-zero fixed vector. Thus, we have an element X in $\mathfrak{a}_1$ such that $X \neq 0$ and $\psi(\phi(X)) = X$. Then we have

$$(X, \phi(X)) = (\psi(\phi(X)), \phi(X)) \in \mathfrak{g} \cap \mathfrak{h} = \{0\} .$$

This is a contradiction. Thus, in this case, either $\mathfrak{g}$ or $\mathfrak{h}$ contains a proper ideal of $\mathfrak{k}$.

(7) *Suppose* $\mathfrak{k}$ is semi-simple and $\mathfrak{k}$ contains at least two simple ideals which are not isomorphic.

Choose a simple ideal $\mathfrak{a}$ of $\mathfrak{k}$ such that $\dim \mathfrak{a}$ is minimal among the simple ideals of $\mathfrak{k}$. Let $\mathfrak{k}_0$ be the direct sum of all simple ideals isomorphic to $\mathfrak{a}$, and $\mathfrak{k}_1$ the complementary ideal, so that we have

$$\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 .$$

Similarly, decompose $\mathfrak{g}$ and $\mathfrak{h}$ as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \quad \text{and} \quad \mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 ,$$

where $\mathfrak{g}_0$ (resp. $\mathfrak{h}_0$) is the direct sum of all simple ideals in $\mathfrak{g}$ (resp. $\mathfrak{h}$) isomorphic to $\mathfrak{a}$.

In virtue of Lemma 2, we see that $\mathfrak{k}_0$ and $\mathfrak{k}_1$ are isomorphic with $\mathfrak{g}_0 \oplus \mathfrak{h}_0$ and $\mathfrak{g}_1 \oplus \mathfrak{h}_1$ respectively. We claim that the ideal $\mathfrak{k}_1$ satisfies the required condition in the sublemma. Let π_0 and π_1 be the projections of $\mathfrak{k}$ onto $\mathfrak{k}_0$ and $\mathfrak{k}_1$ respectively. Consider $\pi_0: \mathfrak{g}_1 \to \mathfrak{k}_0$. From the definitions of $\mathfrak{g}_1$ and $\mathfrak{k}_0$, we see $\pi_0 | \mathfrak{g}_1 = \{0\}$. Thus, $\mathfrak{g}_i \subset \mathfrak{k}_1$. Similarly we have $\mathfrak{h}_1 \subset \mathfrak{k}_1$. Thus, $\mathfrak{k}_1 \supset \mathfrak{g}_1 + \mathfrak{h}_1$. Since $\mathfrak{g}_1 \cap \mathfrak{h}_1 = \{0\}$ and $\dim \mathfrak{k}_1 = \dim \mathfrak{g}_1 + \dim \mathfrak{h}_1$, we conclude that

$$\mathfrak{k}_1 = \mathfrak{g}_1 + \mathfrak{h}_1 \quad \text{(direct sum of vector spaces).}$$

Since $\mathfrak{k}=\mathfrak{g}+\mathfrak{h}$ is a direct sum of vector spaces, we see that $(\mathfrak{k}_1\cap\mathfrak{g})\cap(\mathfrak{k}_1\cup\mathfrak{h})=\{0\}$. On the other hand, $\mathfrak{k}_1\cap\mathfrak{g}\supset\mathfrak{g}_1$ and $\mathfrak{k}_1\cap\mathfrak{h}\supset\mathfrak{h}_1$, and also $\mathfrak{k}_1=\mathfrak{g}_1+\mathfrak{h}_1$ (direct sum of vector spaces). It follows that $\mathfrak{g}_1=\mathfrak{g}\cap\mathfrak{k}_1$ and $\mathfrak{h}_1=\mathfrak{h}\cap\mathfrak{k}_1$ and hence

$$\mathfrak{k}_1=(\mathfrak{g}\cap\mathfrak{k}_1)+(\mathfrak{h}\cap\mathfrak{k}_1)\quad\text{(direct sum of vector spaces).}$$

This proves our claim.

Thus we have completed the proof of Proposition 1.

4. Now we can prove Theorem 1

Proof of Theorem 1. First assume that $\mathfrak{k}$ is semi-simple. Apply Proposition 1 to $\mathfrak{k}$, $\mathfrak{g}$ and $\mathfrak{h}$. We get a direct sum decomposition

$$\mathfrak{k}=\mathfrak{l}\oplus\mathfrak{l}'$$

with the properties:

(i) The projection of $\mathfrak{k}$ onto $\mathfrak{l}$ with respect to the above decomposition induces an isomorphism of $\mathfrak{g}$ onto $\mathfrak{l}$.

(ii) $\mathfrak{k}=\mathfrak{l}+\mathfrak{h}$ (direct sum of vector spaces).

Again apply Proposition 1 to $\mathfrak{k}$, $\mathfrak{h}$ and $\mathfrak{l}$. We have a direct sum decomposition

$$\mathfrak{k}=\mathfrak{m}\oplus\mathfrak{m}'$$

with the properties:

(i′) The projection of $\mathfrak{k}$ onto $\mathfrak{m}$ with respect to this decomposition induces an isomorphisms of $\mathfrak{h}$ onto $\mathfrak{m}$.

(ii′) $\mathfrak{k}=\mathfrak{m}+\mathfrak{l}$ (direct sum of vector spaces).

Since $\mathfrak{m}$ and $\mathfrak{l}$ are both ideals of $\mathfrak{k}$, we have a direct sum

$$\mathfrak{k}=\mathfrak{m}\oplus\mathfrak{l}$$

of Lie algebras. The assumption that $\mathfrak{k}$ is semi-simple implies $\mathfrak{m}=\mathfrak{l}'$. Thus, with respect to the direct sum

$$\mathfrak{k}=\mathfrak{l}\oplus\mathfrak{l}'$$

we see that the projections of $\mathfrak{k}$ onto $\mathfrak{l}$ and $\mathfrak{l}'$ induce isomorphisms of $\mathfrak{g}$ and $\mathfrak{h}$ onto $\mathfrak{l}$ and $\mathfrak{l}'$ respectively. Setting $\mathfrak{g}_1=\mathfrak{l}$, and $\mathfrak{h}_1=\mathfrak{l}'$, we see that the decomposition

$$\mathfrak{k}=\mathfrak{g}_1\oplus\mathfrak{h}_1$$

satisfies the first two properties. The third property follows from $\mathfrak{g}\cap\mathfrak{h}=\{0\}$.

In fact, suppose $\psi(\phi(X))=X$ for $X\in\mathfrak{g}_1$. Then, $(X,\phi(X))=(\psi(\phi(X)),$

$\phi(X))\in\mathfrak{g}\cap\mathfrak{h}=\{0\}$. Thus $X=0$.

Consider the general case. By Corollary 1, we have

$$\mathfrak{s}(\mathfrak{k})=\mathfrak{s}(\mathfrak{g})+\mathfrak{s}(\mathfrak{h}) \quad \text{(direct sum of vector spaces).}$$

Also by Lemma 1, $\dim \mathfrak{c}(\mathfrak{k})=\dim \mathfrak{c}(\mathfrak{g})+\dim \mathfrak{c}(\mathfrak{h})$. It is easily seen that the projection π of $\mathfrak{k}$ onto $\mathfrak{c}(\mathfrak{k})$ induces

$$\mathfrak{c}(\mathfrak{k})=\pi(\mathfrak{c}(\mathfrak{g}))+\pi(\mathfrak{c}(\mathfrak{h})) \quad \text{(direct sum of vector spaces).}$$

From the first rgument, we can choose a direct sum decomposition

$$\mathfrak{s}(\mathfrak{k})=\mathfrak{g}_1'\oplus\mathfrak{h}_1'$$

such that the projections of $\mathfrak{s}(\mathfrak{k})$ onto $\mathfrak{g}_1'$ and $\mathfrak{h}_1'$ induce isomorphisms of $\mathfrak{s}(\mathfrak{g})$ and $\mathfrak{s}(\mathfrak{h})$ onto $\mathfrak{g}_1'$ and $\mathfrak{h}_1'$ respectively. Now put

$$\mathfrak{g}_1=\mathfrak{g}_1'\oplus\pi(\mathfrak{c}(\mathfrak{g}))$$

and

$$\mathfrak{h}_1=\mathfrak{h}_1'\oplus\pi(\mathfrak{c}(\mathfrak{h}))\,.$$

we have a direct sum decomposition

$$\mathfrak{k}=\mathfrak{g}_1\oplus\mathfrak{h}_1\,.$$

We claim that this decomposition satisfies the required properties in Theorem 1. The first two are easy. The last one follows from the first two and $\mathfrak{g}\cap\mathfrak{h}=\{0\}$.

Q.E.D.

REMARK 1. The converse of Theorem 1 holds. Let $\mathfrak{g}_1$ and $\mathfrak{h}_1$ be Lie algebras, and let $\phi:\mathfrak{g}_1\to\mathfrak{h}_1$ and $\psi:\mathfrak{h}_1\to\mathfrak{g}_1$ be Lie algebra homomorphisms such that $\psi\circ\phi$ has no non-zero fixed vector. In the direct sum $\mathfrak{g}_1\oplus\mathfrak{h}_1=\mathfrak{k}$ of Lie algebras, define $\mathfrak{g}$ and $\mathfrak{h}$ by (i) and (ii). Then $\mathfrak{g}$ and $\mathfrak{h}$ are subalgebras and we have

$$\mathfrak{k}=\mathfrak{g}+\mathfrak{h} \quad \text{(direct sum of vector spaces).}$$

REMARK 2. Suppose $M=K/H$ is a homogeneous space space of the type mentioned in the introduction. Then the action of K on K/H is almost effective if and only if ψ is injective.

REMARK 3. Let $M=K/H$ be as above. By the theorem of Borel-Mostow cited before, the Lie algebra homomorphism $\psi\circ\phi=0$ if $\mathfrak{g}$ is simple. Thus we see that if G is simple and the K-action on K/H is almost effective then G is normal. Thus, Ochiai-Takahashi's theorem follows from Theorem 1.

5. Now we consider a homogeneous space of the type mentioned in the introduction. Let $M=K/H$ be a homogeneous space of a connected compact

Lie group K. We assume that a connected Lie subgroup G acts simply transitively on K/H. Since K/H is compact, G is necessarily compact. The composition mapping

$$F: G\times H\to K$$

is a diffeomorphism, so that we have

$$\mathfrak{k} = \mathfrak{g}+\mathfrak{h} \quad \text{(direct sum of vector spaces)},$$

for their Lie algebras. Applying Theorem 1, we have a direct sum decomposition

$$\mathfrak{k} = \mathfrak{g}_1\oplus\mathfrak{h}_1$$

and homomorphisms ϕ: $\mathfrak{g}_1\to\mathfrak{h}_1$ and ψ: $\mathfrak{h}_1\to\mathfrak{g}_1$ such that we have

$$\mathfrak{g} = \{(X, \phi(X))\,|\,X\in\mathfrak{g}_1\}\ ,$$
$$\mathfrak{h} = \{(\psi(Y),\ Y)\,|\,Y\in\mathfrak{h}_1\}\ .$$

Further, as we see from the proof of Theorem 1, we can assume that

$$\mathrm{c}(\mathfrak{g}_1) = \pi(\mathrm{c}(\mathfrak{g}))\ ,$$

where π denotes the projection of $\mathfrak{k}$ onto its center.

Let G_1 be the connected Lie subgroup of K corresponding to the subalgebra $\mathfrak{g}_1$. Since $\mathfrak{g}_1$ is an ideal of $\mathfrak{k}$, G_1 is a normal subgroup of K. Next we claim that G_1 is compact. $\mathrm{s}(G_1)$ is closed in K since it is semi-simple. Thus it suffices to show that $\mathrm{c}(G_1)$ is compact. However, from our construction, $\mathrm{c}(\mathfrak{g}_1)=\pi(\mathrm{c}(\mathfrak{g}))$. Consider the Lie group homomorphism $\tilde{\pi}$: $K\to K/\mathrm{s}(K)$. $\tilde{\pi}\,|\,\mathrm{c}(K)$ is a finite covering map. Thus $\mathrm{c}(G_1)$ is closed in $\mathrm{c}(K)$ if and only if $\tilde{\pi}(\mathrm{c}(G_1))$ is closed. On the other hand, $\mathrm{c}(G)$ is compact, and hence $\tilde{\pi}(\mathrm{c}(G))$ is compact. $\mathrm{c}(\mathfrak{g}_1)=\pi(\mathrm{c}(\mathfrak{g}))$ implies that $\tilde{\pi}(\mathrm{c}(G_1))=\tilde{\pi}(\mathrm{c}(G))$. Thus, $\mathrm{c}(G_1)$ is closed, and hence G_1 is compact. From the property that $\mathfrak{k}=\mathfrak{g}_1\oplus\mathfrak{h}_1$ and $\mathfrak{h}=\{(\psi(Y),\ Y)\,|\,Y\in\mathfrak{h}_1\}$, we have

$$\mathfrak{k}=\mathfrak{g}_1+\mathfrak{h} \quad \text{(direct sum of vector spaces)}.$$

By Lemma 1, the composition mapping

$$G_1\times H\to K$$

defines a covering map. Consequently, G_1 acts transitively on the coset space K/H. Furthermore, fix a point p in K/H. Then the mapping

$$G_1\to K/H$$

defined by $g\to g(p)$ is a covering map. Thus, if $G(\cong K/H)$ is simply connected, then G_1 is also simply connected. Thus we have proved the following:

Theorem 2. *Let K be a connected compact Lie group and H a closed subgroup of K. Assume that a connected Lie subgroup G acts simply transitively on the homogeneous space K/H by the left translation. Then there exists a connected closed normal subgroup G_1 of K such that G_1 acts transitively on K/H and G_1 is locally isomorphic with G as Lie groups.*

Theorem 3. *Under the same assumption as in Theorem 2, assume further that G is simply connected. Then there exists a connected closed normal subgroup G_1 of K such that G_1 is isomorphic with G as Lie groups and G_1 acts simply transitively on K/H.*

6. We give here two examples. The first one shows that the conclusion of Ochiai-Takahashi's theorem does not hold any more if G is not simple.

EXAMPLE 1. Let A be a connected compact semi-simple Lie group and $\mathfrak{a}$ its Lie algebra. We put

$$K = A\times A\times A\,,$$
$$G = \{(x, y, x)\,|\,x, y\in A\}\,,$$
$$H = \{(e, z, z)\,|\,z\in A\}\,.$$

H is a closed subgroup of K. Consider the homogeneous space K/H. We see easily that G acts simply transitively on K/H. G is compact semi-simple and not simple. Choose a K-invariant Riemannian metric ds^2 on K/H. Since K/H can be identified with G, ds^2 is a left -invariant Riemannian metric on G. From the definition, K is contained in the identity-component of isometries of $(K/H=G, ds^2)$. G is not nomal in K, thus G is not normal in the identity-component of isometries.

For this example, an explicit description of Theorem 1 is as follows:

$$\mathfrak{k} = \mathfrak{a}\oplus\mathfrak{a}\oplus\mathfrak{a}\,.$$

Let
$$\mathfrak{g}_1 = \{(X, Y, 0)\,|\,X, Y\in\mathfrak{a}\}\,,$$
$$\mathfrak{h}_1 = \{(0, 0, Z)\,|\,Z\in\mathfrak{a}\}\,.$$

Define ϕ: $\mathfrak{g}_1\rightarrow\mathfrak{h}_1$ by

$$\phi((X, Y, 0)) = (0, 0, X)$$

and ψ: $\mathfrak{h}_1\rightarrow\mathfrak{g}_1$ by

$$\psi((0, 0, Z)) = (0, Z, 0)\,.$$

Then we have

$$\mathfrak{g} = \{(X, \phi(X))\in\mathfrak{g}_1\oplus\mathfrak{h}_1\,|\,X\in\mathfrak{g}_1\}\,,$$
$$\mathfrak{g} = \{(\psi(Y), Y)\in\mathfrak{g}_1\oplus\mathfrak{g}_1\{Y\in\mathfrak{h}_1\}\,.$$

The next example shows that the conclusion of Theorem 3 does not hold if G is not simply connected.

EXAMPLE 2. We choose two simply connected compact Lie groups A and B with the following properties:
1. There exists an injective homomorphism j of A into B.
2. The center $Z(A)$ of A is non-trivial and

$$j(Z(A)) \cap Z(B) = \{e\} .$$

For instance, choose positive integers m and n such that $n>m>2$. Then $A=SU(m)$, $B=SU(n)$ and the canonical injection of $SU(m)$ into $SU(n)$ satisfy the required properties.

Let

$$\begin{aligned} K &= A \times B \times A , \\ G_1 &= A \times B \times \{e\} , \\ G &= \{(a, b, a) \mid a \in A, b \in B\} , \\ H &= \{(e, j(a), a) \mid a \in A\} , \\ \Gamma &= \{(x, e, x) \mid x \in Z(A)\} . \end{aligned}$$

The Lie algebras of A and B are denoted by $\mathfrak{a}$ and $\mathfrak{b}$ respectively. Γ is a finite group contained in the center of K. We consider the quotient group $\bar{K}=K/\Gamma$, and denote by π the canonical projection of K onto $\bar{K}$. $\bar{H}=\pi(H)$ is a closed subgroup of $\bar{K}$. Consider $\bar{K}/\bar{H}$. One can easily show that the group $\bar{G}=\pi(G)$ acts simply transitively on $\bar{K}/\bar{H}$. We claim that no normal subgroup of $\bar{K}$ acts simply transitively on $\bar{K}/\bar{H}$. Suppose a normal subgroup G_1' of $\bar{K}$ acts simply transitively on $\bar{K}/\bar{H}$. Then its Lie algebra $\mathfrak{g}_1'$ satisfies

$$\mathfrak{k}=\mathfrak{g}_1'+\mathfrak{g} \quad \text{(direct sum of vector spaces)},$$

where $\mathfrak{h}=\{(0, j(X), X) \mid X \in \mathfrak{a}\}$. Since $\mathfrak{g}_1'$ is an ideal of $\mathfrak{k}$, we see $\mathfrak{g}_1'=\mathfrak{g}_1=\{(X, Y, 0) \mid X \in \mathfrak{a}, Y \in \mathfrak{b}\}$. It follows that $\pi(G_1)=G_1'$. However, $\pi(G_1)$ is simply connected because $\pi(G_1)=G_1/(G_1 \cap \Gamma) \cong G_1$. This is a contradiction.

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