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SMALL SUBMODULES IN A PROJECTIVE MODULE AND SEMI-T-NILPOTENT SETS

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Let R be a ring with identity element and $(R)_I$ the ring of column-finite matrices over R with infinite degree I. N. Jacobson proposed to determine the Jacobson radical $J((R)_I)$ of $(R)_I$ in his book [5]. Many algebraists have been working on this problem. P.M. Patterson [8], N.E. Sexauer and J.E. Warnock [9] showed $J((R)_I)=(J(R))_I$ if and only if $J(R)$ is right T-nilpotent (cf. [4], Corollary 1 to Proposition 1). On the other hand, W. Liebert [6] gave an exact form of $J((R)_I)$ if R is domain and R. Slover [10,11] and R. Ware and J. Zelmanowitz [12] obtained an exact form of elements in $J((R)_I)$, which involved all results above.

In this note, we shall first give all types of small submodules in a free R-module $M=\sum_I \oplus u_\alpha R$. Since $J((R)_I)$ is determined by small submodules in M, we can obtain their results similarly to [12] and give another forms by means of locally, right semi-T-nilpotent sets of small submodules.

Finally, we shall give a characterization of right perfect module P by means of a structure of $(S)_I/J((S)_I)$, where $S=\mathrm{End}_R(P)$.

1 Jacobson radicals

Throughout we shall assume that R is a ring with identity element and every module is a unitary right R-module. Let A be an R-module and B a submodule of A. B is called *small in A* if a fact: $A=B+T$ for a submodule T of A implies $A=T$. Let $\{M_\alpha\}_I$ be a set of R-modules and $M=\sum_I \oplus M_\alpha$. We put $S_M=\mathrm{End}_R(M)$. We assume the elements in S_M operate on M from the left side. Furthermore, we can express them as the column-summable matrices with entries in $\mathrm{Hom}_R(M_\sigma, M_\tau)$. If $M_\sigma=R$ for all σ, S_M is the ring $(R)_I$ of column-finite matrices over R.

Let $M=\sum_I \oplus u_\alpha R$ and S a small submodule in M. Then $S\subseteq\sum_I \oplus u_\alpha J(R)$. In order to determine a type of S, we shall define a set of right semi-T-nilpotent, right ideals. Let $\{A_\alpha\}_K$ be a set of right ideals in R and K an infinite set. If $\{A_\alpha\}_K$ satisfies the following condition, we call $\{A_\alpha\}_K$ a *right* *semi-T-nilpotent set,* (see a vanishing set of ideals in [12]).

For any countable subset $\{A_{\alpha_i}\}_{i=1}^{\infty}$ *of* $\{A_\alpha\}_K$ *and* $\{a_i \mid \in A_{\alpha_i}\}_{i=1}^{\infty}$, *there exists n, depending on* $\{a_i\}$, *such that* $a_n a_{n-1} \cdots a_1 = 0$.

Let $\{b_\sigma\}_K$ be a set of elements in R. If $\{b_\sigma R\}_K$ is a right semi-T-nilpotent set, we call $\{b_\sigma\}_K$ a *right semi-T-nilpotent set.* If we allow $\alpha_i = \alpha_j$ for $i \neq j$, we call $\{A_\alpha\}$ or $\{b_\alpha\}$ a *right T-nilpotent set.* If $A_\alpha = A$ for all α, then the above concept coincides with one of the usual T-nilpotency.
If K is a finite set, we understand $A_\alpha = 0$ for almost all α, then $\{A_\alpha\}_K$ is always a semi-T-nilpotent set, but not a T-nilpotent set. Now, we shall state the theorem which is substantially due to [12].

Theorem 1 ([12], Theorem 1). *Let* $M = \sum_I \oplus u_\alpha R$ *be a free R-module with infinite basis* u_α *and S a submodule of M. Then the following statements are equivalent.*

1) *S is small in M.*

2) *Let* $p_\alpha \colon M \to u_\alpha R$ *be the projection of M onto* $u_\alpha R$ *and* $A_\alpha = p_\alpha(S)$. *Then* $A_\alpha \subseteq J(R)$ *and* $\{A_\alpha\}_I$ *is a right semi-T-nilpotent set.*

3) *There exists a right semi-T-nilpotent set* $\{A_\alpha\}_I$ *of right ideals in* $J(R)$ *such that* $S \subseteq \sum \oplus u_\alpha A_\alpha$.

We shall prove Theorem 1 in more general forms. First, we shall generalize the concept of right semi-T-nilpotent set of right ideals. Let $\{Q_\alpha\}_I$ be an infinite set of R-modules and $\{S_\beta \mid \subseteq Q_\beta\}_{\beta \in K \subseteq I}$ an infinite set of R-submodules. We take a countable subset $\{Q_{\alpha_i}\}$ of $\{Q_\alpha\}_K$ and a set of homomorphisms $f_i \colon Q_{\alpha_i} \to Q_{\alpha_{i+1}}$ such that $f_i(Q_{\alpha_i}) \subseteq S_{\alpha_{i+1}}$. If for any element t in Q_{α_1} there exists n, depending on t and $\{f_i\}$, such that $f_n f_{n-1} \cdots f_1(t) = 0$, then we call $\{f_i\}$ a locally (semi)-T-nilpotent set of homomorphisms. If for any countable subset $\{Q_{\alpha_i}\}$ and any set of homomorphisms f_i as above, $\{f_i\}$ is always locally (semi-)T-nilpotent, then we call $\{S_\alpha\}_K$ a *locally (right) semi-T-nilpotent set of submodules.*

The following lemma is obtained by [12].

Lemma 1. *Let P be projective. Then* $J(S_P) = \{f \in S_P \mid f(P)$ *is small in* $P\}$.

Lemma 2 ([4], Proposition 1). *Let P be R-projective and S an R-submodule of P. If* $\mathrm{Hom}_R(P, S) \subseteq J(S_P)$, *S is small in P, where* $S_P = \mathrm{End}_R(P)$.

Proof. We assume $P = S + T$ for some submodule T. Then we have a diagram:

$$\begin{array}{ccccccccc} 0 \to & S \cap T & \to & S & \xrightarrow{\nu} & S/S \cap T & \to 0 \\ & & & \nwarrow h & & \wr\!\wr\, f & \\ & & & & & P/T & \\ & & & & & \uparrow \nu' & \\ & & & & & P & \end{array}$$

Since P is projective, we have $h: P\to S$ such that $\nu h=f\nu'$. Hence, $S=h(P)+S\cap T$ and so $P=h(P)+T$. On the other hand, $h(P)$ is small in P from the assumption and Lemma 1. Therefore, $P=T$.

The following proposition implies 1)→2)→3) in Theorem 1.

Proposition 1. *Let $\{M_\alpha\}_I$ be a set of finitely generated R-modules and S a small submodule of $M=\sum_I\oplus M_\alpha$. Then $\{S_\alpha=p_\alpha(S)\}_I$ is a right semi-T-nilpotent set of submodules, where $p_\alpha: M\to M_\alpha$ is the projection.*

Proof. We may assume I is a well ordered, infinite set. Let $\{S_{\alpha_i}\}_1^\infty$ be any subset of $\{S_\alpha\}_I$ and $\{f_i: M_{\alpha_i}\to S_{\alpha_{i+1}}\}_1^\infty$ a given set. Let $\{m_\alpha^{(1)}, m_\alpha^{(2)}, \cdots, m_\alpha^{n_\alpha}\}$ be a generator of M_α. Then for $m_{\alpha_i}^{(k)}$ there exists $s_i^{(k)}$ in S such that

$$s_i^{(k)}=s_{(\beta(i,k),1)}+s_{(\beta(i,k),2)}\cdots+\underbrace{f_i(m_{\alpha_i}^{(k)})}_{\alpha_{i+1}}+\cdots+s_{(\beta(i,k),n_k)} \qquad (*),$$

where $s_{(\beta(i,k),j)}\in S_{(\beta(i,k),j)}$.
Hence, we may assume

$$s_i^{(k)}\in\sum_{j=1}^{m_i}\oplus S_{\beta(i,j)} \qquad \text{for all } k=1, 2, \cdots, n \qquad (**).$$

1 Special case. First, we assume that $\{\beta(i,j)\}_{i=1}^{\infty}\,{}_{j=1}^{m_i}\equiv\{1, 2, \cdots, n, \cdots\}$ in (**) and $\alpha_1<\alpha_2\leqslant m_1<\alpha_3\leqslant m_2<\alpha_4\leqslant\cdots$. We put $M'_{\alpha_i}=\{m_{\alpha_i}+f_i(m_{\alpha_i})\mid m_{\alpha_i}\in M_{\alpha_i}\}\subseteq M_{\alpha_i}\oplus M_{\alpha_{i+1}}$ and $M'=\sum_{\{\alpha\notin\alpha_i\}}M_\alpha+\sum_1^\infty M'_{\alpha_i}+S$. We shall show $M=M'$. For $m_{\alpha_1}^{(k)}$ we have

$$\begin{aligned}M'\supset M'_{\alpha_1}+S&\ni m_{\alpha_1}^{(k)}+f_1(m_{\alpha_1}^{(k)})-s_1^{(k)}\\&=-s_1^{(1,k)}-s_2^{(1,k)}-\cdots+(m_{\alpha_1}^{(k)}-s_{\alpha_1}^{(1,k)})-\cdots-\underbrace{0}_{\alpha_2}-\cdots-s_{m_1}^{(1,k)}\,.\end{aligned}$$

Hence, $m_{\alpha_1}^{(k)}\equiv s_{\alpha_1}^{(1,k)}\pmod{M'}$ and $s_{\alpha_1}^{(1,k)}\in S_{\alpha_1}$. Therefore, $(M_{\alpha_1}+M')/M'=(S_{\alpha_1}+M')/M'$. Since S_{α_1} is small in M_{α_1}, $(S_{\alpha_1}+M')/M'$ is small in $(M_{\alpha_1}+M')/M'$. Accordingly, $M_{\alpha_1}\subseteq M'$. Repeating those arguments we have $M=M'$. Since S is small in M, $M=\sum_{\alpha\notin\{\alpha_i\}}\oplus M_\alpha\oplus\sum_{i=1}^\infty M'_{\alpha_i}$. Hence, there exists n such that $f_nf_{n-1}\cdots f_1(m_{\alpha_1})=0$ for $m_{\alpha_1}\in M_{\alpha_1}$ (see [1], Lemma 9).

2 General case. Since $\beta(i,j)$'s in (**) are countable, we may assume $\{\alpha, \beta(i,j)\}=\{1=\alpha_1, 2, \cdots, n\cdots\}$ after rearranging the order of indices. We shall denote the new index of α_i by $\sigma(\alpha_i)$, namely $\sigma(\alpha_1)=1$. Then

$$s_1^{(k)}(=s_{\alpha_{i2}}^{(k)})=s_1^{(\alpha_{i2},k)}+s_2^{(\alpha_{i2},k)}+\cdots+\underbrace{f_1(m_{\alpha_1}^{(k)})}_{\sigma(\alpha_2)}+\cdots+s_{n(\alpha_{i2},k)}^{(\alpha_{i2},k)}\,.$$

Since $\{f_i\}$ is infinite, there exists $i_3 > i_2$ such that $n(\alpha_{i_3}, k') \geqslant \sigma(\alpha_{i_3}) > \operatorname{Max}_k(n(\alpha_{i_2}, k))$. Repeating those arguments, we obtain $\sigma(\alpha_1) = 1 < \sigma(\alpha_{i_2}) \leqslant \operatorname{Max}_k(n(\alpha_{i_2}, k)) < \sigma(\alpha_{i_3}) \leqslant \operatorname{Max}_{k'}(n(\alpha_{i_3}, k')) < \sigma(\alpha_{i_4}) < \cdots$ and $1 = i_1 < i_2 < i_3 < \cdots$. Put $g_{j-1} = f_{i_j-1} \cdots f_{i_{j-1}}$: $M\alpha_{i_{j-1}} \to S\alpha_{i_j}$, $(g_1 = f_1)$ and consider a countable subset $\{S\sigma(\alpha_{i_j})\}_{j=1}^{\infty}$. Since $g_{j-1}(M\alpha_{i_{j-1}}) \subseteq f_{i_j-1}(M\alpha_{i_j-1})$, $g_{j-1}(m^{(k)}_{\alpha_{i_{j-1}}}) \in p_{\sigma(\alpha_{i_j})}((\sum_{t=1}^{n(\alpha_{i_j},k')} \oplus M_t) \cap S)$. Hence, $\{S\alpha(\alpha_{i_j})\}$ and $\{g_{j-1}\}$ satisfy the conditions of Special case 1. Accordingly, $0 = g_n g_{n-1} \cdots g_1(m_{\alpha_1}) = f_{i_{n+1}-1} \cdots f_1(m_{\alpha_1})$ for some n.

From a special type of the above proof, we have

Corollary 1. *Let $\{N_\alpha\}_I$ be a set of R-modules and $\{T_\alpha | \subseteq N_\alpha\}_I$ a set of submodules. If $\sum_I \oplus T_\alpha$ is a small submodule of $\sum_I \oplus N_\alpha$, then $\{T_\alpha\}_I$ is a locally right semi-T-nilpotent set of submodules.*

Proposition 2. *Let $\{N_\alpha\}_I$ be a set of R-modules and $\{T_\alpha | \subseteq N_\alpha\}_I$. We put $N = \sum_I \oplus N_\alpha$, $T = \sum_I \oplus T_\alpha$ and $S_N = \mathrm{End}_R(N)$. Then $J(S_N) \supseteq \mathrm{Hom}_R(N, T)$ if and only if $\{T_\alpha\}_I$ is a locally right semi-T-nilpotent set of submodules and $\mathrm{Hom}_R(N_\alpha, T_\alpha) \subseteq J(S_{N\alpha})$.*

Proof. We assume $J(S_N) \supseteq \mathrm{Hom}_R(N, T)$. Let $\{N_{\alpha_i}\}$ and $\{f_i: N_{\alpha_i} \to T_{\alpha_{i+1}}\}$ be given sets. We may assume $\alpha_j = j$ for all j. Then

$$\begin{pmatrix} 0 & & & & & 0 \\ f_1 & 0 & & & & \\ & f_2 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & f_n & 0 & \\ 0 & & & & \ddots & \ddots \end{pmatrix}$$

is in $J(S_N)$ from the assumption. Hence, $\{f_i\}$ is locally right T-nilpotent. It is clear that $J(S_{N\alpha}) = e_\alpha J(S_N) e_\alpha \supseteq e_\alpha \mathrm{Hom}_R(N, T) e_\alpha = \mathrm{Hom}_R(N_\alpha, T_\alpha)$, where e_α is the projection of N onto N_α. Conversely, we shall show that $C_{\sigma\tau} = \mathrm{Hom}_R(N_\tau, T_\sigma)$'s satisfy the conditions 1)$\sim$3) in [4], Lemma 5. 1) and 3) are clear from the assumptions and 2) is clear. We note that in the proof of [4], Lemma 5 we only used a fact that $C_{\sigma\tau} \mathrm{Hom}_R(N_\varepsilon, N_\tau) \subseteq C_{\sigma\varepsilon}$. Hence, $J(S_N) \supseteq \mathrm{Hom}_R(N, T)$ from [4], Lemma 5.

The following corollary implies 3)$\to$1) in Theorem 1 and is the converse of Corollary 1 above in a restricted case.

Corollary 1 ([4], Theorem 3). *Let $\{P\}_I$ be a set of R-projectives and $\{S_\alpha | \subseteq P_\alpha\}$ a set of R-submodules. Then $\sum_I \oplus S_\alpha$ is small in $\sum_I \oplus P$ if and only if $\{S_\alpha\}_I$ is a locally, right semi-T-nilpotent set of small submodules S_α in P_α* (see Remark 2 below).

Corollary 2. *Let* $\{P_\alpha\}_I$ *and* $\{S_\alpha\}_I$ *be as above. Then for any set* $\{Q_\beta\}_K$ *such that* $Q_\beta \overset{\varphi_\beta}{\approx} P_{\alpha(\beta)} \in \{P_\alpha\}_I \sum_K \oplus \varphi_\beta^{-1}(S_{\alpha(\beta)})$ *is always small in* $\sum_K \oplus Q_\beta$ *if and only if* $\{S_\alpha\}_I$ *is a locally right T-nilpotent set of submodules.*

Proof. It is enough to show that S_α is small in P_α if $\{S_\alpha\}_I$ is locally right T-nilpotent. $A_\alpha = \mathrm{Hom}_R(P_\alpha, S_\alpha)$ is a right ideal in S_{P_α}. Let $f \in A_\alpha$, then $f' = \sum_{n=0}^{\infty} f^n \in S_{P_\alpha}$ from the assumption. Hence, $(1-f)f'=1$ and so $A_\alpha \subseteq J(S_{P_\alpha})$. Therefore, S_α is small in P_α from Lemma 2.

Corollary 3 ([4, 8, 9, 12, 13]). *Let* $M = \sum_I \oplus u_\alpha R$. *Then* $J(M)$ *is small in* M *if and only if* $J(R)$ *is right T-nilpotent.*

Corollary 4. *Let M be an R-module. If* $\{A_m\}_{m \in M_0}$ *is a right semi-T-nilpotent set of right ideals in* $J(R)$, *then* $\sum_{M_0} mA_m$ *is a small submodule in M. Conversely, if M is projective and S is a small submodule in M, then there exists a semi-T-nilpotent set* $\{A_m\}_{M_0}$ *of right ideals in* $J(R)$ *such that* $S \subseteq \sum_{M_0} mA_m$, *where* M_0 *is any set of generators of M* (see Remark 3).

Proof. Consider an epimorphism φ: $P = \sum_{M_0} \oplus u_m R \to M$; $\varphi(u_m) = m$. Since $\sum \oplus u_m A_m$ is small in P from Corollary 2, $\varphi(\sum \oplus u_m A_m) = \sum mA_m$ is small in M (see [4]). Conversely, we assume M is projective. M is a direct summand of P and so $\varphi i = 1_M$ for monomorphism i. $i(S)$ is also small in P and hence, there exists a right semi-T-nilpotent set $\{A_m\}_{M_0}$ of right ideals in $J(R)$ such that $i(S) \subseteq \sum \oplus u_m A_m$ from Theorem 1. Therefore, $S = \varphi i(S) \subseteq \sum_{M_0} mA_m$.

Corollary 5. *Let P be a projective R-module. Then the following statements are equivalent.*

1) $\{J(P)\}$ *is itself a locally, right T-nilpotent set of submodules.*
2) $J(S_P)$ *is locally right T-nilpotent.*
3) *Any set of small submodules in P is a locally, right T-nilpotent set.*

Proof. 1)→2). $\mathrm{Hom}_R(P, J(P)) \subseteq J(S_P)$ from the proof of Corollary 2. Hence, $\mathrm{Hom}_R(P, J(P)) = J(S_P)$ is locally right T-nilpotent.
2)→3). Let $\{S_i\}_1^\infty$ be a set of small submodules in P. Then $\mathrm{Hom}_R(P, S_i) \subseteq J(S_P)$ from Lemma 1. Hence, $\{S_i\}$ is a locally, right-T-nilpotent set.
3)→1). First, we shall show that the union of small submodules $\{S_\alpha\}_I$ is also small in P. Consider the natural epimorphism: $\sum_I \oplus P_\alpha \to P \to 0$, $P_\alpha = P$. Then $\sum_I \oplus S_\alpha$ is small in $\sum_I \oplus P_\alpha$ from 3) and Corollary 1. Hence, $\bigcup_I S_\alpha$ is small in P. It is easily seen that pR is small in P, where $p \in J(P)$. Therefore, $J(P)$ is

small in P from the above.

The proof above shows that if $J(P)$ is not small in P, then there exists a locally, right non-T-nilpotent set of small submodules $\{S'\}_I$.

Now, we shall give a general form of Theorem 1.

Theorem 1′. *Let $\{P_\alpha\}_I$ be a set of R-projectives. Let $M=\sum_I \oplus P_\alpha$ and S a submodule of M. Then the following statements are equivalent.*

1) *S is small in M.*

2) *Let p_α: $M \to P_\alpha$ be the projection of M onto P_α and $S_\alpha = p_\alpha(S)$. Then $\{S_\alpha\}_I$ is a locally, right semi-T-nilpotent set of small submodules.*

3) *There exists a locally, right semi-T-nilpotent set $\{S_\alpha\}_I$ of small submodules S_α in P_α such that $S \subseteq \sum_I \oplus S_\alpha$.*

Proof. 2)→3)→1). It is is clear from Corollary 1 to Proposition 2.
1)→2). We shall prove it in a general form:

Lemma 3. *In Proposition 1, we assume every M_α is a summand of a direct sum Q_α of finitely generated R-modules $M_{\alpha\beta}$. Then the statement in Proposition 1 is valid.*

Proof. Put $Q_\alpha = \sum_{\beta \in I_\alpha} \oplus M_{\alpha\beta}$ and $M^* = \sum_I \oplus Q_\alpha$. Then M is a summand of M^*. Let i be the injection of M into M^*. Then $i(S)$ is small in M^* and $\sum_I \sum_{I_\alpha} \oplus p_{\alpha\beta}(i(S)) \supset \sum_I \oplus i(S_\alpha)$ and $i(S_\alpha) \subset \sum_{I_\alpha} \oplus p_{\alpha\beta}(i(S))$. Now, $\{p_{\alpha\beta}(i(S))\}_{I, I_\alpha}$ is a locally semi-T-nilpotent set from Proposition 1. Let $\{M_{\alpha_i}\}_1^\infty$ and f_i: $M_{\alpha_i} \to S_{\alpha_{i+1}}\}_1^\infty$ be given sets. Then we can extend f_i to f_i': $Q_{\alpha_i} \to i(S_{\alpha_{i+1}})$ by sending a direct complement to zero. We shall denote f_i' by a column-finite matrix $(a^{(i)}_{\sigma\tau})$, where $a^{(i)}_{\sigma\tau} \in \mathrm{Hom}_R(M_{\alpha_i\tau}, p_{\alpha_{i+1}\sigma}(i(S))$. Let m be in M_{α_1} and $i(m) = \sum_{j=1}^t m_{\alpha_1\beta_j}$, $m_{\alpha_1\beta_j} \in M_{\alpha_1\beta_j}$. Then $f_1(m) = f_1'(i(m)) = \sum_{j=1}^t \sum_k a^{(1)}_{\sigma_k\beta_j}(m_{\alpha_1\beta_j})$, where $a^{(1)}_{\sigma_k\beta_j} = 0$ for almost all k.

$$f_2 f_1(m) = f_2' f_1'(i(m)) = \sum_{j=1}^t \sum_{k'} \sum_k a^{(2)}_{k'\sigma_k} a^{(1)}_{\sigma_k\beta_j}(m_{\alpha_1\beta_j})$$

Since $p_{\alpha\beta}(i(S))$ is locally semi-T-nilpotent, we obtain n such that $f_n f_{n-1} \cdots f_1(m) = 0$ from Konig Graph Theorem.

Let M be an R-module. We can correspond (not necessarily unique) any element in $S_M = \mathrm{End}_R(M)$ to a column-finite matrix $(a_{\sigma\tau})$ over R by making use of generators.

Theorem 2. *Let P be R-projective. Then $f \in J(S_P)$ if and only if f corresponds to a matrix above such that $\{\sum_\tau a_{\sigma\tau} R\}_\sigma$ is a right semi-T-nilpotent set of right ideals in $J(R)$ (cf. [12]).*

It is clear from Corollary 4 to Proposition 2.

Theorem 2′ *Let* $\{P_\alpha\}_I$ *be a set of R-projective modules,* $P=\sum_I \oplus P_\alpha$ *and* $S_P=\mathrm{End}_R(P)$. *Then*

$$J(S_P)=\bigcup_{\{S_\sigma\}}\begin{pmatrix} [P_1, S_1][P_2, S_1]\cdots[P_\sigma, S_1]\cdots \\ [P_1, S_2][P_2, S_2]\cdots[P_\sigma, S_2]\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ [P_1, S_\sigma][P_2, S_\sigma]\cdots[P_\sigma, S_\sigma]\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{pmatrix}$$

where matrices are column-summable, $\{S_\alpha \mid \subseteq P_\alpha\}$ *runs through all locally, right semi-T-niplotent sets of small submodules* S_α *in* P_α *and* $[P_\sigma, S_\tau]=\mathrm{Hom}_R(P_\sigma, S_\tau)$.

Proof. It is clear $J(S_P)\supseteq([P_\sigma, S_\tau])$ from Lemma 1 and Corollary 1 to Proposition 2. Let $f\in J(S_P)$ and $f=(f_{\sigma\tau}), f_{\sigma\tau}\in[P_\tau, P_\sigma]$. Then $p_\sigma(f(P))=\sum_\tau f_{\sigma\tau}(P_\tau)\ (=S_\sigma)$. Since $f(P)$ is small in P, $\{S_\sigma\}_I$ is a right semi-T-nilpotent set of submodules from Therom 1′.

Corollary 1.

$$J((R)_I)=\bigcup_{\{A_\sigma\}}\begin{pmatrix} A_1, A_1, \cdots \\ A_2, A_2, \cdots \\ \cdots\cdots \\ A_\sigma, A_\sigma, \cdots \end{pmatrix}$$

where $\{A_\sigma\}_I$ *runs through all the right semi-T-nilpotent sets of right ideals in* $J(R)$ *and all permutations* $\{A_{\pi(\sigma)}\}$ *of* $\{A_\sigma\}$.

Corollary 2 ([10, 11, 12]). *Let* $(a_{\sigma\tau})$ *be in* $(R)_I$. *Then the following statements are equivalent.*

1) $(a_{\sigma\tau})\in J((R)_I)$.

2) $\{\sum_\tau a_{\sigma\tau}R\}_\sigma$ *is a right semi-T-nilpotent set.*

3) *Any set* $\{a_{\sigma\tau}\}$ *is a right semi-T-nilpotent set, where almost all σ's are distinct.*

Proof. 3)→2) We can prove it from Konig Graph Theorem. Other implications are clear from Corollary 1.

By $J_f((R)_I)$ we shall denote the set of matrices in $(J(R))_I$ almost all of whose rows are zero. On the other hand, we denote a small submodule $\sum_1^n \oplus u_{\alpha_i}J(R)$ in $M=\sum_I \oplus u_\alpha R$ by $J(\alpha_1, \alpha_i, \cdots, \alpha_n)(M)$. Then we have

Corollary 3. *The following statements are equivalent.*

1) $J((R)_I)=J_f((R)_I)$

2) *Every small submodule in M is contained in some* $J(\alpha_1, \alpha_2, \cdots, \alpha_n)(M)$.

3) *There are no non-trivial, infinite right semi-T-nilpotent sets of elements in* $J(R)$ (cf. [6], Theorem 1).

Proof. 1)$\rightarrow$2). We assume 2) is not satisfied. Then there exists a small submodule S in M which is not contained in any $J(\alpha_1', \alpha_2', \cdots, \alpha_n')(M)$. Hence, for a suitable sequence $\{\alpha_i, \alpha_i \neq \alpha_j \text{ for } i \neq j\}$, there exist elements s_i^* in S such that

$$s_i^* = \cdots + u_{\alpha_i} s_{\alpha_i i} + \cdots , \; s_{\alpha_i i} \neq 0 \in J(R) .$$

We define f in S_M by setting

$$f(u_i) = s_i^* \text{ and } f(u_\alpha) = 0 \text{ for } u_\alpha \notin \{u_i\}_1^\infty .$$

Since $f(M) \subseteq S$, $f \in J((R)_I) = J_f((R)_I)$ from lemma 1. Therefore, $f(M) \subseteq J(\beta_1, \beta_2, \cdots, \beta_m)(M)$, which is a contradiction. Other implications are clear.

REMARKS 1. If $\{T_\alpha\}_I$ is locally, right T-nilpotent in Proposition 2, $J(S_N) \supseteq \mathrm{Hom}_R(N, T)$ (see the proof of Corollary 2 to Proposition 2).

2. Let Z be the ring of integers and p prime. Put $N_\alpha = Z_{p^\infty}$ for all α in Proposition 2. Then $\mathrm{End}_Z(Z_{p^\infty}) = \hat{Z}_p$; the ring of p-adic completions and S_N is the ring of column-summable matrices $(a_{\sigma\tau})$ over $\hat{Z}_p$. Furthermore, $J(S_N) = \{(a_{\sigma\tau}) \mid a_{\sigma\tau} \in p\hat{Z}_p\}$ from [1], Theorem 9 and Proposition 10. Let $A_n = \{a \in Z_{p^\infty} \mid ap^n = 0\}$. Then $\{T_\alpha\}$ is a locally semi-T-nilpotent if and only if $T_\alpha = A_{n(\alpha)}$ for almost all α. On the other hand, $\mathrm{Hom}_Z(Z_{p^\infty}, A_n) = 0$. Hence, $\mathrm{Hom}_Z(N, T) = 0$ if $T_\alpha = A_{n(\alpha)}$ for all α and so $J(S_N) \neq \bigcup_T \mathrm{Hom}_Z(N, T)$ (cf. Theorem 2′). Furthermore, let $A = \sum_1^\infty \oplus A_i$ and $M = \sum_1^\infty \oplus Z_{p^\infty} \xrightarrow{\varphi} Z_{p^\infty}$ the natural epimorphism. Then $\varphi(A) = Z_{p^\infty}$ and so A is not small in M (cf. Corollary 1 to Proposition 2). Hence, every small submodule in M is of a type $A^{(n)} = \{m \in M \mid mp^n = 0\}$ (use the similar argument above and the proof of Proposition 1).

3. Let Q be the rationals. Then Q is an injective and flat Z-module. It is clear that Z is a small submodule in Q. Put $A = \sum_1^\infty \oplus Q_i$: $Q_i = Q$ and φ: $A \rightarrow Q$ by setting $\varphi(q_i) = (1/i)q_i$; $q_i \in Q_i$. Since $\mathrm{Hom}_Z(Q, Z) = 0$, $\{Z\}$ is a locally T-nilpotent set of small submodules. However, $\varphi(\sum \oplus Z) = Q$ and so $\sum \oplus Z$ is not small in A (see Corollary 1 to Proposition 2). Furthermore, $J(Z) = 0$ and so Z is not of a form in Corollary 4 to Proposition 2.

4. If R is a right perfect ring, $MJ(R) = J(M)$ is a unique maximal one among small submodules in an R-module M. Hence, every set of small submodules is a locally, right semi-T-nilpotent set and so almost results above are trivially valid without any assumptions:finitely generated and projective.

5. It is clear that

$$J((R)_I)=\begin{pmatrix} A_1, A_1\cdots \\ A_2, A_2\cdots \\ \cdots\cdots\cdots \\ A_\sigma, A_\sigma\cdots \\ \cdots\cdots\cdots \end{pmatrix}$$

for a right semi-T-nilpotent set of right ideals A_σ if and only if $A_\sigma=J(R)$ for all σ and $J(R)$ is right T-nilpotent.

2 Perfect modules

We shall add here a characterization for a finitely generated projective module to be perfect.

Theorem 3. *Let P be a finitely generated projective module and $M=\sum_1^\infty \oplus P$. Then P is perfect if and only if $S_M/J(S_M)$ is a regular ring in the sense of Von Neumann and every idempotent in $S_M/J(S_M)$ is lifted to S_M* (cf. [3], Theorem 1).

Proof. If P is perfect, the statements are obtained by [7]. Conversely, Let $S_P=\mathrm{End}_R(P)$. Then $S_M=(S_P)_I$. Let $\bar{e}$ be an idempotent in $(J(S_P))_I/J(S_M)$. We may assume e is idempotent in $(J(S_P))_I$ from the assumtion. Since $J(S_P)=\mathrm{Hom}_R(P, J(P))$ from Lemma 1, $e(M)\subseteq\sum\oplus J(P)=\sum\oplus PJ(R)=MJ(R)$. Hence, $e(M)=e(M)J(R)$. Therefore, $e=0$. On the other hand, $S_M/J(S_M)$ is regular and so $J(S_M)=(J(S_P))_I$. Accordingly, $J(S_P)$ is right T-nilpotent and $S_P/J(S_P)$ is semi-simple artinian from [4], Corollary to Lemma 2. Thus, $P=\sum_1^n\oplus P_i$ and $\mathrm{End}_R(P_i)$ is a local ring, which implies P is perfect from [2], Theorem 6.

Corollary 1. *Let R be a semi-simple artinian ring if and only if $J(R)$ contains no non-trivial right semi-T-nilpotent sets and $S_M/J(S_M)$ is a regular ring, where $M=\sum\oplus u_iR$.*

Proof. If $J(R)$ contains no right semi-T-nilpotent sets, then $J(S_M)=J_f(S_M)$. For any elements $(a_{\sigma\tau})$, $(b_{\sigma\tau})$ in S_M, $(a_{\sigma\tau})\equiv(b_{\sigma\tau})\pmod{J(S_M)}$ implies $a_{\sigma\tau}=b_{\sigma\tau}$ for almost all σ. Let aE be in S_M and $a\in R$, where E is the identity matrix in S_M. Then there exists $(b_{\sigma\tau})$ in S_M such that $aE(b_{\sigma\tau})aE\equiv aE \pmod{J(S_M)}$. Hence, there exists σ such that $ab_{\sigma\sigma}a=a$ from the above. Therefore, R is regular and $J(R)=0$. Since $(R)_I=(R)_I/(J(R))_I$ is regular, R is artinian from [4], Corollary to Lemma 2.

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