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ON π-SOLVABLE GROUPS WHOSE CHARACTER DEGREES ARE π-NUMBERS

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1. Introduction. Let π be a set of primes, and let $n=p_1\cdot p_2\cdot\cdots\cdot p_t$ be a positive integer, where the p_i are (not necessarily distinct) primes. Then we say that the total exponent (shortly T-exponent) of n is t and write $e(n)=t$. If $p_i\in\pi$ for $i=1, 2, \cdots, t$ with the above notation, then n is said to be a π-number.

Let $Irr(G)$ be the set of irreducible complex characters of a group G. We say that a group G has *c.d.π* (*character degrees π*) if $\chi(1)$ is a π-number for any $\chi\in Irr(G)$, a group G has *r.x.e* (*representation exponent e*) if $e(\chi(1))\leqq e$ for any $\chi\in Irr(G)$, and G has *r.x.e for π* (*representation exponent e for π*) if G has *c.d.π* and *r.x.e*.

In this paper we shall prove the following theorems.

Theorem I. *Let G have r.x.e for π. Suppose G is π-solvable when $|\pi|\geqq 3$. Then G has a normal series*

$$G=A_e\triangleright B_{e-1}\triangleright A_{e-1}\triangleright\cdots\triangleright B_0\triangleright A_0$$

and there exists some prime $p_i\in\pi$ for any i such that

(1) *A_i has r.x.i for π,*

(2) *A_i/B_{i-1} is a cyclic π_i-group, where $\pi_i=\pi-\{p_i\}$,*

(3) *B_{i-1}/A_{i-1} is an elementary abelian p_i-group, and*

(4) *$|A_i\colon A_{i-1}|$ is a π-number with $e(|A_i\colon A_{i-1}|)\leqq 2i+1$.*

In particular G has a subnormal abelian subgroup A_0 whose index is a π-number with $e(|G\colon A_0|)\leqq e(e+2)$.

This theorem generalizes the result of I.M. Isaacs and D.S. Passman [5] in the case $\pi=\{p\}$. In the case $\pi=\{p\}$, indeed, $p_1=p_2=\cdots=p_e=p$ and the π_i are empty with the above notation. Thus $A_i=B_{i-1}$, that is, the normal series in Theorem I has elementary abelian factor groups.

In Theorem I G may have, however, larger subnormal abelian subgroups. We shall show the existence of such subgroups. First we make the following definition.

Let f_s (resp. f_n) be a function with the following property. If G is a solvable (resp. nilpotent) group with *r.x.e*, then G has a subnormal abelian subgroup A with $e(|G:A|)\leqq f_s(e)$ (resp. $f_n(e)$). Moreover we assume that f_s (resp. f_n) is the smallest such function. Let $f_{(p)}$ be the corresponding function for the class of groups with *r.x.e* for a prime p.

In what follows, we denote the largest integer $\leqq x$ by $[x]$.

In [6] we know the existence of $f_{(p)}$ for any prime p. Actually $f_{(p)}(0)=0$ and

$$2e\leqq f_{(p)}(e)\leqq[4e-\log_2 4e] \qquad \text{when } e\geqq 1 .$$

In this paper we have:

Theorem II. *The functions f_s and f_n exist and satisfy*

(1) $f_n(0)=0, f_n(1)=2$ *and*

$2e\leqq f_n(e)\leqq[4e-\log_2 8e]$ *when* $e\geqq 2$.

(2) $f_n(e)\leqq f_s(e)\leqq e(e+3)/2$.

This yields in particular

$f_n(0)=f_s(0)=0, f_n(1)=f_s(1)=2, f_n(2)=4$ *and* $f_s(2)=4$ *or* 5.

All groups in this paper are assumed to be finite unless otherwise stated. Let $N\lhd G$. If $\chi\in Irr(G/N)$, then χ may be viewed as a character of G. For example $\hat{G}=Irr(G/G')$, where G' is the commutator subgroup of G, is the set of linear characters of G. In what follows an irreducible character means an irreducible complex character. If G is a group, then $Z(G)$ and $\Phi(G)$ denote the center and Frattini subgroup of G respectively. If S is a set, then $|S|$ denotes the cardinality of S. We write

$\pi(G)=\{\text{primes } p \mid p \text{ divides } |G|\}$,

$\pi'=\{\text{primes } p \mid p\notin\pi\}$, and

$p'=\{p\}'$.

Let χ be a character. We denote simply $e(\chi(1))$ by $e(\chi)$. If $e(\chi)=e$, then we say that χ is a character with total exponent e (shortly T-exponent e). All the other notation can be seen in [3] or [6].

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2. Groups with *c.d.π*. The following theorem is a slight extension of the Burnside's p^aq^b-Theorem, (see [3] 4.3.3).

Theorem 2.1. *Let G have c.d.π. If $|\pi|\leqq 2$, then G is solvable.*

Proof. Since any normal subgroup or factorgroup of G satisfies the same assumption, the theorem follows at once by induction on $|G|$ if G is not simple. So we may assume G is simple. Therefore we may also assume $p\in\pi\subseteq\{p, q\}$ and G has a nontrivial Sylow p-subgroup P. Choose $1\neq x\in Z(P)$. Let $1_G\neq\chi\in Irr(G)$. If $\chi(1)$ is a power of p, then the simplicity of G and Burnside's lemma (see [3] 4.3.1) imply $\chi(x)=0$. Thus by orthogonality relations,

$$0=\sum_{\chi\in Irr(G)}\chi(1)\chi(x)=1+q\alpha$$

where α is an algebraic integer. So $\alpha=-1/q$, which is clearly imposible.

There exists no extension of Theorem 2.1 to the case $|\pi|\geqq 3$ as $SL(2,5)$ shows.

The following results on groups with c.d.p' for a prime p are shown in [8] and [1].

Proposition A (N. Ito). *If G is a solvable group with c.d.p', then G has a normal abelian Sylow p-subgroup.*

Proposition B (P. Fong). *If G is a p-slovable group with c.d.p', then G has a normal abelian Sylow p-subgroup.*

The latter includes the former. We shall extend these propositions in Theorem 2.5. We start with some lemmas.

If a π-number n is also a π'-number, then $n=1$. Therefore the following lemma is immediate.

Lemma 2.2. *If G is a π'-group with c.d.π, then G is abelian.*

Lemma 2.3 (P. X. Gallagher [2], Theorem 8). *Suppose G is a π-separable group with a Hall π'-subgroup H. If the degree of any irreducible constituent of $(1_H)^G$ is a π-number, then $H\lhd G$.*

REMARK. In [2] the term "π-solvable" seems to be used in the sense of "π-separable".

The following lemma is proved by using the Schur-Zassenhaus Theorem, (see [3] 6.3.5).

Lemma 2.4. *If G is π-separable, then G possesses a Hall π'-subgroup.*

We are now ready to extend Proposition B. If G is a π-separable group with c.d.π, then G has a Hall π'-subgroup H by Lemma 2.4 and hence Lemma 2.3 is applicable. Therefore $H\lhd G$ and H is a π'-group with c.d.π. So H is abelian by Lemma 2.2. By combining Theorem 2.1 and Ito's Theorem we have:

Theorem 2.5. *Suppose G is π-separable when $|\pi|\geqq 3$. Then G has a normal abelian Hall π'-subgroup if and only if G has c.d.π.*

The following corollary is useful in the proof of Theorem I in section 3.

Corollary 2.6. *Let G have c.d.π. Suppose G is π-solvable when $|\pi|\geqq 3$.*

Then G is solvable.

Proof. By the theorem G has a normal abelian Hall π'-subgroup H. Then G/H is a π-solvable π-group, and hence G/H is solvable. Therefore G is also solvable.

Now it is clear the following corollary holds for subnormal subgroups of arbitrary groups.

Corollary 2.7. *Let G have c.d.π. Suppose G is π-separable when $|\pi|\geqq 3$. Then every subgroup of G has also c.d.π.*

Proof. Let G be as above. By the theorem G has a normal abelian Hall π'-subgroup H. Let K be a subgroup of G. Then $H\cap K$ is a normal abelian Hall π'-subgroup of K and hence the theorem implies the corollary.

3. Groups with *r.x.e* for π. In this section we shall prove Theorem I.

The following properties of the total exponent immediately follow from our definition.

Lemma 3.1. (1) *$e(m)\geqq 0$, and $e(m)=0$ if and only if $m=1$.*
(2) *$e(mn)=e(m)+e(n)$.*
In particular these yield:
(3) *When s divides t, $e(s)\leqq e(t)$, and the equality holds if and only if $s=t$.*

If G has $r.x.0$, then G has no nonlinear irreducible characters and hence G is abelian. We know that groups with $r.x.1$ are solvable ([7] Theorem 6.1), but groups with $r.x.2$ are not necessarily solvable. Indeed the simple group A_5, the alternating group on 5 letters, has character degrees 1, 3, 2^2, 5.

By using Frobenius Reciprocity Theorem, Clifford's Theorem and our definition, we have the following immediately.

Lemma 3.2. *Let N be subnormal in G where G has r.x.e for π. Then N has r.x.e for π.*

The following lemma will be useful in applying induction on the total exponent.

Lemma 3.3. *Let $N\lhd G$ where G has r.x.e for π. If G/N is nonabelian, then N has r.x.$(e-1)$ for π.*

Proof. By Lemma 3.2, it will be sufficient to show that N has no irreducible characters with T-exponent e. Assume that N has an irreducible character θ with $e(\theta)=e$. Let χ be an irreducible constituent of θ^G. Then $\theta(1)$ divides $\chi(1)$ and hence

$$e=e(\theta)\leqq e(\chi)\leqq e$$

for G has *r.x.e.* We have the equality throughout so that $e(\chi)=e$ and $\chi|_N=\theta\in Irr(N)$. Since G/N is nonabelian, there exists $\varphi\in Irr(G/N)$ such that $\varphi(1)>1$. Then $\varphi\chi\in Irr(G)$ (see [2] Theorem 2), and hence

$$e = e(\chi)<e(\varphi)+e(\chi) = e(\varphi\chi)\leqq e\,.$$

This is a contradiction.

We remark that in the proof of Lemma 3.3 above we obtained the following result.

Corollary 3.4. *Let $N\lhd G$ where G and N have r.x.e. Suppose $\theta\in Irr(N)$ with $e(\theta)=e$. If χ is an irreducible constituent of θ^G, then $e(\chi)=e$ and $\chi|_N=\theta\in Irr(N)$.*

The following proposition generalizes Lemma 2.7 in [5] however it will not be used in this paper.

Proposition 3.5. *Let $N\lhd G$ where G has c.d.π. Suppose G/N is a π'-group. Then we have:*

(1) *Any irreducible character of N is G-invariant and $\chi|_N\in Irr(N)$ for any $\chi\in Irr(G)$.*

(2) *If N has r.x.e for π, then so does G.*

Proof. Let $\chi\in Irr(G)$. By Clifford's Theorem, $\chi|_N=e\sum_{i=1}^{t}\theta_i$ where the θ_i are distinct irreducible constituents and $\chi(1)=et\theta_1(1)$. Then et is a π-number since G has *c.d.π*. Now et divides $|G\colon N|$ which is a π'-number. Thus we have $e=t=1$. Since χ is arbitrary, (1) and (2) follow from Frobenius Reciprocity Theorem.

Before going on to another result, we state here the result by Isaacs and Passman, which will be needed.

Lemma 3.6 ([5] Proposition 2.5). *Let $N\lhd G$ with G/N nilpotent. Suppose $\chi\in Irr(G)$ with $\chi|_N$ reducible. Then there exists a normal subgroup T of G of prime index such that $N\subseteq T$ and $\chi=\psi^G$ for some $\psi\in Irr(T)$.*

The following lemma generalizes Lemma 2.8 in [5].

Lemma 3.7. *Let $N\lhd G$ with G/N nilpotent. Let G have r.x.e for π and N have r.x.$(e-1)$ for π. If F is the inverse image of $\Phi(G/N)$ in G, then F has r.x.$(e-1)$ for π.*

Proof. $F\lhd G$ and thus by Lemma 3.2 F has *r.x.e* for π. Therefore it would be sufficient for our purpose to show that F has no irreducible character with T-exponent e. Suppose $\theta\in Irr(F)$ satisfies $e(\theta)=e$. Let χ be an irreducible constituent of θ^G. By Corollary 3.4, $e(\chi)=e$ and $\chi|_F$ is irreducible. Since N has $r.x.(e-1)$ for π, $\chi|_N$ is reducible, and by Lemma 3.6 there exists a subgroup T maximal in G and containing N with $\chi=\psi^G$ for some $\psi\in Irr(T)$. Therefore ψ is a constituent of $\chi|_T$ which is thus reducible. Consequently $\chi|_F$ must be reducible for $F\subseteq T$. This is a contradiction and the result follows.

The following lemma is a part of the result appearing in [6], which is extremely useful in proving our main theorems. We will call it Isaacs-Passman's Lemma in this paper.

Lemma 3.8 (Isaacs-Passman's Lemma). *Let E be a group such that $E''=1<E'$ and $E'\subseteq K$ for all K with $1<K\lhd E$. Then we have one of the following.*

Case P. (1) *E is a p-group for some prime p.*

(2) *$Z(E)$ is cyclic.*

(3) *Every nonlinear irreducible character has degree $|E: Z(E)|^{1/2}$.*

Case Q. (4) *E is a Frobenius group with a cyclic complement and elementary abelian q-group Q as kernel.*

(5) *Every nonlinear irreducible character has degree $|E: Q|$.*

(6) *For any $\lambda\in\hat{Q}$ and any $x\in E-Q$, there exists $\mu\in\hat{Q}$ with $\lambda=\mu^x\mu^{-1}$.*

Let N be normal and maximal with respect to G/N being nonabelian. We note that if G is solvable then $E=G/N$ satisfies of Isaacs-Passman's Lemma.

We are now ready for the proof of Theorem I.

Proof of Theorem I. We prove the result by induction on e. When $e=0$, the result is trivial. Suppose $e\geqq 1$. It will be sufficient to show that G has a normal series $G\rhd B_{e-1}\rhd A_{e-1}$ and there exists some prime $p_1\in\pi$ such that

(1)′ A_{e-1} has $r.x.(e-1)$ for π,

(2)′ G/B_{e-1} is a cyclic π_1-group where $\pi_1=\pi-\{p_1\}$,

(3)′ B_{e-1}/A_{e-1} is an elementary abelian p_1-group, and

(4)′ $e(|G: A_{e-1}|)\leqq 2e+1$.

We know that G is solvable by Corollary 2.6. We may assume G is nonabelian. Then there exists $N\lhd G$ which is maximal with G/N nonabelian. Now $E=G/N$ satisfies the hypotheses of Isaacs-Passman's Lemma. Thus E has a unique nonlinear irreducible character degree m, which is also a character degree of G. So m is a π-number with $e(m)\leqq e$, because G has $r.x.e$ for π. Since E is nonabelian, N has $r.x.(e-1)$ for π by Lemma 3.3.

We consider two cases according to Isaacs-Passman's Lemma, which we apply to E.

Case P. E is a p-group for some prime p. Then p divides m and thus $p\in\pi$. Let A_{e-1} be the inverse image of $\Phi(G/N)$ in G. By Lemma 3.7 A_{e-1} has $r.x.(e-1)$ for π, and satisfies (1)′. Since $Z(E)$ is cyclic and $|E: Z(E)|=m^2$,

$$\begin{aligned} e(|G\colon A_{e-1}|) &= e(|E\colon \Phi(E)|) \leqq e(|E\colon \Phi(E)\cap Z(E)|) \\ &= e(|E\colon Z(E)|)+e(|Z(E)\colon \Phi(E)\cap Z(E)|) \leqq 2e(m)+1 \leqq 2e+1 . \end{aligned}$$

Thus we get (4)$'$. Let $B_{e-1}=G$ and $p_1=p$. Then (2)$'$ and (3)$'$ hold, and the result follows for this case.

Case Q. E is a Frobenius group with a cyclic complement and elementary abelian q-group Q as kernel. Let K be the inverse image of Q in G. Since K has *r.x.e* by Lemma 3.2, we may consider the following two cases.

Case Q-1. K has *r.x.*$(e-1)$ for π. Let A_{e-1} be the inverse image of $\Phi(G/K)$ in G. Now $G/K\cong E/Q$ is a cyclic group of order m, therefore by Lemma 3.7 A_{e-1} has *r.x.*$(e-1)$ for π and satisfies (1)$'$. Since $|G\colon A_{e-1}|$ divides m, (4)$'$ follows for $e(m)\leqq e\leqq 2e+1$. Choose a prime divisor p_1 of $|G\colon A_{e-1}|$, which is a square-free π-number, and let B_{e-1} be the inverse image of a Sylow p_1-subgroup of G/A_{e-1} in G. Then (2)$'$ and (3)$'$ follow.

Case Q-2. K has *r.x.e* for π but not *r.x.*$(e-1)$ for π. Then there exists $\theta\in Irr(K)$ such that $e(\theta)=e$. By Corollary 3.4 θ is G-invaraiant. Let $g\in G-K$. For any $\mu\in\hat{Q}$, $\mu\theta\in Irr(K)$ and $e(\mu\theta)=e(\theta)=e$. Thus similarly $\mu\theta$ is G-invariant, so that

$$\theta\mu=(\theta\mu)^g=\theta^g\mu^g=\theta\mu^g$$

and $\theta=\theta\mu^g\mu^{-1}$. Hence θ vanishes off $\mathrm{Ker}(\mu^g\mu^{-1})$. By (6) of Isaacs-Passman's Lemma, for any character $\lambda\in\hat{Q}$ we can find a character $\mu\in\hat{Q}$ and an element $g\in G-K$ with $\lambda=\mu^g\mu^{-1}$. Thus θ vanishes Ker λ. Now $Q=K/N$ has a subgroup of index q. Let A_{e-1} be its inverse image in G. A_{e-1} is the kernel of $(1_{A_{e-1}})^K$ which is a sum of linear characters of Q. So θ vanishes off A_{e-1}. Let $\theta|_{A_{e-1}}=a\sum_{i=1}^t\varphi_i$ where φ_i are distinct. Then

$$a^2t=(\theta|_{A_{e-1}},\theta|_{A_{e-1}})_{A_{e-1}}=\frac{|K|}{|A_{e-1}|}(\theta,\theta)_K=q .$$

Hence $a=1$ and $t=q$. Thus $q=\theta(1)/\varphi(1)\in\pi$ and $\theta|_{A_{e-1}}$ is reducible. For any irreducible character of K with T-exponent e, similarly its restriction to K is reducible. Therefore we have (1)$'$. Let $B_{e-1}=K$, $p_1=q$ and $\pi_1=\pi-\{p_1\}$. Since q is relatively prime to $m=|E\colon Q|=|G\colon K|$, (2)$'$ and (3)$'$ are satisfied. Now

$$e(|G\colon A_{e-1}|)=e(|G\colon K|)+e(|K\colon Q|)=e(m)+1\leqq e+1\leqq 2e+1 ,$$

and hence (4)$'$ is also satisfied. This proves the theorem.

As consequences of Theorem I we have the following.

Corollary 3.9. *Assume that G satisfies the hypotheses of Theorem I. Then we have*:

(1) *G has the derived length* $\leqq 2e+1$, *and Sylow p-subgroup of G has the derived length* $\leqq e+1$.

(2) *G has a subnormal abelian subgroup* A_0 *with* $|G: A_0| \leqq r^{e(e+2)}$, *where r is the biggest prime of* $\pi(G) \cap \pi$.

(3) *If G has an abelian Hall* π*-subgroup, then G has a normal series*

$$G = A_e \triangleright A_{e-1} \triangleright \cdots \triangleright A_0$$

such that (i) A_i *has r.x.i for* π *and* (ii) A_i/A_{i-1} *is a cyclic* π*-group of square-free order, whose T-exponent* $\leqq i$.

Proof. (1) and (2) immediately follow from Theorem I. We consider (3). Any section of G which is a π-group must be abelian. Theorefore only Case Q–1 in the proof of Theorem I can occur. Hence the result follows.

The above (1) may be of interest as the analogy to the following result appearing in [4]. A Sylow p-subgroup of a solvable group G has the derived length $\leqq 2m+1$, where m is the biggest integer such that p^m divides $\chi(1)$ for some $\chi \in Irr(G)$.

Let G be a (not necessarily finite) group, and we suppose every irreducible $\boldsymbol{C}[G]$-module is of finite dimension over $\boldsymbol{C}$, where $\boldsymbol{C}$ is the field of complex numbers. Then we may use the terminology "*r.x.e* for π" as in the case of finite groups.

The following consequence of Theorem I generalizes Theorem I of [5].

Corollary 3.10. *Let G be (not necessarily finite) finitely generated group with r.x.e for* π. *Moreover suppose* $|\pi|$ *is finite when G is not finite. Then G has a normal series*

$$G = A_e \triangleright B_{e-1} \triangleright A_{e-1} \triangleright \cdots \triangleright B_0 \triangleright A_0$$

and there exists some prime $p_i \in \pi$ *for any i such that*

(1) A_0 *is abelian,*

(2) A_i/B_{i-1} *is a cyclic* π_i*-group where* $\pi_i = \pi - \{p_i\}$,

(3) B_{i-1}/A_{i-1} *is an elementary abelian* p_i*-group, and*

(4) $|A_i: A_{i-1}|$ *is a* π*-number with T-exponent* $\leqq 2i+1$.

In particular $|G: A_0|$ *is a* π*-number with T-exponent* $\leqq e(e+2)$ *and hence* $|G: A_0| \leqq r^{e(e+2)}$, *where* $r = \max(\pi(G) \cap \pi)$.

Proof. Let G be a finitely generated group which satisfies the above hypotheses. By the assumption there exists a prime r such that $r \geqq s$ for any $s \in \pi(G) \cap \pi$. There are only finitely many subgroups of G with index $\leqq r^{e(e+2)}$ by M. Hall's Theorem (see [9] p. 56 or [6] p. 901). Suppose that $L_1, L_2, \cdots, L_t$ are all of those which are nonabelian. Choose $x_i, y_i \in L_i$ with the commutator $z_i = [x_i, y_i] \neq 1$. By Passman's Theorem ([10] Theorem V), G is a subdirect product of finite groups. Thus we can find a normal subgroup N of finite index in G such that $z_i \notin N$ for $i=1, 2, \cdots, t$. Then G/N is a finite group with *r.x.e* for π and thus there exists a normal series

$$G = A_e \triangleright B_{e-1} \triangleright A_{e-1} \triangleright \cdots \triangleright B_0 \triangleright A_0 \triangleright N$$

such that (2), (3) and (4) hold, A_0/N is abelian and $|G: A_0| \leqq r^{e(e+2)}$ by Theorem I. By the choise of N, A_0 is abelian and hence the result is proved.

4. Large subnormal abelian subgroups.

In this section we shall prove Theorem II.

We note that the function f_s exists and satisfies $f_s(e) \leqq e(e+2)$ by Theorem I. Thus there exists f_n and clearly $f_n(e) \leqq f_s(e)$.

In order to improve the upper bounds, we start with lemmas which correspond to the results in [6]. The following lemma is due ultimately to Isaacs and Passman.

Lemma 4.1. *Let G have r.x.e. Suppose $N \triangleleft G$ with $E=G/N$ being as in Case P of Isaacs-Passman's Lemma. Let Z be the complete inverse image of $Z(E)$ in G. Let $\beta \in Irr(E)$ with $\beta(1)>1$. Then we have:*

(1) *Given any character $\varphi \in Irr(Z)$, if χ_1 is an irreducible constituent of φ^G and if χ_1 is an irreducbile constituent of $\chi\beta$, then*

$$e(\chi)+e(\chi_1) \geqq e(\beta)+e(t)+2e(\varphi)$$

where t is the number of distinct conjugates of φ.

(2) *Z has r.x.$[e-e(\beta)/2]$.*

(3) *Moreover if $e(\beta)$ is even, then G has a normal subgroup B with the following properties: $B>Z$, $e(|B: Z|)=1$ and B has r.x.$(e-e(\beta)/2)$.*

Proof. (1) Let χ be an irreducible constituent of φ^G. Then since $Z \triangleleft G$, $\chi|_Z = a\sum_{i=1}^t \varphi_i$ $\varphi_1=\varphi$. Let $\beta|_Z=\beta(1)\lambda$, where $\lambda \in \widehat{Z/N}$. Let $(\varphi\lambda)^G=\sum a_i\chi_i$. By the proof of Lemma 3.5 of Isaacs-Passman [6], $a_1at/\beta(1)=(\chi\beta, \chi_1)$, $\chi(1)=at\varphi(1)$ and $\chi_1(1)=a_1t\varphi(1)$. Hence

$$\chi(1)\chi_1(1) = a_1at^2\varphi(1)^2 = (\chi\beta, \chi_1)\beta(1)t\varphi(1)^2$$

and

$$e(\chi)+e(\chi_1) \geqq e(\beta)+e(t)+2e(\varphi)$$

as desired.

(2) Since G has *r.x.e*, $e(\chi)$ and $e(\chi_1)$ are $\leqq e$. By (1), therefore, $e(\varphi) \leqq e-e(\beta)/2$. Since φ is an arbitrary character of Z, Z has *r.x.* $[e-e(\beta)/2]$, and (2) follows.

(3) Let $e(\beta)$ be even. Since G/Z is a p-group for some prime p, there exists B such that $Z<B\lhd G$ and $|B:Z|=p$. We may show B has $r.x.(e-e(\beta)/2)$. Suppose that there exists an irreducible character θ of B with $e(\theta)>e-e(\beta)/2$. By (2) Z has $r.x.(e-e(\beta)/2)$ and hence $\theta|_Z$ is reducible. By Lemma 3.6 there exists $\varphi\in Irr(Z)$ with $\theta=\varphi^B$. So $e(\theta)=e(\varphi)+1$, and we have

$$e-e(\beta)/2\leqq e(\theta)-1 = e(\varphi)\leqq e-e(\beta)/2 .$$

Thus we have $e(\varphi)=e-e(\beta)/2$. Now φ has p conjugates in B. Hence if φ has t conjugates in G, we have $t\geqq p>1$ and $e(t)>0$. Thus by (1),

$$2e-e(\beta) = 2e(\varphi)\leqq 2e-e(\beta)-e(t)<2e-e(\beta) .$$

This is a contradiction. Therefore B has $r.x.(e-e(\beta)/2)$.

Lemma 4.2. *$f_s(0)=0$ and*

$$f_s(e)\leqq\max\{f_s(e-1)+e+1, f_s(e-(m+1)/2)+2m, f_s(e-n/2)+2n-1 \mid m \text{ is an odd integer with } 0<m\leqq e \text{ and } n \text{ is an even integer with } 0<n\leqq e\} .$$

Proof. A group with $r.x.0$ is abelian and hence $f_s(0)=0$. Let v be the right-hand side of the above inequality. The proof is by induction on $|G|$. We may assume that G is a nonabelian group with $r.x.e$ and that $e\geqq 1$. Since G is solvable, we can choose $N\lhd G$ with $E=G/N$ being a group as in Isaacs-Passman's Lemma.

We consider three cases according to the cases of the proof of Theorem I. First we consider the case Q-1.

Case Q-1. K has $r.x.(e-1)$, where K is as in the proof of Theorem I. Then K has a subnormal abelian subgroup A such that $e(|K:A|)\leqq f_s(e-1)$. Since $K\lhd G$ and $e(|G:K|)\leqq e$, A is a subnormal abelian subgroup of G such that

$$e(|G:A|) = e(|G:K|)+e(|K:A|)\leqq e+f_s(e-1)<v .$$

Case Q-2. K has $r.x.e$ but not $r.x.(e-1)$. Let A_{e-1} be as in the proof of that theorem. Then A_{e-1} is a subnormal subgroup with $r.x.(e-1)$ and with $e(|G:A_{e-1}|)\leqq e+1$. By induction A_{e-1} has a subnormal abelian subgroup A with $e(|A_{e-1}:A|)\leqq f_s(e-1)$. Therefore A is a subnormal abelian subgroup of G such that

$$e(|G:A|)\leqq e+1+f_s(e-1)\leqq v .$$

Case P. E is a p-group for some prime p. Let Z be the inverse image of $Z(E)$ in G. Let $\beta\in Irr(E)$ with $\beta(1)>1$. We know that $|G:Z|=\beta(1)^2$ and that $0<e(\beta)\leqq e$.

Moreover there exist two cases to consider.

Case P-1. $e(\beta)$ is odd. Then

$$[e-e(\beta)/2] = e-(e(\beta)+1)/2 \leqq e-1 .$$

By Lemma 4.1 (2), Z has $r.x.(e-(e(\beta)+1)/2)$. By induction Z has a subnormal abelian subgroup A with $e(|Z: A|) \leqq f_s(e-(m+1)/2)$, where $m=e(\beta)$. Thus A is a subnormal abelian subgroup of G with

$$e(|G: A|) \leqq 2m+f_s(e-(m+1)/2) \leqq v .$$

Case P-2. $e(\beta)$ is even. Then let B be as in Lemma 4.1 (3). Since B has $r.x.(e-e(\beta)/2)$ and $e(\beta) \geqq 2$, B has a subnormal abelian subgroup A with $e(|B: A|) \leqq f_s(e-n/2)$, where $n=e(\beta)$. Thus A is a subnormal abelian subgroup of G with

$$e(|G: A|) \leqq 2n-1+f_s(e-n/2) \leqq v .$$

In any case G has a subnormal abelian subgroup A with $e(|G: A|) \leqq v$, and hence $f_s(e) \leqq v$. This completes the proof of our lemma.

From the proof of Theorem A in [6], we have immediately (2) of the following lemma.

Lemma 4.3 (Isaacs-Passman). *For any prime p there exists $f_{(p)}$, which satisfies*

(1) $f_{(p)}(0)=0$, $f_{(p)}(1)=2$, $f_{(p)}(2)=4$ *and*

(2) $2e \leqq f_{(p)}(e)$

$\leqq \max\{f_{(p)}(e-(m+1)/2)+2m, f_{(p)}(e-n/2)+2n-1 \mid$ *m is an odd integer with $0<m \leqq e$ and n is an even integer with $0<n \leqq e\}$.*

The equality $f_{(p)}(2)=4$ of (1) is seen in [11], and the other equalities of (1) are seen in [6].

We remark that clearly $f_{(p)}(e) \leqq f_n(e) \leqq f_s(e)$ for any prime p.

Corollary 4.4. $f_{(p)}(e) = 2e$ *for* $e \leqq 1$.

$$f_{(p)}(e) \leqq 4e-[\log_2 8e] \qquad \text{for } e \geqq 2 .$$

Proof. By Lemma 4.3 (1), $f_{(p)}(0)=0$, $f_{(p)}(1)=2$ and $f_{(p)}(2)=4$, therefore by (2)

$$f_{(p)}(3) \leqq 4 \cdot 3-[\log_2 8 \cdot 3] .$$

Thus the result holds for $e \leqq 3$. We may suppose $e \geqq 4$. Our inequality will be proved by induction on e. By Lemma 4.3 (2), it would be sufficient to show the following two inequalities.

(i) If m is any odd integer with $0<m\leqq e$, then

$$f_{(p)}(e-(m+1)/2)+2m\leqq 4e-[\log_2 8e]\ .$$

(ii) If n is any even integer with $0<n\leqq e$, then

$$f_{(p)}(e-n/2)+2n-1\leqq 4e-[\log_2 8e]\ .$$

Proof of (i). Let m be as in (i). Then since $e\geqq 4$, $2\leqq e-(m+1)/2\leqq e-1$. Hence induction is applicable. Write

$$A=\{4e-[\log_2 8e]\}-\{f_{(p)}(e-(m+1)/2)+2m\}\ .$$

By induction,

$$\begin{aligned}A&\geqq 4e-[\log_2 8e]-\{4(e-(m+1)/2)-[\log_2 8(e-(m+1)/2)]+2m\}\\&=[\log_2 4(e-(m+1)/2)]-[\log_2 e]\ .\end{aligned}$$

Since $m\leqq e$, $4(e-(m+1)/2)\geqq 2(e-1)\geqq e$ for $e\geqq 4$. Therefore we get $A\geqq 0$, and hence (i) also follows.

Proof of (ii). Let n be as in (ii). Then $2\leqq e-n/2\leqq e-1$ for $e\geqq 4$. Thus by using induction we get

$$\begin{aligned}&\{4e-[\log_2 8e]\}-\{f_{(p)}(e-n/2)+2n-1\}\\&\geqq[\log_2 2(e-n/2)]-[\log_2 e]\geqq 0\ ,\end{aligned}$$

because $n\leqq e$, $2(e-n/2)\geqq e$. Thus (ii) is proved, and hence the result follows.

We will need the following elementary inequality in (3).

Lemma 4.5. (1) $[x]+[y]+1\geqq[x+y]$.

(2) $[x]-[y]\geqq[x-y]$.

(3) *We define a function z on all of nonnegative integers as follows.*

$$z(x)=\begin{cases}2x & \text{if } x=0 \text{ or } 1\ ,\\ 4x-[\log_2 8x] & \text{if } x\geqq 2\ .\end{cases}$$

Then we have

$$z(x+y)\geqq z(x)+z(y)\qquad \textit{for any } x, y\ .$$

and thus $z(\sum_{i=1}^r x_i)\geqq\sum_{i=1}^r z(x_i)$.

Proof. The inequalities (1) and (2) are well-known.

(3) By induction on r the last inequality follows from the first inequality. We consider three cases.

Case 1. $x\leqq 1$ and $y\leqq 1$. Then since $z(2)=4$, $z(x+y)\geqq z(x)+z(y)$.

Case 2. Either x or y is $\leqq 1$. We may assume that $x\geqq 2$ and $y=1$. Then we have

$$z(x+y)-z(x)-z(y) = z(x+1)-z(x)-2$$
$$= 2+[\log_2 x]-[\log_2(x+1)] \geqq [\log_2(4x)/(x+1)] > 0 .$$

The first inequality follows from (2) and the last inequality follows from the fact that $(4x)/(x+1)>2$ for $x\geqq 2$.

Case 3. $x\geqq 2$ and $y\geqq 2$. Then we have

$$\begin{aligned} &z(x+y)-z(x)-z(y) \\ &= 3+[\log_2 x]+[\log_2 y]-[\log_2(x+y)] \\ &\geqq 2+[\log_2 xy]-[\log_2(x+y)] \\ &\geqq [\log_2(4xy)/(x+y)] > 0 . \end{aligned}$$

The first (resp. the second) inequality follows from (1) (resp. (2)) and the last inequlity follows from the fact that $(4xy)/(x+y)>2$ for $x\geqq 2$ and $y\geqq 2$.

Now we are ready to prove our second main theorem.

Proof of Theorem II. By the first remark in this section, we may prove (1) and, (2)$'$: $f_s(e)\leqq e(e+3)/2$.
We discuss (2)$'$ first. Use induction on e. By Lemma 4.2, it would be sufficient to show that the following inequalities:

(i) $f_s(e-1)+e+1\leqq e(e+3)/2$ for $e\geqq 1$.

(ii) If m is any odd intetger with $0<m\leqq e$, then

$$f_s(e-(m+1)/2)+2m\leqq e(e+3)/2 .$$

(iii) If n is any even integer with $0<n\leqq e$, then

$$f_s(e-n/2)+2n-1\leqq e(e+3)/2 .$$

Proof of (i). By induction,

$$f_s(e-1)+e+1\leqq (e-1)\{(e-1)+3\}/2+e+1 = e(e+3)/2 .$$

Proof of (ii). Let m be as in (ii). Since $0\leqq e-(m+1)/2\leqq e-1$, induction is applicable. Thus

$$\begin{aligned} &f_s(e-(m+1)/2)+2m \\ &\leqq \frac{1}{2}\Big(e-\frac{m+1}{2}\Big)\Big(e-\frac{m+1}{2}+3\Big)+2m \\ &= \frac{1}{2}e(e+3)+\frac{1}{8}(m+1)^2-\frac{1}{4}(2e+3)(m+1)+2m \\ &\leqq \frac{1}{2}e(e+3) , \end{aligned}$$

because m and e are integers with $0<m\leqq e$.

Proof of (iii). Let n be as in (iii). Since $0<e-n/2\leqq e-1$, by induction we can prove (iii) similarly.

Next we discuss (1). By Lemma 4.3 and the remark following that lemma, it would be sufficient to show that

$$(1)' \quad f_n(0)=0,\ f_n(1)\leqq 2 \text{ and}$$
$$f_n(e)\leqq 4e-[\log_2 8e] \qquad \text{for } e\geqq 2\,.$$

Now $f_n(0)=0$ is trivial. Let G be a nilpotent group with *r.x.e*, and write $G=P_1\times P_2\times\cdots\times P_r$, where P_i is a Sylow p_i-subgroup of G. Suppose that P_i has $r.x.e_i$ but not $r.x.(e_i-1)$. Then G has $r.x.\sum_{i=1}^r e_i$ and hence $\sum_{i=1}^r e_i\leqq e$. We define $z(x)$ as in Lemma 4.5 (3). By Corollary 4.4 we know $f_{(p_i)}(e_i)\leqq z(e_i)$. Thus P_i has a subnormal abelian subgroup A_i with $e(|P_i:A_i|)\leqq z(e_i)$. If $A=A_1\times A_2\times\cdots\times A_r$, then A is a subnormal abelian subgroup of G, and

$$e(|G:A|)=\sum_{i=1}^r e(|P_i:A_i|)\leqq\sum_{i=1}^r z(e_i)\leqq z(\textstyle\sum_{i=1}^r e_i)$$
$$\leqq z(e)\,.$$

The second and the last inequalities follow from Lemma 4.5 (3). We have, therefore, $f_n(e)\leqq z(e)$, and prove $(1)'$. This completes the proof of Theorem II.

5. A remark on a result of Issacs-Passman. A group G is said to have *r.b.n* (*representation bound n*) if $\chi(1)\leqq n$ for any $\chi\in Irr(G)$.

The following result appears as Theorem D of [6]. Let h_2 be the function with the following property. If G is a solvable group with *r.b.n*, then G has a subnormal abelian subgroup of index $\leqq h_2(n)$. Moreover we assume that h_2 is the smallest such function. Then

$$h_2(n)\leqq n^{3/2\ \log_2 2n}\,.$$

In this section we remark that the above upper bound may be slightly improved as follows.

Theorem 5.1. $h_2(n)\leqq n^{\log_2 2n}$.

Proof. If G is abelian, the result is trivial, so we may assume that G is nonabelian. As usual, choose $N\lhd G$ with G/N being a group of Isaacs-Passman's Lemma. There are three cases in the proof of Theorem D of [6].

Case P. G has a normal subgroup of index $\leqq n^2$ with $r.b.(n/2)$. ·········(1)

Case Q-1. G has a normal subgroup Q of index $\leqq n$ with $r.b.(n/2)$. ······(2)

Case Q-2. G has a normal subgroup Q of index $\leqq n$ with *r.b.n* but not $r.b.(n/2)$, and Q/N is an abelian Sylow q-subgroup of G/N for some prime q. In this case, moreover, it is known that if $\theta\in Irr(Q)$ with $\theta(1)>n/2$ then θ vanishes off N. We consider this case more precisely.

Now Q/N has a subgroup of index q. Let D be its inverse image in G. Then θ vanishes off D and $D \lhd Q$. Let $\theta|_D = a\sum_{i=1}^{t}\varphi_i$. Then

$$a^2 t = (\theta|_D, \theta|_D) = \frac{|Q|}{|D|}(\theta, \theta) = q\,.$$

Hence $a=1$ and $t=q$. Thus

$$q \leqq at\varphi_1(1) = \theta(1) \leqq n\,,$$

because Q has *r.b.n.* So

$$|G: D| \leqq nq \leqq n^2$$

and $\varphi_1(1) = \theta(1)/q \leqq n/2$. Since θ is an arbitrary character of Q with $\theta(1) > n/2$, D has *r.b.*$(n/2)$ by Frobenius Reciprocity Theorem. Thus we have:

G has a subnormal subgroup D of index $\leqq n^2$ with *r.b.*$(n/2)$. ············(3)

We now apply induction on n. (1), (2) and (3) imply that G has a subnormal subgroup M of index $\leqq n^2$ with *r.b.*$(n/2)$. By induction M has a subnormal abelian subgroup A with

$$|M: A| \leqq (n/2)^{\log_2 n}\,.$$

Then A is subnormal in G with

$$|G: A| \leqq n^2(n/2)^{\log_2 n} = n^{\log_2 2n}\,,$$

and the result follows.

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