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LIMIT THEOREMS FOR CERTAIN PARABOLIC EQUATION OF BOLTZMANN TYPE AND THEIR ASSOCIATED MARKOV PROCESSES*

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1. Introduction

Consider the equation of Boltzmann type for a gas model with discrete velocity states:

$$
\text{(I)}\qquad \begin{cases} \dfrac{\partial}{\partial t}u(t,i,x)=d_i\dfrac{\partial^2}{\partial x^2}u(t,i,x)+v_i\dfrac{\partial}{\partial x}u(t,i,x) \\ \qquad\qquad\qquad +\sum_{j,k=1}^{n}a_{j,k}^{i}u(t,j,x)u(t,k,x)\,, \\ u(0,i,x)=u(i,x)\,, \\ \qquad\qquad\qquad i\in\langle 1,n\rangle,\ t\geqq 0,\ x\in R^1\,, \end{cases}
$$

where $\langle 1,n\rangle$ denotes the set $\{1,2,\cdots,n\}$,

$$
(1.1)\qquad \begin{cases} d_1\geqq 0,\ \cdots,\ d_n\geqq 0\,, \\ v_i \text{ and } a_{j,k}^{i} \text{ are real numbers,} \end{cases}
$$

and $\{a_{j,k}^{i}\}$ satisfies:

$$
(1.2)\qquad \begin{cases} \sum_{i=1}^{n}a_{j,k}^{i}=0 & j,k\in\langle 1,n\rangle\,, \\ a_{j,k}^{i}\geqq 0 & \text{if } i \text{ equals neither } j \text{ nor } k. \end{cases}
$$

This equation has been considered by Mimura [6], Yamaguchi [12], Conner [1] and Kolodner [4].

In this paper, we shall first discuss the asymptotic behavior of the solution $u(t,i,x)$ of equation (I) and, second, construct the temporally inhomogeneous Markov process (X_t) associated with an appropriate solution $u(t,i,x)$ of equation (I) and, finally, investigate the limiting property of the process (X_t).

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When we consider a Markov process associated with equation (I), the second condition of (1.2) is strengthened as follows:

(1.3) $\qquad a_{j,k}^{i} \geqq 0 \qquad \text{if } i \neq j\,.$

The following typical example of equation (I) is taken from Conner [1]. Let the velocity space be $\langle 1, n\rangle$, and assume that molecules travel R^1 with velocity v_i if they are in the state i. Let $u(t, i, x)dx$ be the density of molecules at time t with state i whose positions belong to dx. Let $\nu_{j,k}=\nu_{k,j}\geqq 0$ be the collision rate between molecules of state j and k over the epoch dt, and $0\leqq\Gamma_{j,k}^{i}\leqq 1$ be the probability for a molecule of state j to have its velocity state scattered from j to i through collision with a molecule of state k. Then we have,

$$\begin{aligned}&\frac{\partial}{\partial t}u(t, i, x)+v_i\frac{\partial}{\partial x}u(t, i, x)\\ &=\sum_{j,k=1}^{n}(\Gamma_{j,k}^{i}\nu_{j,k}u(t, j, x)u(t, k, x)-\Gamma_{i,k}^{j}\nu_{i,k}u(t, i, x)u(t, k, x))\,.\end{aligned}$$

Conner defined

$$B_{j,k}^{i} = \frac{1}{2}\nu_{j,k}(\Gamma_{j,k}^{i}+\Gamma_{k,j}^{i}-(\delta_{i,k}+\delta_{i,j})\sum_{l=1}^{n}\Gamma_{j,k}^{l}) = \frac{1}{2}\nu_{j,k}(\Gamma_{j,k}^{i}+\Gamma_{k,j}^{i}-\delta_{i,j}-\delta_{i,k})$$

and represented the above equation in the form

$$\frac{\partial}{\partial t}u(t, i, x) = -v_i\frac{\partial}{\partial x}u(t, i, x)+\sum_{j,k=1}^{n}B_{j,k}^{i}u(t, j, x)u(t, k, x)\,.$$

This $B_{j,k}^{i}$ is symmetric in j and k for each i and satisfies condition (1.2) but does not satisfy condition (1.3). However if one defines

$$\begin{aligned}a_{j,k}^{i} &= \Gamma_{j,k}^{i}\nu_{j,k} & i \neq j\,,\\ &= \Gamma_{i,k}^{i}\nu_{i,k}-\sum_{l=1}^{n}\Gamma_{i,k}^{l}\nu_{i,k} & i = j\,,\end{aligned}$$

then the equation also is of the form

$$\frac{\partial}{\partial t}u(t, i, x) = -v_i\frac{\partial}{\partial x}u(t, i, x)+\sum_{j,k=1}^{n}a_{j,k}^{i}u(t, j, x)u(t, k, x)\,,$$

with the $a_{j,k}^{i}$ satisfying condition (1.2) and (1.3).

Throughout the paper, we shall discuss the case when the initial condition $u(0,i,x)$ is periodic with respect to x. Let $u(0,i,x)$ be a continuous, nonnegative periodic function on R^1 with period ω. It then follows that the solution $u(t,i,x)$ of equation (I) is nonnegative, continuous and periodic with period ω on R^1 as well as $u(0,i,x)$, and the equation

(1.4) $$\sum_{i=1}^{n}\int_0^{\omega}u(t, i, x)\,dx = \sum_{i=1}^{n}\int_0^{\omega}u(0, i, x)\,dx \qquad t \geqq 0\,,$$

is satisfied whenever the solution exists.

We shall also discuss separately the spatially homogeneous case, i.e.,

$$\text{(II)}\qquad \begin{cases} \dfrac{d}{dt}u(t,i)=\sum_{j,k=1}^{n} a_{j,k}^{i}u(t,j)u(t,k)\,, \\ u(0,i)=u(i)\,. \end{cases}$$

Jenks [3] has studied the asymptotic behavior of the equation (II). Hence in this case, we confine ourselves to the problem on Markov processes.

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2. Summary

We are concerned with spatially homogeneous cases in section 3 and section 4, and spatially inhomogeneous cases from section 5 to 9.

In section 3, we construct a Markov process corresponding to the equation (II) when a nonnegative solution $u(t,i)$ is given. McKean [5] introduced a class of Markov processes associated with certain nonlinear parabolic equations, and Tanaka [9] and Ueno [11] continued its investigation in somewhat special cases. The process we consider here is the Markov process associated with (II) in the sense of McKean [5]. In section 4, we investigate the asymptotic behavior of the process.

In section 5, we study the equation (I) in a neighborhood of the fixed point $(u_1^0, \cdots, u_n^0)$, i.e., the solution of equation $\sum_{j,k=1}^{n} a_{j,k}^{i}u_j^0u_k^0=0$, $i\in\langle 1, n\rangle$. We take $(u_1^0, \cdots, u_n^0)$ to be a probability vector. Suppose that the initial data $u(0,i,x)$ is continuous and periodic in x with period ω and $\frac{1}{\omega}\sum_{i=1}^{n}\int_0^{\omega} u(0,i,x)dx=1$. Then if $\max_{i,x}|u(0,i,x)-u_i^0|$ is sufficiently small, equation (I) has a global solution $u(t,i,x)$ which converges to u_i^0 exponentially fast. In section 6, under certain additional conditions, we shall show that the above result is valid for every positive initial data $u(0,i,x)$. In section 7, we construct a Markov process corresponding to the equation (I) with an appropriate solution $u(t,i,x)$ being fixed. This is also a Markov process of the type introduced by McKean [5]. Ogawa [7] [8] has proved a similar result. In section 8, we study the asymptotic behavior of the Markov process. In section 9, we treat the case $n=2$ in more details.

3. Markov processes corresponding to the equation (II)

Let $u(t,i)$ be a solution of equation (II):

$$\text{(II)} \qquad \begin{cases} \dfrac{d}{dt}u(t,i) = \sum_{j,k=1}^{n} a_{j,k}^{i} u(t,j)u(t,k)\,, \\ u(0,i) = u(i)\,, \end{cases}$$

where, $\{a_{j,k}^{i}\}$ satisfies (1.2) and (1.3).

It is easy to see that if $u(0,1)\geqq 0, \cdots, u(0,n)\geqq 0$, then $u(t,1)\geqq 0, \cdots, u(t,n)\geqq 0$ for every $t\geqq 0$. (see. Yamaguchi [12]). The equation (II) implies that $\dfrac{d}{dt}\sum_{i=1}^{n} u(t,i)=0$, and $\sum_{i=1}^{n} u(t,i)=$ constant. Therefore the solution of (II) remains nonnegative and bounded.

In this paper, Ω denotes the set of probability n-vectors, i.e.,

$$\Omega = \{(u_1, \cdots, u_n) \colon u_1 \geqq 0, \cdots, u_n \geqq 0, \sum_{i=1}^{n} u_i = 1\}\,.$$

Lemma 3.1. *There exists* $(u_1^0, \cdots, u_n^0)$ *in* Ω *satisfying*

$$\text{(3.1)} \qquad \sum_{j,k=1}^{n} a_{j,k}^{i} u_j^0 u_k^0 = 0 \qquad i \in \langle 1, n \rangle\,.$$

Proof. Choose $L>0$ large enough to satisfy

$$u_i + \frac{1}{L}\sum_{j,k=1}^{n} a_{j,k}^{i} u_j u_k \geqq 0$$

for any $(u_1, \cdots, u_n)$ in Ω and i in $\langle 1, n\rangle$.

Then the mapping

$$\Omega \ni (u_i) \to (u_i + \frac{1}{L}\sum_{j,k=1}^{n} a_{j,k}^{i} u_j u_k) \in R^n$$

is continuous, and the range is contained in Ω. Therefore, applying Brouwer's fixed point theorem, there exists (u_i^0) in Ω such that

$$u_i^0 + \frac{1}{L}\sum_{j,k=1}^{n} a_{j,k}^{i} u_j^0 u_k^0 = u_i^0 \qquad i \in \langle 1, n \rangle\,,$$

$$\text{i.e.,} \quad \sum_{j,k=1}^{n} a_{j,k}^{i} u_j^0 u_k^0 = 0 \qquad i \in \langle 1, n \rangle\,.$$

Fix a nonnegative solution $u(t,i)$ of the equation (II), and put

$$\text{(3.2)} \qquad \begin{cases} a_{i,j}(t) = \sum_{k=1}^{n} a_{i,k}^{j} u(t,k)\,, \\ A(t) = (a_{i,j}(t))\,. \end{cases}$$

Secondly, construct the fundamental solution $(U_{s,t}(i,j))_{s\leqq t}$ of the differential equation,

$$(3.3)\qquad \begin{cases} \dfrac{d}{dt}\phi(t) = \phi(t)A(t)\,, \\ \phi(t) = (\phi(t, 1), \cdots, \phi(t, n))\,. \end{cases}$$

That is $(U_{s,t}(i,j))$ satisfies

$$(3.4)\qquad \begin{cases} U_{s,s} = I = (\delta_{i,j})\,, \\ \dfrac{d}{dt}U_{s,t} = U_{s,t}A(t)\,, \\ U_{s,t}U_{t,u} = U_{s,u} \qquad s\leqq t\leqq u\,. \end{cases}$$

Since $u(t)$ is a solution of equation (II), and since $A(t)$ is defined by (3.2), it follows that $U_{s,t}$ depends only on $u(s)$ and $t-s$. Hence we may define

$$(3.5)\qquad P^{u(s)}_{t-s}(i,j) = U_{s,t}(i,j)\,.$$

Theorem 3.1. *$(P^{u(s)}_{t}(i,j))$ has the following properties:*

$$(3.6)\qquad \sum_{k=1}^{n} P^{u(s)}_{t-s}(i,k)P^{u(t)}_{u-t}(k,j) = P^{u(s)}_{u-s}(i,j) \qquad s\leqq t\leqq u\,,$$

$$(3.7)\qquad P^{u(s)}_{0}(i,j) = \delta_{i,j}\,,$$

$$(3.8)\qquad P^{u(s)}_{t-s}(i,j)\geqq 0 \qquad s\leqq t\,,$$

$$(3.9)\qquad \sum_{j=1}^{n} P^{u(s)}_{t-s}(i,j) = 1 \qquad \textit{for any } i\in\langle 1, n\rangle\,,$$

(3.10) *If $\phi(t, i)$ is a solution of* (3.3), *then*

$$\phi(t,j) = \sum_{k=1}^{n} \phi(s,k)P^{u(s)}_{t-s}(k,j)\,,$$

especially, $u(t,j)=\sum_{k=1}^{n} u(s,k)P^{u(s)}_{t-s}(k,j)\,.$

Proof. (3.6), (3.7) and (3.10) are properties of $(U_{s,t})$. Because $\{A(t)\}_{t\geqq 0}$ are nonnegative offdiagonal matrices, (3.8) is obvious. (3.9) follows from $\dfrac{d}{dt}\sum_{i=1}^{n} \phi(t,i)=0$.

By this theorem, there exists a (in general) temporally inhomogeneous Markov process $(X_t)_{t\geqq 0}$, whose transition probability is $U_{s,t}(i,j)=P^{u(s)}_{t-s}(i,j)$, and with state space $\langle 1, n\rangle$. Let $P_{s,i}(\cdot)$ denote the measure of the process starting at i at time s.

4. Asymptotic behavior of the process (X_t) associated with equation (II)

In this section, we consider the limit distribution of the process (X_t). We shall discuss the following three cases separately.

(A.1) There exists $u_1^0>0, \cdots, u_n^0>0$ and $m>0$ such that

$$\lim_{t\to\infty} u(t, i) = mu_i^0 \qquad i\in\langle 1, n\rangle,$$

$$\text{and } \sum_{1=1}^{n} u_i^0 = 1 .$$

(A.2) There exists $(u_1^0, \cdots, u_n^0)$ in Ω and $m>0$ such that

$$\lim_{t\to\infty} u(t, i) = mu_i^0 \qquad \text{for any } i\in\langle 1, n\rangle \text{ and } \int_0^\infty ||u(t)-mu^0||dt<\infty .$$

For $x=(x_1, \cdots, x_n)$ in R^n, $||x||$ means $\sum_{i=1}^{n}|x_i|$.

(A.3) There exists $T>0$ such that $u(t, k)=u(t+T, k)$ for any k and $t\geqq 0$ and there exists t_0 in $[0, T]$ such that $u(t_0, k)>0$ for any k.

4.1. Case (A.1)

$$a_{i,j} = m\sum_{k=1}^{n} a_{i,k}^{j} u_k^0 \tag{4.1}$$

satisfies

$$\begin{cases} \sum_{j=1}^{n} a_{i,j} = 0 , \\ \sum_{i=1}^{n} u_i^0 a_{i,j} = 0 , \\ a_{i,j}\geqq 0 \qquad \text{if } i\neq j . \end{cases} \tag{4.2}$$

Therefore the matrix $A=(a_{i,j})$ is a generator of a continuous time Markov chain having a positive invariant measure $(u_1^0, \cdots, u_n^0)$. Therefore, if necessary, changing the order of rows and columns, we may assume that A has the form

$$\begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_l \end{pmatrix} \tag{4.3}$$

where $A_1, \cdots, A_l$ are irreducible, and A_p is *a* $k_p\times k_p$ square matrix $(1\leqq p\leqq l)$ and $\sum_{p=1}^{l} k_p=n$.

In general, for a nonnegative offdiagonal matrix A, there exists $\lambda>0$ such that $A_\lambda=A+\lambda I$ is a nonnegative matrix. A nonnegative offdiagonal matrix A is said to be *irreducible* if for any i and j in $\langle 1, n\rangle$, there exists $k(i,j)>0$ such that the (i,j) component of $A_\lambda^{k(i,j)}$ is positive. Obviously, if A is irreducible, then e^{hA} $(h>0)$ is a positive matrix, and the converse is also true.

$A(t)$ can be written in the form

$$(4.4)\qquad \begin{pmatrix} A_1(t) & & & 0 \\ & A_2(t) & & \\ & & \ddots & \\ 0 & & & A_l(t) \end{pmatrix}$$

because $a_{i,j}=0$ implies $a_{i,j}(t)=0$. Therefore, we may consider each block separately. Hence we first assume that A is irreducible.

Put

$$(4.5)\qquad \mathcal{F} = \{\phi = (\phi_1, \cdots, \phi_n) : \sum_{i=1}^{n} \phi_i = 0\} .$$

Obviously, for ϕ in R^n, it holds that

$$(4.6)\qquad ||\phi e^{tA}|| \leqq ||\phi|| \qquad t \geqq 0 .$$

For $h>0$, let $e^{hA}=(\alpha_{i,j}(h))$, $c(h)=\min\limits_{1\leqq i,j\leqq n} \alpha_{i,j}(h)>0$. If ϕ belongs to $\mathcal{F}$, then

$$\begin{aligned}(4.7)\qquad ||\phi e^{hA}|| &= \sum_{j=1}^{n} |\sum_{i=1}^{n} \phi_i \alpha_{i,j}(h)| \\ &= \sum_{j=1}^{n} |\sum_{i=1}^{n} \phi_i(\alpha_{i,j}(h)-c(h))| \\ &\leqq \sum_{j=1}^{n} \sum_{i=1}^{n} |\phi_i| (\alpha_{i,j}(h)-c(h)) \\ &= (1-nc(h))||\phi|| .\end{aligned}$$

These two results implies that there exists $K>0$ and $\rho>0$ satisfying

$$(4.8)\qquad ||\phi e^{tA}|| \leqq K e^{-\rho t} ||\phi|| \qquad \phi \in \mathcal{F} .$$

We can write equation (3.3) as

$$\frac{d}{dt}\phi(t) = \phi(t)A + \phi(t)\,(A(t)-A) ,$$

or equivalently

$$(4.9)\qquad \phi(t)-\phi(0)e^{tA} = \int_0^t \phi(s)\,(A(s)-A)e^{(t-s)A} ds .$$

Taking norms both sides of (4.9),

$$||\phi(t)-\phi(0)e^{tA}|| \leqq \int_0^t ||\phi(s)\,(A(s)-A)|| K e^{-\rho(t-s)} ds \rightarrow 0 .$$

This implies

$$(4.10)\qquad \lim_{t\to\infty} \phi(t) = \lim_{t\to\infty} \phi(0)e^{tA} = \phi(0) \begin{pmatrix} u_1^0, & \cdots, & u_n^0 \\ \cdots & \cdots & \cdots \\ u_1^0, & \cdots, & u_n^0 \end{pmatrix} .$$

In the general case, put I_p be the set $\langle k_1+\cdots+k_{p-1}+1,\ k_1+\cdots+k_p \rangle$. Then

$$\lim_{t\to\infty} e^{tA_p} = \frac{1}{\sum\limits_{j\in I_p} u_j^0} \begin{pmatrix} u^0_{k_1+\cdots+k_{p-1}+1}, & \cdots, & u^0_{k_1+\cdots+k_p} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ u^0_{1_1+\cdots+k_{p-1}+1}, & \cdots, & u^0_{k_1+\cdots+k_p} \end{pmatrix}$$

Therefore, for i in I_p and j in I_q, we get

$$\lim_{t\to\infty} P_{0,i}(X_t = j) = \begin{cases} 0 & p \neq q\,, \\ \dfrac{u_j^0}{\sum\limits_{k\in I_p} u_k^0} & p = q\,. \end{cases} \tag{4.11}$$

4.2. Case (A.2)

Let $A=(a_{i,j})$ be the matrix defined by (4.1). By the general theory of Markov chain, there always exists

$$\Pi = \lim_{t\to\infty} e^{tA}\,. \tag{4.12}$$

Rewrite (4.9),

$$\begin{aligned} \phi(t) = \phi(0)e^{tA} &+ \int_0^t \phi(s)\,(A(s)-A)\Pi\, ds \\ &+ \int_0^t \phi(s)\,(A(s)-A)\,(e^{(t-s)A}-\Pi)ds\,. \end{aligned} \tag{4.13}$$

Therefore, by the Assumption (A.2)

$$\begin{aligned} &||\phi(t)-\phi(0)e^{tA} - \int_0^t \phi(s)\,(A(s)-A)\Pi\, ds|| \\ &\qquad \leqq \int_0^t ||\phi(s)\,(A(s)-A)||\cdot||e^{(t-s)A}-\Pi||ds \to 0 \end{aligned}$$

where, for $A=(a_{i,j})$, $||A||$ means $\max\limits_{i\in\langle 1,n\rangle} (\sum\limits_{j=1}^{n} |a_{i,j}|)$.

Since $\int_0^\infty ||u(t)-mu^0||dt<\infty$, the integral $\int_0^\infty \phi(s)\,(A(s)-A)\Pi\, ds$ converges. Therefore,

$$\lim_{t\to\infty} \phi(t) = \phi(0)\Pi + \int_0^\infty \phi(s)\,(A(s)-A)\Pi\, ds\,.$$

Since we can write $\phi(t)=\phi(0)P^{u(0)}_t$, this means that

$$\lim_{t\to\infty} P^{u(0)}_t = (c^{u(0)}(i,j))$$

exists. Hence, we obtain

$$\lim_{t\to\infty} P_{0,i}(X_t = j) = c^{u(0)}(i,j) \qquad i,j\in\langle 1, n\rangle\,. \tag{4.14}$$

4.3. Case (A.3)

$U_{t_0t_0+2T} = U_{t_0,t_0+T}U_{t_0+T,t_0+2T} = (U_{t_0,t_0+T})^2$ implies that

$$(4.15)\qquad U_{t_0,t_0+mT}=(U_{t_0,t_0+T})^m \qquad m\geqq 1 .$$

Let

$$(4.16)\qquad U_{t_0,t_0+T}=(c_{i,j}),$$

then

$$(4.17)\qquad \begin{cases}\sum_{j=1}^{n} c_{i,j}=1, \\ \sum_{i=1}^{n} u(t_0, i)c_{i,j}=u(t_0,j) \qquad j\in\langle 1, n\rangle, \\ c_{i,j}\geqq 0,\end{cases}$$

are satisfied. Changing, if necessary, the order of rows and columns of $A(t_0)$, we may write

$$(4.18)\qquad A(t_0)=\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ * & & A_l\end{pmatrix}$$

where $A_1, \cdots, A_l$ are irreducible. By the continuity of $A(t)$ it follows that there exists $\rho>0$, $\mu>0$ and $\varepsilon>0$ such that

$$(4.19)\qquad A(t)\geqq\begin{cases}\rho A(t_0)-\mu I & t_0\leqq t\leqq t_0+\varepsilon \\ -\mu I & t_0+\varepsilon\leqq t\leqq t_0+T .\end{cases}$$

This implies that $U_{t_0,t_0+T}\geqq e^{\varepsilon\rho A(t_0)-\mu T}$. Therefore, $U_{t_0,t_0+T}=\begin{pmatrix} U_1 & & 0 \\ & \ddots & \\ * & & U_l\end{pmatrix}$, where $U_1, \cdots, U_l$ are positive matrices. From this we shall show that U_{t_0,t_0+T} must be of the form

$$(4.20)\qquad U_{t_0,t_0+T}=\begin{pmatrix} U_1 & & 0 \\ & \ddots & \\ 0 & & U_l\end{pmatrix}.$$

We prove the case $l=2$, since the general case follows by induction. We write $U_{t_0,t_0+T}=\begin{pmatrix} U_1, & 0 \\ U, & U_2\end{pmatrix}$. If $U\neq 0$, the Perron-Frobenius root ρ_2 of U_2 is strictly smaller than 1. This contradicts (4.17).

We first consider the case that U_{t_0,t_0+T} is a single positive matrix. (4.15) and (4.17) implies

$$(4.21)\qquad \lim_{h\to\infty} U_{t_0,t_0+hT}=\begin{pmatrix} u(t_0) \\ \vdots \\ u(t_0)\end{pmatrix}\frac{1}{\sum_{k=1}^{n} u(0, k)} .$$

Therefore, using $P^{u(t_0+hT)}_t=P^{u(t_0)}_t$ and $u(t_0)P^{u(t_0)}_t=u(t_0+t)$, we have

$$\lim_{h\to\infty} P^{u(0)}_{t+t_0+hT}=\lim_{h\to\infty} P^{u(0)}_{t_0} U_{t_0,t_0+hT} P^{u(t_0+hT)}_t$$

$$= P_{t_0}^{u(0)} \lim_{h\to\infty} U_{t_0,t_0+hT} P_t^{u(t_0)}$$

$$= \begin{pmatrix} u(t_0+t) \\ \vdots \\ u(t_0+t) \end{pmatrix} \frac{1}{\sum_{k=1}^{n} u(0, k)} .$$

In general, let U_p be a $k_p \times k_p$ square matrix for each $p \in \langle 1, l\rangle$ and $\sum_{p=1}^{l} k_p = n$ and I_p be the set $\langle k_1 + \cdots + k_{p-1} + 1, k_1 + \cdots + k_p\rangle$. Then

$$\lim_{h\to\infty} (U_p)^h = \begin{pmatrix} u(t_0, k_1+\cdots+k_{p-1}+1), \cdots, u(t_0, k_1+\cdots+k_p) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ u(t_0, k_1+\cdots+k_{p-1}+1), \cdots, u(t_0, k_1+\cdots+k_p) \end{pmatrix} \frac{1}{\sum_{g\in I_p} u(0, g)}$$

and therefore for i in I_p and j in I_q,

$$\lim_{r\to\infty} P_{0,i}(X_{t+rT} = j) = \begin{cases} 0 & p \neq q , \\ \dfrac{u(t,j)}{\sum_{h\in I_p} u(t, h)} & p = q . \end{cases} \tag{4.22}$$

5. The basic theorem on the asymptotic behavior of the solution $u(t, i, x)$ of equation (I)

Hereafter we shall be concerned with equation (I) under the condition (1.1) and (1.2) and always assume that either of the following condition is satisfied.

(I.a) $\quad d_1 + \cdots + d_n > 0$,

(I.b) $\quad d_1 = \cdots = d_n = 0$ and there exists i and j such that $v_i \neq v_j$.

If neither (I.a) nor (I.b) is satisfied, equation (I) is reduced to equation (II) by a simple change of variables.

Let $\omega > 0$ be fixed, $S^1 = R^1/\omega Z$, where Z denotes the set of all integers, and $E = \langle 1, n\rangle \times S^1$. Let μ be the counting measure on $\langle 1, n\rangle$:

$$\mu(di) = \sum_{j=1}^{n} \delta_j(di) ,$$

where $\delta_j(A)$ is the delta measure at j.

We use the notation $C(E)$ to denote the set of all bounded continuous functions on E and the norm $\|\phi\|_\infty = \max_{(i,x)\in E} |\phi(i, x)|$, then $C(E)$ becomes a Banach space. As was pointed out in section 1, if the initial condition $u(0, i, x)$ is a function of $C(E)$, then the solution $u(t,i,x)$ of equation (I) is in $C(E)$ for every $t \geqq 0$. Hence, we may consider equation (I) as an evolution equation in $C(E)$.

For each i in $\langle 1, n\rangle$, let $\{T_t^i(x, dy)\}$ be the Markovian semigroup, whose generator is given by the differential operator $d_i\frac{\partial^2}{\partial x^2}+v_i\frac{\partial}{\partial x}$.

Then we can transform equation (I) into the integral equation:

$$\text{(I)}' \qquad u(t, i, x) = T_t^i u(0, i, x)+\int_0^t T_{t-s}^i\Big(\sum_{j,k=1}^n a_{j,k}^i u(s, j, \cdot)u(s, k, \cdot)\Big)(x)ds\,.$$

If $u(0,i,x)$ is twice continuously differentiable with respect to x, then the solution $u(t,i,x)$ of equation (I)′ is also such a function (see Conner [1]), and therefore satisfies equation (I). For this reason, we say that $u(t,i,x)$ is the solution of equation (I) whenever $u(t,i,x)$ is the solution of the integral equation (I)′.

As we shall see below, we investigate a transformed function $v(t,i,x)$ rather than $u(t,i,x)$. The same situation for $u(t,i,x)$ is valid for the differential equation and the corresponding integral equation satisfied by $v(t,i,x)$.

If $\phi(x)$ is a periodic function on R^1 with period ω, then

$$(5.1) \qquad T_t^i\phi(x)= T_t^i\phi(x+m\omega) \qquad m\in Z$$

is satisfied. Therefore, we may consider that the semigroup acts on the function defined on S^1.

Assumption 5.1. There exists $u^0=(u_1^0, \cdots, u_n^0)$ in Ω such that $u_1^0>0, \cdots, u_n^0>0$ and $\sum_{j,k=1}^n a_{j,k}^i u_j^0 u_k^0=0$ are satisfied for i in $\langle 1, n\rangle$. Let $b_{i,j}$ be $\sum_{k=1}^n (a_{j,k}^i+a_{k,j}^i)u_k^0$, and B be the matrix $(b_{i,j})$, then B is a nonnegative offdiagonal irreducible matrix.

REMARK. Assumption 5.1 (except the part of the irreducibility of B) was introduced by Jenks [3] to prove the asymptotic stability of the solution of equation (II).

Let $1=\frac{1}{\omega}\int_E u(0,i,x)\mu(di)dx$ and $u(t,i,x)=u_i^0+v(t,i,x)u_i^0$, then equation (I) is transformed into the following equation:

$$(5.2) \qquad \frac{\partial}{\partial t}v(t, i, x) = d_i\frac{\partial^2}{\partial x^2}v(t, i, x)+v_i\frac{\partial}{\partial x}v(t, i, x)+\sum_{j=1}^n c_{i,j}v(t, j, x) + \sum_{j,k=1}^n b_{j,k}^i v(t, j, x)v(t, k, x)$$

where $c_{i,j}=(u_i^0)^{-1}b_{i,j}u_j^0$ and $b_{j,k}^i=(u_i^0)^{-1}a_{j,k}^i u_j^0 u_k^0$.

We first consider the linearized equation for $v(t,i,x)$,

$$(5.3) \qquad \frac{\partial}{\partial t}v(t, i, x) = d_i\frac{\partial^2}{\partial x^2}v(t, i, x)+v_i\frac{\partial}{\partial x}v(t, i, x)+\sum_{j=1}^n c_{i,j}v(t, j, x)\,.$$

$C=(c_{i,j})$ is irreducible and satisfies

$$(5.4)\qquad \begin{cases} \sum_{i=1}^{n} u_i^0 c_{i,j} = 0\,, \\ \sum_{j=1}^{n} c_{i,j} = 0, \\ c_{i,j} \geqq 0 \qquad \text{if } i \neq j\,. \end{cases}$$

We choose $q>0$ large enough to satisfy $q+c_{i,i}\geqq 0$ for all i in $\langle 1,n\rangle$. For this q, (5.3) is equivalent to

$$(5.3)'\qquad \frac{\partial}{\partial t}v(t,i,x) = d_i\frac{\partial^2}{\partial x^2}v(t,i,x)+v_i\frac{\partial}{\partial x}v(t,i,x)-qv(t,i,x) +\sum_{j=1}^{n} p_{i,j}v(t,j,x)\,,$$

where $p_{i,j}=c_{i,j}+q\delta_{i,j}$.

Let $\{T_t\}$ be the semigroup generated by equation (5.3)′ i.e., $(T_t v)(i,x)=\int_E T_t(i,x\colon dj,dy)v(j,y)$, where $T_t(i,x\colon dj,dy)$ is the solution of the equation

$$(5.5)\qquad T_t(i,x\colon dj,dy) = e^{-qt}T_t^i(x,dy)\delta_i(dj) +\int_0^t ds e^{-q(t-s)}\sum_{k=1}^{n} p_{i,k}\int_{S^1} T_{t-s}^i(x,dz)T_s(k,z\colon dj,dy)\,.$$

Then the solution $v(t,i,x)$ of (5.3)′ is given by

$$(5.6)\qquad v(t,i,x) = (T_t v(0))(i,x)\,.$$

Lemma 5.1. *If Assumption* 5.1 *is satisfied, then there exists* $t_0>0$ *and* $c>0$ *such that*

$$(5.7)\qquad T_{t_0}(i,x\colon dj,dy)\geqq c\mu(dj)dy$$

holds for any (i,x) *in* E.

Proof. (i) Case (I.a).

Choose l_0 satisfying $d_{l_0}\neq 0$, then for each $t>0$, there exists $c(t)>0$ such that

$$T_t(l_0,x\colon l_0,dy)\geqq c(t)dy$$

holds. For any i and j, we can choose $k_1,\cdots,k_{m-1}$ to satisfy

$$p_{i,k_1}p_{k_1,k_2}\cdots p_{k_l,l_0}p_{l_0,k_{l+1}}\cdots p_{k_{m-1},j}>0\,.$$

Then, if we choose t and u large enough,

$$(5.8)\qquad T_{t+s+u}(i,x\colon j,A) \geqq \int_{S^1} T_t(i,x\colon l_0,dz)\int_{S^1} T_s(l_0,z\colon l_0,dw)T_u(l_0,w\colon j,A)$$

$$\geqq T_t(i, x: l_0, S^1) c(s) \int_{S^1} dw T_u(l_0, w: j, A)$$
$$\geqq K \int_A dy$$

holds for some positive constant K.

(ii) Case (I.b).

Let $d_i=0$ for every i and let $v_{i_0} \neq v_{j_0}$ for some $i_0 \neq j_0$. Using (5.5), we shall prove for any $m \geqq 1$,

$$T_t(i, x: dj, dy) \geqq \int_0^t \int_0^{s_1} \cdots \int_0^{s_{m-1}} e^{-qt} ds_1 \cdots ds_m \sum_{k_1, \cdots, k_m=1}^{n} p_{i,k_1} p_{k_1,k_2} \cdots p_{k_{m-1},k_m} \delta_{k_m}(dj)$$
$$\delta(x+v_i(t-s_1)+v_{k_1}(s_1-s_2)+\cdots+v_{k_{m-1}}(s_{m-1}-s_m)+v_{k_m} s_m, dy) . \tag{5.9}$$

Here and after we denote, by $\delta_x(dy)$ or $\delta(x, dy)$, the delta measure at x. At first by (5.5), $T_t(i,x:dj,dy) \geqq e^{-qt} \delta_{x+v_i t}(dy) \delta_i(dj)$. Substitute this into (5.5),

$$T_t(i, x: dj, dy)$$
$$\geqq \int_0^t dse^{-q(t-s)} \sum_{k=1}^{n} p_{i,k} \int_{S^1} \delta_{x+v_i(t-s)}(dz) e^{-qs} \delta_{z+v_k s}(dy) \delta_k(dj)$$
$$= e^{-qt} \int_0^t \sum_{k=1}^{n} p_{i,k} \delta_{x+v_i(t-s)+v_k s}(dy) \delta_k(dj) ds ,$$

and the case $m=1$ is proved. We assume that (5.8) has been shown for $1, 2, \cdots, m$ and prove the case $m+1$. By (5.5),

$$T_t(i, x: dj, dy) \geqq \int_0^t dse^{-q(t-s)} \sum_{k=1}^{n} p_{i,k} \int_{S^1} \delta_{x+v_i(t-s)}(dz) T_s(k, z: dj, dy)$$
$$\geqq \int_0^t dse^{-q(t-s)} \sum_{k=1}^{n} p_{i,k} \int_{S^1} \delta_{x+v_i(t-s)}(dz) \int_0^s \int_0^{s_1} \cdots \int_0^{s_{m-1}} e^{-qs} ds_1 \cdots ds_m \sum_{k_1, \cdots, k_m=1}^{n} p_{k_1,k_2}$$
$$\cdots p_{k_{m-1},k_m} \delta_{k_m}(dj) \delta(z+v_k(s-s_1)+v_{k_1}(s_1-s_2)+\cdots+v_{k_{m-1}}(s_{m-1}-s_m)$$
$$+v_{k_m} s_m, dy)$$
$$= \text{the right hand side of (5.9) for } m+1.$$

Therefore, for a nonnegative function ϕ on E,

$$\int_E T_t(i, x: dj, dy) \phi(j, y)$$
$$\geqq \int_0^t \cdots \int_0^{s_{m-1}} e^{-qt} ds_1 \cdots ds_m \sum_{k_1, \ldots, k_m=1}^{n} p_{i,k_1} \cdots p_{k_{m-1},k_m} \phi(k_m, x+v_i(t-s_1)$$
$$+v_{k_1}(s_1-s_2)+\cdots+v_{k_{m-1}}(s_{m-1}-s_m)+v_{k_m} s_m) \tag{5.9$'$}$$

is satisfied. By Assumption 5.1, for any i and j, we can find m and $k_1, \cdots, k_{m-1}$ such that

$$p_{i,k_1} \cdots p_{k_h,i_0} p_{i_0,k_{h+1}} \cdots p_{k_l,j_0} p_{j_0,k_{l+1}} \cdots p_{k_{m-1},j} > 0 .$$

Let $r_0=\max\{r: v_j \neq v_{k_r}\}$. Such r_0 exists since $v_{i_0}\neq v_{j_0}$. We now calculate the integral

$$(5.10)\qquad \int_0^{s_{r_0}}\cdots\int_0^{s_{m-1}} ds_{r_0+1}\cdots ds_m\phi(j,\, x+v_i(t-s_1)+v_{k_1}(s_1-s_2)\cdots+v_js_m)$$

Set $\theta=x+v_it+(v_{k_1}-v_i)s_1+\cdots+(v_{k_{r_0}}-v_{k_{r_0-1}})s_{r_0}$. By a simple calculation, we have

$$\begin{aligned}(5.10)\quad &= \int_0^{s_{r_0}}\cdots\int_0^{s_{m-1}} ds_{r_0+1}\cdots ds_m\phi(j,\, \theta+(v_j-v_{k_{r_0}})s_{r_0+1})\\ &= \int_0^{s_{r_0}} \frac{s_{r_0+1}^{m-1-r_0}}{(m-1-r_0)!}\phi(j,\, \theta+(v_j-v_{k_{r_0}})\, s_{r_0+1})ds_{r_0+1}\\ &= \int_0^{(v_j-v_{k_{r_0}})s_{r_0}} \frac{y^{m-1-r_0}}{(m-1-r_0)!}\phi(j,\theta+y)\frac{dy}{(v_j-v_{k_{r_0}})^{m-r_0}}\end{aligned}$$

If $s_{r_0}>\dfrac{\omega}{|v_j-v_{k_{r_0}}|}$, there exists $K>0$ such that for every θ,

$$(5.10)\qquad \geqq K\int_0^{\omega}\phi(j,y)dy\,.$$

From this, we can easily show that $T_t(i,x:j,dy)\geqq Ldy$ for sufficiently large t and every i,j and x.

Next we set

$$\mathcal{E}=\{\phi(i,x):\ \phi\in C(E), \int_E\phi(i,x)u_i^0\mu(di)dx=0\}\,,$$

then $\mathcal{E}$ is a Banach space under the norm $\|\cdot\|_\infty$.

Lemma 5.2. *T_t maps $\mathcal{E}$ into $\mathcal{E}$.*

Proof. For any function ϕ on S^1, it is satisfied that

$$(5.11)\qquad \int_{S^1}dx\int_{S^1}T_t^i(x,dy)\phi(y)=\int_{S^1}dx\phi(x)\qquad i\in\langle 1,n\rangle\,.$$

From (5.4), for any ϕ in $\mathcal{E}$, we obtain

$$\begin{aligned}(5.12)\quad &\sum_{i=1}^n u_i^0\int_{S^1}dxT_t\phi(i,x)\\ &= e^{-qt}\sum_{i=1}^n u_i^0\int_{S^1}dx\int_{S^1}T_t^i(x,dy)\phi(i,y)\\ &+\int_{S^1}dx\int_0^t dse^{-q(t-s)}\sum_{i,j=1}^n u_i^0(c_{i,j}+q\delta_{i,j})\int_{S^1}T_{t-s}^{\,i}(x,dz)\int_{S^1}T_s(j,z:dk,dy)\phi(k,y)\\ &= \int_0^t dse^{-q(t-s)}\sum_{j=1}^n qu_j^0\int_{S^1}dxT_s\phi(j,x)\,.\end{aligned}$$

Therefore, $\psi(t)=\int_E T_t\phi(i,x)u_i^0\mu(di)dx$ satisfies

$$\psi(t) = q\int_0^t e^{-q(t-s)}\psi(s)ds\,.$$

This means $\psi(t)=0$.

Lemma 5.3. *There exist $N>0$ and $\rho>0$ such that for any ϕ in $\mathcal{E}$*

$$(5.13)\qquad \|T_t\phi\|_\infty \leqq Ne^{-\rho t}\|\phi\|_\infty\,.$$

Proof. By Lemma 5.1, for any ϕ in $\mathcal{E}$,

$$\begin{aligned}|T_{t_0}\phi(i, x)| &\leqq c\left|\sum_{j=1}^n u_j^0\int_{S^1}\phi(j, x)dx\right| + \left|\int_{S^1}\sum_{j=1}^n (T_{t_0}(i, x: j, dy) - cu_j^0 dy)\phi(j,y)\right| \\ &\leqq \|\phi\|_\infty(1-c\omega)\,.\end{aligned}$$

That is $\|T_{t_0}\phi\|_\infty \leqq \|\phi\|_\infty(1-c\omega)$. Iteration implies $\|T_{mt_0}\phi\|_\infty \leqq \|\phi\|_\infty(1-c\omega)^m$. Combining this and $\|T_t\phi\|_\infty \leqq \|\phi\|_\infty$ for any ϕ in $C(E)$, we obtain the result.

We now turn to the nonlinear equation (5.2). We define a mapping B: $C(E)\to C(E)$ by

$$(5.14)\qquad B[v]\,(i, x) = \sum_{j,k=1}^n b_{j,k}^i v(j, x)v(k, x)\,,$$

then it is easy to see that $B[v]$ belongs to $\mathcal{E}$ and there exists $K>0$ such that

$$(5.15)\qquad \|B[v]\|_\infty \leqq K\|v\|_\infty^2\,.$$

We can transform equation (5.2) into the following integral equation:

$$(5.16)\qquad v(t, i, x) = T_t v(0, i, x) + \int_0^t ds\int_E T_s(i, x: dj, dy)B[v(t-s, \cdot)]\,(j, y)\,.$$

Taking norms both sides of (5.16), we get

$$(5.17)\qquad \|v(t)\|_\infty \leqq Ne^{-\rho t}\|v(0)\|_\infty + \int_0^t ds Ne^{-\rho s}K\|v(t-s)\|_\infty^2\,.$$

If $\|v(0)\|_\infty < \frac{\rho}{N^2K}$ then (5.16) has a global solution and

$$(5.18)\qquad \|v(t)\|_\infty \leqq \frac{\rho}{NK + \frac{\rho - N^2K\|v(0)\|_\infty}{N\|v(0)\|_\infty}e^{\rho t}}$$

is satisfied. Summing up the above results, we obtain

Theorem 5.1. *Under the Assumption* 5.1, *let $N>0$, $\rho>0$ and $K>0$ be the constants defined in Lemma* 5.3 *and* (5.15) *respectively. Suppose that $u(0,i,x)\in C(E)$, $\frac{1}{\omega}\int_E u(0,i,x)\mu(di)dx=1$ and $\sup_{(i,x)\in E}\left(\frac{|u(0,i,x)-u_i^0|}{u_i^0}\right) < \frac{\rho}{N^2K}$. Then equation* (I) *has a global solution $u(t,i,x)$ in $C(E)$ tending to u_i^0 exponentially fast.*

REMARK. Let m be a positive constant. If $m=\frac{1}{\omega}\int_E u(0, i, x)\mu(di)dx$, then we transform $u(t,i,x)=mu_i^0+v(t,i,x)u_i^0$, and we get similar results as in Lemma 5.1~Lemma 5.3. Therefore, if $||u(0,i,x)-mu_i^0||_\infty$ is sufficiently small, then equation (I) has a global solution $u(t,i,x)$ and $u(t,i,x)$ converges to mu_i^0 exponentially fast.

6. Further results on the asymptotic behavior of the solution $u(t, i, x)$ of equation (I)

In this section, we assume the following condition as well as Assumption 5.1.

Assumption 6.1. There exists $c>0$ such that for every $(\eta_1, \cdots, \eta_n)$ in $[0, 1]^n$,

$$(6.1) \qquad 0\leqq \sum_{j,k=1}^n b_{j,k}^i\eta_j\eta_k+c\eta_i\leqq c \qquad i\in\langle 1, n\rangle,$$

is satisfied, where $b_{j,k}^i=(u_i^0)^{-1}a_{j,k}^i u_j^0 u_k^0$ as before.

REMARK. Broadwell's model (see [6]) does not satisfy Assumption 5.1 and Assumption 6.1.

By the following remarks, it is seen that this condition assures the boundedness of solutions of equation (I).

If Assumption 6.1 is satisfied, then for any $M>0$ and $(\eta_1, \cdots, \eta_n)$ in $[0, M]^n$,

$$(6.1)' \qquad 0\leqq \sum_{j,k=1}^n b_{j,k}^i\eta_j\eta_k+cM\eta_i\leqq cM^2 \qquad i\in\langle 1, n\rangle$$

is satisfied. Conversely, if (6.1)′ is valid for some $M>0$ and any $(\eta_1, \cdots, \eta_n)$ in $[0, M]^n$, then Assumption 6.1 is satisfied.

Assumption 6.1 is not easy to be verified. However one can show that Assumption 6.1 holds if

$$(6.2) \qquad \sum_{k=1}^n b_{j,k}^i+b_{i,j}^i\geqq 0 \qquad \text{for every } i\neq j\,.$$

In fact let $a_{i,j}=\sum_{k=1}^n b_{j,k}^i$ and $c_{j,k}^i=b_{j,k}^i-a_{i,j}\delta_{i,k}$, then for $(\eta_1, \cdots, \eta_n)$ in $[0, 1]^n$, we have

$$\sum_{j,k=1}^n b_{j,k}^i\eta_j\eta_k=\sum_{j\neq i}\Big(\sum_{k=1}^n c_{j,k}^i\eta_k\Big)\eta_j+\sum_{j=1}^n\Big(\sum_{k=1}^n b_{j,k}^i+b_{i,j}^i-a_{i,i}\delta_{i,j}\Big)\eta_j\eta_i\,.$$

Since $\sum_{k=1}^n c_{j,k}^i=0$ and $c_{j,k}^i\geqq 0$ if $i\neq j$ and $i\neq k$, the first term on the right is majorized by $c(1-\eta_i)$, where $c\geqq -\sum_{j\neq i} c_{j,i}^i\geqq 0$ for every i. By virtue of (6.2), it follows that second term on the right is also majorized by $c(1-\eta_i)$, where $c\geqq -(\sum_{k=1}^n b_{i,k}^i+b_{i,j}^i-a_{i,i})\geqq 0$ for every i.

Lemma 6.1. *Suppose that Assumption* 6.1 *is satisfied, then the equation*

$$(6.3)\qquad \frac{\partial}{\partial t}v(t,i,x) = d_i\frac{\partial^2}{\partial x^2}v(t,i,x)+v_i\frac{\partial}{\partial x}v(t,i,x)+\sum_{j,k=1}^{n} b_{j,k}^{i}v(t,j,x)v(t,k,x)$$

is a confinement system for $[0,M]^n$ *for every* $M>0$. *Conversely, if equation* (6.3) *is a confinement system for* $[0,M]^n$ *for some* $M>0$, *then Assumption* 6.1 *is satisfied.*

Equation (6.3) is said to be a *confinement system* for $[0,M]^n$ if for any initial value $0\leqq v(0,i,x)\leqq M(1\leqq i\leqq n)$, we have $0\leqq v(t,i,x)\leqq M(1\leqq i\leqq n)$ for every $t\geqq 0$.

Proof. If Assumption 6.1 is satisfied, rewrite (6.3) as

$$(6.3)'\qquad \frac{\partial}{\partial t}v(t,i,x) = d_i\frac{\partial^2}{\partial x^2}v(t,i,x)+v_i\frac{\partial}{\partial x}v(t,i,x)-cMv(t,i,x) +\Big(\sum_{j,k=1}^{n} b_{j,k}^{i}v(t,j,x)v(t,k,x)+cMv(t,i,x)\Big)$$

and transform $(6.3)'$ into the integral equation

$$(6.3)''\qquad v(t,i,x) = e^{-cMt}T_t^i v(0,i,x) +\int_0^t\int_{S^1} e^{-cM(t-s)}T_{t-s}^{i}(x,dy)\Big(\sum_{j,k=1}^{n} b_{j,k}^{i}v(s,j,y)v(s,k,y)+cMv(s,i,y)\Big)ds\,.$$

If we solve $(6.3)''$ in the iteration scheme, it follows that equation (6.3) is a confinement system for $[0,M]^n$ for every $M>0$.

Conversely, suppose that equation (6.3) is a confinement system for $[0,M]^n$ for some $M>0$. Let

$$f_i(\phi_1,\cdots,\phi_n)=\sum_{j,k=1}^{n} b_{j,k}^{i}\phi_j\phi_k$$

for $\phi=(\phi_1,\cdots,\phi_n)$ in $[0,M]^n$. It is obvious that $f_i(\phi_1,\cdots,\phi_n)\geqq 0$ if $\phi_i=0$. On the other hand, we have

$$(6.4)\qquad f_i(\phi_1,\cdots,\phi_n)\leqq 0 \quad\text{when}\quad \phi_i=M\,,$$

since equation (6.3) is a confinement system. Then expand f_i at $\phi_i=M$,

$$f_i(\phi_1,\cdots,\phi_n)=b_{i,i}^{i}(M-\phi_i)^2+g_i(\phi_1,\cdots,\phi_{i-1},\phi_{i+1},\cdots,\phi_n)(M-\phi_i) +f_i(\phi_1,\cdots,\phi_{i-1},M,\phi_{i+1},\cdots,\phi_n)\,.$$

From (6.4), there exists $K>0$ such that

$$f_i(\phi_1,\cdots,\phi_n)\leqq K(M-\phi_i)\qquad 0\leqq\phi_1,\cdots,\phi_n\leqq M\,.$$

It then follows that there exists $K>0$ satisfying

$$-K\phi_i \leqq \sum_{j,k=1}^{n} b_{j,k}^{i}\phi_j\phi_k \leqq K(M-\phi_i)$$

$$i\in\langle 1, n\rangle,\ 0\leqq\phi_1, \cdots, \phi_n\leqq M\,,$$

which implies Assumption 6.1.

Theorem 6.1. *Suppose that Assumption* 5.1 *and* 6.1 *are satisfied. If* $u(0,i,x)$ *is in* $C(E)$ *and*

$$\text{(6.5)}\qquad \min_{(i,x)\in E}\frac{u(0,i,x)}{u_i^0}=\delta>0$$

then the solution $u(t,i,x)$ *of equation* (I) *converges to* mu_i^0 *exponentially fast, where* $m=\dfrac{1}{\omega}\displaystyle\int_E u(0,i,x)\mu(di)dx$

Proof. Let $v(t,i,x)=\dfrac{u(t,i,x)}{u_i^0}$, then equation (I) becomes

$$\text{(6.6)}\qquad \frac{\partial}{\partial t}v(t,i,x)=d_i\frac{\partial^2}{\partial x^2}v(t,i,x)+v_i\frac{\partial}{\partial x}v(t,i,x)+\sum_{j,k=1}^{n}b_{j,k}^{i}v(t,j,x)v(t,k,x)\,,$$

and the assumption becomes $\min_{(i,x)\in E} v(0,i,x)=\delta>0$, and it suffices to show that $v(t,i,x)$ converges to m exponentially fast.

(i) $m(t)=\min_{(i,x)\in E} v(t,i,x)$ converges to m.

Let $w(t,i,x)=v(t,i,x)-\delta$, then (6.6) becomes

$$\text{(6.7)}\qquad \frac{\partial}{\partial t}w(t,i,x)=d_i\frac{\partial^2}{\partial x^2}w(t,i,x)+v_i\frac{\partial}{\partial x}w(t,i,x)+\delta\sum_{j=1}^{n}c_{i,j}w(t,j,x)$$
$$+\sum_{j,k=1}^{n}b_{j,k}^{i}w(t,j,x)w(t,k,x)\,.$$

Set $M=\max_{(i,x)\in E} v(0,i,x)$. By the remark following Assumption 6.1, we have

$$\frac{\partial}{\partial t}w(t,i,x)\geqq d_i\frac{\partial^2}{\partial x^2}w(t,i,x)+v_i\frac{\partial}{\partial x}w(t,i,x)+\delta\sum_{j=1}^{n}c_{i,j}w(t,j,x)-cMw(t,i,x)\,.$$

If $z(t,i,x)$ is the solution of the euqation

$$\begin{cases}\dfrac{\partial}{\partial t}z(t,i,x)=d_i\dfrac{\partial^2}{\partial x^2}z(t,i,x)+v_i\dfrac{\partial}{\partial x}z(t,i,x)+\delta\displaystyle\sum_{j=1}^{n}c_{i,j}z(t,j,x)\,,\\ z(0,i,x)=w(0,i,x)\,,\end{cases}$$

then $w(t,i,x)\geqq e^{-cMt}z(t,i,x)$, especially $v(t,i,x)\geqq\delta$ and we can see that $m(t)$ is an increasing function in t. By Lemma 5.1, there exists $t_0>0$ and $k>0$ satisfying

$$z(t_0, i, x)\geqq k(m-\delta)\,.$$

Therefore, there exists $\rho>0$ such that

$$w(t_0, i, x)\geqq\rho(m-\delta)\,.$$

This implies

$$m(t_0)-m(0)\geqq\rho(m-m(0))\,,$$

i,.e., $m-m(t_0)\leqq(1-\rho)\,(m-m(0))$.
By the iteration for any $k\geqq1$,

$$m-m(kt_0)\leqq(1-\rho)^k(m-m(0))\,.$$

This means that $m(t)$ converges to m exponentially fast.

(ii) $M(t)=\max_{(i,x)\in E} v(t, i, x)$ converges to m.

By step (i), we may assume $v(t, i, x)\geqq\delta>0$ for any t, i and x. By Lemma 6.1, $M(t)$ is a decreasing function. By Lemma 5.1, there exists $t_1>0$ and $c>0$ such that

$$T_{t_1}(i, x\colon j, dy)\geqq c dy \qquad \text{for all } i, j \text{ and } x.$$

We rewrite (6.7),

$$(6.7)' \qquad \frac{\partial}{\partial t}w(t, i, x) = d_i\frac{\partial^2}{\partial x^2}w(t, i, x)+v_i\frac{\partial}{\partial x}w(t, i, x) + \sum_{j=1}^{n} c_{i,j}w(t, j, x)-c(M(nt_1)-\delta)w(t, i, x) +(\sum_{j,k=1}^{n} b_{j,k}^{i}w(t, j, x)w(t, k, x)+c(M(nt_1)-\delta)w(t, i, x))$$

for $t\geqq nt_1$. For $t=(n+1)t_1$, (6.7)′ becomes

$$(6.7)'' \qquad w((n+1)t_1, i, x) = e^{-c(M(nt_1)-\delta)t_1}T_{t_1}w(nt_1, i, x) +\int_{nt_1}^{(n+1)t_1} ds e^{-c(M(nt_1)-\delta)((n+1)t_1-s)}\int_E T_{(n+1)t_1-s}(i, x\colon dj, dy) (\sum_{j,k=1}^{n} b_{j,k}^{i}w(s, j, y)w(s, k, y)+c(M(nt_1)-\delta)w(s, i, y))\,.$$

Using Assumption 6.1 and $w(s,j,y)=v(s,j,y)-\delta\leqq M(nt_1)-\delta$ for $nt_1\leqq s$, we have

$$\sum_{j,k=1}^{n} b_{j,k}^{i}w(s, j, y)w(s, k, y)+c(M(nt_1)-\delta)\leqq c(M(nt_1)-\delta)^2\,.$$

Therefore, there exists $1>a>0$, independent of n, such that

$$M((n+1)t_1)-\delta\leqq a(m-\delta)+(1-a)\,(M(nt_1)-\delta)\,,$$

that is to say

$$M((n+1)t_1) \leqq am+(1-a)M(nt_1).$$

$M(\infty)=\lim_{n\to\infty} M(nt_1)$ satisfies $M(\infty)\leqq am+(1-a)M(\infty)$, *i.e.*, $m=M(\infty)$.

(iii) Combining step (i) and step (ii) and the remark following to Theorem 5.1, we obtain the result.

We next show that condition (6.5) of Theorem 6.1 is weakened under some additional assumption. Following Jenks [3], the system $\{a_{j,k}^i\}$ is said to be *irreducible*, if for any partition of $\langle 1, n\rangle$ into I and J, there always exists some i in I and (j, k) in $J\times J$ satisfying $a_{j,k}^i>0$.

Lemma 6.2. *Suppose that either* (a) *or* (b) *is satisfied*:

(a) *Assumption* 6.1 *and the following condition are satisfied*:

There exists $c>0$ and a nonnegative, irreducible matrix $(e_{i,j})$ such that

$$(6.8)\qquad \sum_{j,k=1}^n b_{j,k}^i \eta_j \eta_k + c\eta_i \geqq \sum_{j=1}^n e_{i,j}\eta_j^2 \qquad i\in\langle 1, n\rangle$$

is satisfied for all $0\leqq \eta_1, \cdots, \eta_n \leqq 1$.

(b) *The system $\{a_{j,k}^i\}$ is irreducible, and $d_1>0, \cdots, d_n>0$, and the Assumption* 6.1 *is satisfied.*

Then for any nonnegative initial value $u(0,i,x)$ in $C(E)$ satisfying $\sum_{i=1}^n u(0,i,x) \not\equiv 0$, there exists $t_1>0$ such that

$$(6.9)\qquad \min_{(i,x)\in E} u(t_1, i, x)>0.$$

Proof. We first prove case (b). Fix $t_1>0$ and put $J=\{j\in\langle 1, n\rangle: \min_{x\in S^1} u(t_1, j, x)>0\}$. If J is a proper subset of $\langle 1,n\rangle$, then let I be the set $\langle 1, n\rangle - J$. Define $v(t, i, x)=\dfrac{u(t,i,x)}{u_i^0}$ as before. Noting that the system $\{b_{j,k}^i\}$ is also irreducible, choose i_0 in I and (j_0, k_0) in $J\times J$ such that $b_{j_0,k_0}^{i_0}>0$. Since equation (6.6) is a confinement system, there exists $M>0$ such that

$$\begin{aligned}(6.10)\qquad &\sum_{(j,k)\neq(j_0,k_0)} b_{j,k}^{i_0} v(t,j,x)v(t,k,x)+cMv(t,i_0,x)\\ &= \sum_{\substack{(j,k)\neq(j_0,k_0)\\ j\neq i_0,\, k\neq i_0}} b_{j,k}^{i_0} v(t,j,x)v(t,k,x)\\ &\quad + v(t,i_0,x)\Big[cM+\sum_{k=1}^n (b_{i_0,k}^{i_0}+b_{k,i_0}^{i_0})v(t,k,x)-b_{i_0,i_0}^{i_0}v(t,i_0,x)\Big]\geqq 0.\end{aligned}$$

One rewrite (6.6) in the form

$$\frac{\partial}{\partial t}v(t,i_0,x) = d_{i_0}\frac{\partial^2}{\partial x^2}v(t,i_0,x)+v_{i_0}\frac{\partial}{\partial x}v(t,i_0,x)-cMv(t,i_0,x)$$

$$+b_{j_0,k_0}^{i_0}v(t, j_0, x)v(t, k_0, x)$$
$$+(\sum_{(j,k)\neq(j_0,k_0)} b_{j,k}^{i_0}v(t, j, x)v(t, k, x)+cMv(t, i_0, x))\,.$$

As before, this equation is transformed into the integral equation

$$\begin{aligned} v(t_1, i_0, x) &= e^{-cMt_1}T_{t_1}^{i_0}v(0, i_0, x) \\ &+\int_0^{t_1} e^{-cM(t_1-s)}T_{t_1-s}^{i_0}(b_{j_0,k_0}^{i_0}v(s, j_0, \cdot)v(s, k_0, \cdot))\,(x)ds \\ &+\int_0^{t_1} e^{-cM(t_1-s)}T_{t_1-s}^{i_0}(\sum_{(j,k)\neq(j_0,k_0)} b_{j,k}^{i_0}v(s, j, \cdot)v(s, k, \cdot)+cMv(s, i_0, \cdot))\,(x)ds\,. \end{aligned}$$

From (6.10), it follows that the right side is not less than

$$b_{j_0,k_0}^{i_0}\int_0^{t_1} e^{-cM(t_1-s)}T_{t_1-s}^{i_0}(v(s, j_0, \cdot)v(s, k_0, \cdot))\,(x)ds>0$$

which is a contradiction. This means $J=\langle 1, n\rangle$.

Next we proceed to case (*a*). For the moment we further assume that $0\leqq v(0, i, x)\leqq 1$. Take a probability matrix $(\tilde{e}_{i,j})$ *i.e..,* $\sum_{j=1}^{n}\tilde{e}_{i,j}=1$ such that $\tilde{e}_{i,j}=0$ iff $e_{i,j}=0$ and a small $k>0$ such that $e_{i,j}\geqq k\tilde{e}_{i,j}$ for all (i, j) in $\langle 1, n\rangle\times\langle 1, n\rangle$. Then by (6.6), we have

$$\begin{aligned} \frac{\partial}{\partial t}v(t, i, x) &= d_i\frac{\partial^2}{\partial x^2}v(t, i, x)+v_i\frac{\partial}{\partial x}v(t, i, x)-cv(t, i, x) \\ &\quad+(\sum_{j,k=1}^{n} b_{j,k}^{i}v(t, j, x)v(t, k, x)+cv(t, i, x)) \\ &\geqq d_i\frac{\partial^2}{\partial x^2}v(t, i, x)+v_i\frac{\partial}{\partial x}v(t, i, x)-cv(t, i, x) \\ &\quad+\sum_{j=1}^{n} e_{i,j}v(t, j, x)^2 \\ &\geqq d_i\frac{\partial^2}{\partial x^2}v(t, i, x)+v_i\frac{\partial}{\partial x}v(t, i, x)-cv(t, i, x) \\ &\quad+k\sum_{j=1}^{n}\tilde{e}_{i,j}v(t, j, x)^2\,. \end{aligned}$$

Let (X_t) be the branching process corresponding to the equation

$$(6.11)\quad \begin{cases} \dfrac{\partial}{\partial t}w(t, i, x) = d_i\dfrac{\partial^2}{\partial x^2}w(t, i, x)+v_i\dfrac{\partial}{\partial x}w(t, i, x)-cw(t, i, x) \\ \qquad\qquad +k\sum_{j=1}^{n}\tilde{e}_{i,j}w(t, j, x)^2\,, \\ w(0, i, x) = v(0, i, x)\,, \end{cases}$$

with state space $S=\bigcup_{n=0}^{\infty}E^n$ (where E^n is the symmetric direct product of E for

$n>0$, and E^0 is a single point $\{\partial\}$), and we write $P_{(i,x)}$ for its measure starting at (i, x) in E, and $E_{(i,x)}$ for its expectation. Function ϕ on E with $0 \leqq \phi(i, x) \leqq 1$ can be extended on S by

$$\hat{\phi}((i_1, x_1), \cdots, (i_k, x_k)) = \phi(i_1, x_1) \cdots \phi(i_k, x_k) \qquad k \geqq 1\,,$$
$$\hat{\phi}(\partial) = 1\,.$$

It is known that

$$w(t, i, x) = E_{(i,x)}[\hat{v}(0, X_t)]\,.$$

Since $v(t, i, x) \geqq w(t, i, x)$, it is sufficient to show $w(t_1, i, x)>0$ for some $t_1>0$. Let $\{T_t(i, x\colon d(j_1, y_1) \cdots d(j_k, y_k))\}$ be the branching semigroup of (X_t), then

$$(6.11)' \qquad T_t\hat{\phi}(i, x) = e^{-ct}T_t^i\phi(i, x) + k\sum_{j=1}^{n} \tilde{e}_{i,j}\int_0^t e^{-c(t-s)}T_{t-s}^{\,i}[T_s\hat{\phi}(\cdot)]^2(j, x)ds\,.$$

We first consider the case (I.b) in the beginning of section 5. In general for $m \geqq 1$, we have

$$(6.12) \qquad T_t(i, x\colon d(j, y_0) \cdots d(j, y_{2^m-1}))$$
$$\geqq \int_0^t e^{-c(t-s)}ds k\tilde{e}_{i,k_m} \prod_{h_m=0}^{1} \int_0^s e^{-c(s-s_{m-1}^{h_m})} ds_{m-1}^{h_m} k\tilde{e}_{k_m,k_{m-1}}$$
$$\prod_{h_{m-1}=0}^{1} \int_0^{s_{m-1}^{h_m}} e^{-c(s_{m-1}^{h_m} - s_{m-2}^{h_{m-1}})} ds_{m-2}^{h_{m-1}} k\tilde{e}_{k_{m-1},k_{m-2}}$$
$$\vdots$$
$$\prod_{k_2=0}^{1} \int_0^{s_2^{h_3}} e^{-c(s_2^{h_3} - s_1^{h_2})} ds_1^{h_2} e^{-2cs_1^{h_2}} k\tilde{e}_{k_2,k_1}$$
$$\prod_{h_1=0}^{1} \delta(k_1, x+v_i(t-s)+v_{k_m}(s-s_{m-1}^{h_m})+\cdots+v_{k_2}(s_2^{h_3}-s_1^{h_2})+v_{k_1}s_1^{h_2}\colon d(j, y_{h_m\cdots h_1}))\,,$$

where each k_m is an arbitrary integer in $\langle 1, n\rangle$ and $y_{h_m\cdots h_1}$ denotes $y_{h_m 2^{m-1}+\cdots+h_1}$. In fact $T_t(i, x\colon d(j, y_0) \cdots d(j, y_{2^m-1}))$ equals the summation of the right side of (6.12) over all $k_m, \cdots, k_1$ in $\langle 1, n\rangle$.
Therefore, there exists $c_m(t)>0$ for each $t>0$ and $m>1$ such that

$$(6.12)' \qquad T_t\hat{v}(0, i, x) \geqq c_m(t)\int_0^t k\tilde{e}_{i,k_m} ds \prod_{h_m=0}^{1} \int_0^s k\tilde{e}_{k_m,k_{m-1}} ds_{m-1}^{h_m} \cdots$$
$$\prod_{h_2=0}^{1} \int_0^{s_2^{h_3}} ds_1^{h_2} k\tilde{e}_{k_2,k_1} v(0, k_1, x+v_i(t-s)+\cdots+v_{k_1}s_1^{h_2})^2\,.$$

Fix a j such that $v(0, j, x) \not\equiv 0$ and suppose that $v_{i_0} \neq v_{j_0}$. For any i, we can choose $k_2, k_3, \cdots, k_{m-1}$ such that

$$\tilde{e}_{i,k_{m-1}}\cdots\tilde{e}_{k_h+1,i_0}\tilde{e}_{i_0,k_h}\cdots\tilde{e}_{k_l+1,j_0}\tilde{e}_{j_0,k_l}\cdots\tilde{e}_{k_2,j}>0$$

since $(\tilde{e}_{i,j})$ is irreducible. Similar consideration as in the proof of Lemma 5.1 implies that, when $k_1=j$, the right side of (6.12) is strictly positive for t large enough. This proves that $w(t_1, i, x)>0$ for some t_1 which does not depend on i and x.

Next we consider the case (I.a) in the beginning of section 5. Suppose that $d_{i_0}>0$. As before, fix a j such that $v(0, j, x)\not\equiv 0$. In general for $m\geqq 1$, it follows from the preceding property that

$$(6.13)\quad \begin{aligned} &T_t(i, x: d(j, y_0)\cdots d(j, y_{2^m-1}))\\ &\geqq\int_0^t e^{-c(t-s)}T_{t-s}^i(x, dy)dsk\tilde{e}_{i,k_m}\\ &\prod_{h_m=0}^1\int_0^s e^{-c(s-s_{m-1}^{h_m})}ds_{m-1}^{h_m}T_{s-s_{m-1}^{h_m}}^{k_m}(y, dy_{h_m})k\tilde{e}_{k_m k,m-1}\\ &\qquad\vdots\\ &\prod_{h_2=0}^1\int_0^{s_2^{h_3}} e^{-c(s_2^{h_3}-s_1^{h_2})}ds_1^{h_2}T_{s_2^{h_3}-s_1^{h_2}}^{k_2}(y_{h_m\cdots h_3}, dy_{h_m\cdots h_2})e^{-2c_1^{h_2}}k\tilde{e}_{k_2,j}\\ &\prod_{h_1=0}^1 T_{s_1^{h_2}}^j(y_{h_m\cdots h_2}, dy_{h_m\cdots h_1}),\end{aligned}$$

where $y_{h_m\cdots h_1}$ denotes $y_{h_m 2^{m-1}+\cdots+h_1}$.
Therefore, there exists $c_m(t)>0$ for each $t>0$ and $m\geqq 1$, such that

$$(6.14)\quad \begin{aligned} &T_t\hat{v}(0, i, x)\geqq c_m(t)\int_0^t T_{t-s}^i(x, dy)\,dsk\tilde{e}_{i,k_m}\\ &\prod_{h_m=0}^1\int_0^t ds_{m-1}^{h_m}T_{s-s_{m-1}^{h_m}}^{k_m}(y, dy_{h_m})k\tilde{e}_{k_m,k_{m-1}}\cdots\\ &\int_0^{s_2^{h_3}} ds_1^{h_2}T_{s_2^{h_3}-s_1^{h_2}}^{k_2}(y_{h_m\cdots h_3}, dy_{h_m\cdots h_2})k\tilde{e}_{k_2,j}[T_{s_1^{h_2}}^j v(0, j, y_{h_m\cdots h_2})]^2 .\end{aligned}$$

We can choose $k_2, k_3, \cdots, k_{m-1}$ such that

$$\tilde{e}_{i_0,k_{m-1}}\cdots\tilde{e}_{k_2,j}>0 .$$

We shall show that $T_{t_0}\hat{v}(0, i_0, x)>0$ for large enough t_0, by the induction on m. It is easy to see this in the case $m=1$. We assume that we have shown $1, 2, \cdots, m-1$ and prove the case m. If some d_{k_l} or d_j is positive, then this is reduced to the case that we assumed to have shown. Therefore, we may assume $d_{k_2}=\cdots=d_{k_m}=0$. Inequality (6.14) becomes

$$\begin{aligned} T_t\hat{v}(0, i_0, x)\geqq c_m(t)\int_0^t T_{t-s}^{i_0}(x, dy)dsk\tilde{e}_{i_0,k_m}\prod_{h_m=0}^1\int_0^s ds_{m-1}^{h_m}k\tilde{e}_{k_m,k_{m-1}}\cdots\\ \prod_{h_2=0}^1\int_0^{s_2^{h_3}} ds_1^{h_2}k\tilde{e}_{k_2,j}v(0, j, y+v_{k_m}(s-s_{m-1}^{h_m})+\cdots+v_j s_j^{h_2})^2 .\end{aligned}$$

If $v_{k_1}=\cdots=v_{k_m}$, then $v(0, j, y+v_{k_m}(s-s_{m-1}^{h_m})+\cdots+v_j s_1^{h_2})^2=v(0, j, y+v_j s)^2$, so that

$$T_t\hat{v}(0, i_0, x)\geqq c_m(t)\int_0^t T_{t-s}^{i}(x, dy)ds\tilde{e}_{i_0,k_m}v(0, j, y+v_j s)^{2^m}$$

$$\prod_{h_m=0}^{1}\int_0^s ds_{m-1}^{h_m}k\tilde{e}_{k_m,k_{m-1}}\cdots\prod_{h_2=0}^{1}\int_0^{s_2^{h_3}} ds_1^{h_2}k\tilde{e}_{k_2,j}>0\,.$$

If $(v_j, v_{k_1}, \cdots, v_{k_m})$ contains at least two different numbers, then this is reduced to the case (I.b) which we have proved previously.

Set $J=\{j\in\langle 1, n\rangle: \inf_{x\in S^1} v(2t_0, j, x)>0\}$. If J is a proper subset of $\langle 1, n\rangle$, then let I be the set $\langle 1, n\rangle - J$. Since $(\tilde{e}_{i,j})$ is irreducible, there exists i in I and J such that $\tilde{e}_{i,j}>0$. Then by equation (6.11)′, we have

$$T_{2t_0}\hat{v}(0, i, x)\geqq k\tilde{e}_{i,j}\int_0^{2t_0} e^{-c(2t_0-s)}T_{2t_0-s}^{i}[T_s\hat{v}(0, \cdot)]^2(j, x)ds>0$$

for each x in S^1, which is a contradiction. Therefore $J=\langle 1, n\rangle$ and we have proved the lemma in case $0\leqq v(0, i, x)\leqq 1$.

We next delete the restriction $0\leqq v(0, i, x)\leqq 1$. If $0\leqq v(0, i, x)\leqq M$, set $\bar{v}(t, i, x)=\dfrac{v(t, i, x)}{M}$, then $\bar{v}(t, i, x)$ satisfies the equation

$$\frac{\partial}{\partial t}\bar{v}(t, i, x) = d_i\frac{\partial^2}{\partial x^2}\bar{v}(t, i, x)+v_i\frac{\partial}{\partial x}\bar{v}(t, i, x)$$

$$+M\sum_{j,k=1}^{n} b_{j,k}^{i}\bar{v}(t, j, x)\bar{v}(t, k, x)\,.$$

It follows that $\tilde{b}_{j,k}^{i}=Mb_{j,k}^{i}$ satisfies Assumption 6.2 with c and $b_{i,j}$ replaced by Mc and $Mb_{i,j}$. By the preceding result, it follows that $\inf_{(i,k)\in E}\bar{v}(t_1, i, x)>0$ for some t_1. Hence we have $\inf_{(i,k)\in E} v(t_1, i, x)>0$. Thus we have shown Lemma 6.2.

Summing up Theorem 6.1 and Lemma 6.2, we obtain

Theorem 6.2. *Suppose that either* (a) *or* (b) *in Lemma* 6.2 *is satisfied, then for any nonnegative* $u(0, i, x)$ *in* $C(E)$, *the solution* $u(t, i, x)$ *of equation* (I) *converges to* mu_i^0 *exponentially fast, where* $m=\dfrac{1}{\omega}\int_E u(0, i, x)\mu(di)dx$.

7. Markov processes corresponding to equation (I)

In section 7 and section 8, condition (1.3) is always assumed. In this section, we also assume that equation (I) has a continuous nonnegative solution $u(t,i,x)$. Fix this $u(t, i, x)$ and consider the following equation for $s\leqq t$

$$(7.1)\qquad \begin{cases} \dfrac{\partial}{\partial t}\phi(t, i, dx) = \phi(t, i, dx)\left(d_i\dfrac{\partial^2}{\partial x^2} - v_i\dfrac{\partial}{\partial x}\right) \\ \qquad\qquad + \sum_{j,k=1}^{n} a_{j,k}^{i}\phi(t, j, dx)u(t, k, x)\,, \\ \phi(s, di, dx) \text{ is a bounded (signed) measure on } E. \end{cases}$$

Here the precise meaning of (7.1) shoud be understood as

$$(7.1)'\qquad \frac{\partial}{\partial t}\int_E \phi(t, di, dx)f(i, x)$$
$$= \int_E \phi(t, di, dx)\left(d_i\frac{\partial^2}{\partial x^2} - v_i\frac{\partial}{\partial x}\right)f(i, x)$$
$$+ \int_E \phi(t, dj, dx)\sum_{i,k=1}^{n} a_{j,k}^{i}u(t, k, x)f(i, x)\,,$$

for all $f \in C^\infty(E)$.

For each $T < \infty$, let $q_T = \max_{\substack{(i,x)\in E \\ 0\leq t\leq T}} \{\sum_{k=1}^{n} |a_{i,k}^{i}u(t, k, x)|\}$. We denote $\{\tilde{T}_t^i(x, dy)\}$ to be the Markovian semigroup whose generator is the differential operator $d_i\frac{\partial^2}{\partial x^2} - v_i\frac{\partial}{\partial x}$. Then we can transform equation (7.1) into the integral equation

$$(7.2)\qquad \phi(t, i, dx) = \int_{S^1}\phi(s, i, dy)\tilde{T}_{t-s}^{i}(y, dx)e^{-q_T(t-s)}$$
$$+\int_s^t e^{-q_T(t-r)}dr\int_E \phi(r, dj, dy)\left(\sum_{k=1}^{n} a_{j,k}^{i}u(r, k, y) + q_T\delta_{i,j}\right)\tilde{T}_{t-r}^{i}(y, dx)\,,$$
$$0 \leqq s \leqq t \leqq T\,.$$

Let $\phi(t) = \phi(t, E)$. Then from (7.2),

$$\phi(t) = \phi(s)e^{-q_T(t-s)} + \int_s^t e^{-q_T(t-r)}q_T\phi(r)dr$$

i.e., $\phi(t) = \phi(s)$, and therefore the solution preserves the total measure. If $\phi(s, di, dx)$ is a nonnegative measure, then the solution $\phi(t, di, dx)$ of (7.2) is also a nonnegative measure. If we solve (7.2) in the iteration scheme, then we get a minimal solution $\phi(t, di, dx)$. Because of the preservation of total measure, (7.1) has a unique solution. Since (7.2) is a linear equation in ϕ, (7.2) (*i.e.*, (7.1)) always has a unique solution. We denote $\{P_{s,t}(i, x\colon dj, dy)\}_{s\leqq t}$ the solution of (7.1) when $\phi(s, dj, dy) = \delta_{(i,x)}(dj, dy)$.

Theorem 7.1. *$\{P_{s,t}(i, x\colon dj, dy)\}_{s\leqq t}$ has the following properties*:

$$(7.3)\qquad \int_E P_{s,t}(i, x\colon dk, dz)P_{t,u}(k, z\colon dj, dy) = P_{s,u}(i, x\colon dj, dy)$$
$$s \leqq t \leqq u\,.$$

(7.4) $\quad P_{s,t}(i, x\colon dj, dy)$ *is a nonnegative measure.*

(7.5) $\quad P_{s,t}(i, x\colon E) = 1\,.$

Let $\phi(t, di, dx)$ *be the solution of* (7.1), *then*

(7.6) $$\int_E \phi(s, di, dx) P_{s,t}(i, x\colon dj, dy) = \phi(t, dj, dy)\,.$$

Let $\psi(t, i, x)$ *be the solution of equation*

(7.7) $$\begin{cases} \dfrac{\partial}{\partial t}\psi(t, i, x) = d_i \dfrac{\partial^2}{\partial x^2}\psi(t, i, x) + v_i \dfrac{\partial}{\partial x}\psi(t, i, x) \\ \qquad\qquad + \displaystyle\sum_{j,k=1}^{n} a_{j,k}^{i}\psi(t, j, x)u(t, k, x)\,, \\ \psi(s, i, x) = \psi(i, x)\,, \end{cases}$$

and let $\phi(t, di, dx)$ *be the solution of* (7.1) *with* $\phi(s, di, dx) = \psi(i, x)\mu(di)dx$, *then it follows that*

(7.8) $$\phi(t, di, dx) = \psi(t, i, x)\mu(di)dx\,,$$

especially, from (7.6) *and* (7.7),

(7.9) $$\int_E u(s, i, x)\mu(di)dx P_{s,t}(i, x\colon dj, dy) = u(t, j, y)\mu(dj)dy\,.$$

Proof. It is sufficient to show our theorem for each $T < \infty$ and $0 \leqq s \leqq t \leqq u \leqq T$. (7.3) and (7.6) follow from the uniqueness of solution of equation (7.1). If we solve (7.2) in the iteration scheme, then (7.4) is clear. (7.5) follows from the preservation of total measure. Next we prove (7.8). Let $\phi(t, di, dx) = \psi(t, i, x)\mu(di)dx$, then for any $v(i, x)$ in $C^\infty(E)$,

$$\begin{aligned} &\frac{\partial}{\partial t}\int_E \phi(t, di, dx)v(i, x) \\ &= \int_E \phi(t, di, dx)\left(d_i \frac{\partial^2}{\partial x^2} - v_i \frac{\partial}{\partial x}\right)v(i, x) \\ &\quad + \int_E \phi(t, dj, dx)\sum_{i,k=1}^{n} a_{j,k}^{i} u(t, k, x)v(i, x) \end{aligned}$$

and $\phi(s, di, dx) = \psi(i, x)\mu(di)dx$. By the uniqueness of solution of (7.1), $\phi(t, di, dx) = \psi(t, i, x)\mu(di)dx$.

By Theorem 7.1, we can construct a temporally inhomogeneous Markov process (X_t) with transition probability $(P_{s,t})$ and the state space E. Let $P_{s,(i,x)}(\cdot)$ denote the measure of the process starting at (i, x) at time s.

8. Asymptotic behavior of the process (X_t) associated with equation (I)

Let $u(i, x)$ be a twice continuously differentiable and strictly positive function, satisfying

$$d_i \frac{\partial^2}{\partial x^2} u(i, x) + v_i \frac{\partial}{\partial x} u(i, x) + \sum_{j,k=1}^{n} a_{j,k}^{i} u(j, x) u(k, x) = 0 .$$

Suppose that $u(t, i, x)$ is a nonnegative solution of equation (I) such that

$$\lim_{t \to \infty} \|u(t) - u\|_\infty = 0 . \tag{8.1}$$

Throughout in this section, we shall fix such $u(i, x)$ and $u(t, i, x)$ and assume the following condition:

Assumption 8.1. Each nonnegative offdiagonal matrix $(\sum_{k=1}^{n} a_{j,k}^{i} u(k,x))$, $x \in S^1$, is irreducible.

Under these hypothesis, we shall study the asymptotic behavior of the process (X_t) defined in the preceding section.

Let $\{S_t\}$ be the semigroup generated by the following differential equation,

$$\frac{\partial}{\partial t} \psi(t, i, x) = d_i \frac{\partial^2}{\partial x^2} \psi(t, i, x) + v_i \frac{\partial}{\partial x} \psi(t, i, x) + \sum_{j,k=1}^{n} a_{j,k}^{i} \psi(t, j, x) u(k, x) . \tag{8.2}$$

Since each matrix $(\sum_{k=1}^{n} a_{j,k}^{i} u(k, x))$, $x \in S^1$, is nonnegative offdiagonal, $S_t \psi \geqq 0$ for $\psi \geqq 0$.

Put
$$A(t)\psi(i, x) = \sum_{j,k=1}^{n} a_{j,k}^{i} \psi(j, x) u(t, k, x) ,$$

$$A\psi(i, x) = \sum_{j,k=1}^{n} a_{j,k}^{i} \psi(j, x) u(k, x) ,$$

and consider the equation

$$\frac{\partial}{\partial t} \psi(t, i, x) = \left(d_i \frac{\partial^2}{\partial x^2} + v_i \frac{\partial}{\partial x} + A\right) \psi(t, i, x) + (A(t) - A) \psi(t, i, x) , \tag{8.3}$$

or, equivalently

$$\psi(t) = S_t \psi(0) + \int_0^t S_{t-s} (A(s) - A) \psi(s) ds . \tag{8.3$'$}$$

Set

$$\mathcal{F} = \{\phi : \int_E \phi(i, x) \mu(di) dx = 0\} \subset L^1(E, \mu(di)dx) = L^1 ,$$

and $||\phi||_1=\int_E|\phi(i,x)|\mu(di)dx$, then $\mathcal{F}$ becomes a Banach space under the norm $||\cdot||_1$. We can easily show that

$$S_t(\mathcal{F})\subset\mathcal{F},$$
$$(A(t)-A)\psi\in\mathcal{F}\qquad\text{for any }\psi\text{ in }L^1,$$

are satisfied.

Lemma 8.1. *There exists $K>0$ and $\rho>0$ satisfying*

$$||S_t\phi||_1\leqq Ke^{-\rho t}||\phi||_1\qquad\textit{for any }\phi\textit{ in }\mathcal{F}.$$

Proof. At first, we remark that

$$\int_E\mu(di)dxS_t(i,x:dj,dy)=\mu(dj)dy$$

is satisfied. As in the same method as in the proof of Lemma 5.1, we can show that there exists $t_1>0$ and $c>0$ satisfying

$$S_{t_1}(i,x:dj,dy)\geqq c\mu(dj)dy\qquad\text{for any }(i,x)\text{ in }E,$$

using Assumption 8.1.
By (8.2), we can write

$$(S_{t_1}\phi(0))(i,x)=\int_E(S_{t_1}(i,x:dj,dy)-c\mu(dj)dy)\phi(0,j,y),$$

and therefore,

$$\sum_{i=1}^n\int_{S^1}|S_{t_1}\phi(0))(i,x)|dx$$
$$\leqq\sum_{i=1}^n\int_{S^1}dx\int_E(S_{t_1}(i,x:dj,dy)-c\mu(dj)dy)|\phi(0,j,y)|$$
$$=(1-cn\omega)\sum_{j=1}^n\int_{S^1}|\phi(0,j,y)|dy,$$

i.e., $||S_{t_1}\phi||_1\leqq(1-cn\omega)||\phi||_1$. Combining this and $||S_t\phi||_1\leqq||\phi||_1$, we get the result.

Theorem 7.1 implies the L^1-boundedness of the solution $\psi(t,i,x)$ of equation (8.3). Therefore

$$||\psi(t)-S_t\psi(0)||_1=||\int_0^tS_{t-s}(A(s)-A)\psi(s)ds||_1$$
$$\leqq\int_0^tKe^{-\rho(t-s)}||(A(s)-A)\psi(s)||_1ds\to0,\ \textit{as }t\to\infty,$$

since $||(A(s)-A)\psi(s)||_1\leqq\max_{1\leqq j\leqq n}(\sum_{i,k=1}^n|a_{j,k}^i|)||u(s)-u||_\infty||\psi(s)||_1$.

Let $m=\dfrac{\int_E \psi(0, i, x)\mu(di)dx}{\int_E u(i, x)\mu(di)dx}$, then $\eta(0)=\psi(0)-mu$ belongs to $\mathcal{F}$, therefore,

$$\lim_{t\to\infty} \psi(t) = \lim_{t\to\infty} S_t\psi(0) = \lim_{t\to\infty} (S_t\eta(0)+mS_tu) = mu\,,$$

in the L^1 sense. This means that

$$\lim_{t\to\infty}\int_E \psi(t, i, x)f(i, x)\mu(di)dx = m\int_E u(i, x)f(i, x)\mu(di)dx \tag{8.4}$$

holds for any bounded Borel measurable function $f(i, x)$.

Theorem 8.1. *Under the Assumption* 8.1, *it holds that for any Borel set* A *in* S^1, *and* (i, x) *in* E,

$$\lim_{t\to\infty} P_{0,(i,x)}(X_t\in(j, A)) = \frac{\int_A u(j, y)dy}{\int_E u(j, y)\mu(dj)dy}\,. \tag{8.5}$$

By (7.8) in Theorem 7.1, (8.5) is nothing but (8.4) when the initial measure has a density with respect to the Lebesgue measure. In the general case, we fix i_0 and x_0 arbitrarily and divide

$$P_{0,t}(i_0, x_0\colon j, dy) = p_{0,t}(i_0, x_0\colon j, y)dy+P^{(s)}_{0,t}(i_0, x_0\colon j, dy) \tag{8.6}$$

where $P^{(s)}_{0,t}(\cdot\,:\cdot)$ is a singular part for the Lebesgue measure.
Let $\phi(t)=\int_E p_{0,t}(i_0, x_0\colon j, y)\mu(dj)dy$.

Lemma 8.2. $\phi(t)$ *increases to* 1.

Proof. Since $\int_E p_{0,t}(i_0, x_0\colon k, z)\mu(dk)dzP_{t,t+s}(k, z\colon j, dy)$ is absolutely continuous for the Lebesgue measure it follows that $p_{0,t+s}(i_0, x_0\colon j, y)\mu(dj)dy\geqq\int_E p_{0,t}(i_0, x_0\colon k, z)\mu(dk)dzP_{t,t+s}(k, z\colon dj, dy)$ and $\phi(t+s)\geqq\phi(t)$. This means that $\phi(t)$ is an increasing function. We show that there exists $s_0>0$, $s_1>0$ and $c>0$ such that

$$P_{t,t+s_0}(i, x\colon j, dy)\geqq cdy \qquad \text{for any } i, j, x \text{ and } t\geqq s_1\,. \tag{8.7}$$

Since $u(i, x)$ is strictly positive, and by (8.1), there exists $s_1>0$, $m>0$ and $M<\infty$ such that

$$M\geqq \sup_{\substack{t\geqq S_1\\(i,x)\in E}} u(t, i, x)\geqq \inf_{\substack{t\geqq S_1\\(i,x)\in E}} u(t, i, x)\geqq m\,.$$

Let $\qquad a_{j,i} = \sum_{k=1}^{n} a_{j,k}^{i} m \quad j\neq i\,,$

$$a_{i,i} = \sum_{k=1}^{n} a_{i,k}^{i} M ,$$

then for $t \geqq s_1$,

$$(\sum_{k=1}^{n} a_{j,k}^{i} u(t, k, x)) \geqq (a_{j,i}) .$$

It is easy to see that the matrix $(a_{j,i})$ is irreducible. Let $q = \max_{1 \leqq i \leqq n} |a_{i,i}|$, and $p_{j,i} = a_{j,i} + q\delta_{j,i}$, then for $t \geqq s \geqq s_1$,

$$\begin{aligned} &P_{s,t}(i, x: j, dy) \\ &= \delta_{i,j} \tilde{T}_{t-s}^{i}(x, dy) e^{-q(t-s)} \\ &\quad + \int_s^t e^{-q(t-r)} dr \int_E P_{s,r}(i, x: dl, dz) (\sum_{k=1}^{n} a_{l,k}^{j} u(r, k, z) + q\delta_{l,j}) \tilde{T}_{t-r}^{j}(z, dy) \\ &\geqq \delta_{i,j} \tilde{T}_{t-s}^{i}(x, dy) e^{-q(t-s)} + \int_s^t e^{-q(t-r)} dr \int_E P_{s,r}(i, x: dl, dz) p_{l,j} \tilde{T}_{t-r}^{j}(z, dy) . \end{aligned}$$

This means that for all i, x, j and $t \geqq r \geqq s_1$, it holds that

$$P_{r,t}(i, x: j, dy) \geqq Q_{t-r}(i, x: j, dy) ,$$

where $Q_{t-r}(\cdot:\cdot)$ is the solution of

$$\begin{cases} Q_0(i, x: j, dy) = \delta_{i,j} \delta_x(dy) , \\ Q_{t-s}(i, x: j, dy) = \delta_{i,j} \tilde{T}_{t-s}^{i}(x, dy) e^{-q(t-s)} \\ \qquad\qquad + \int_s^t e^{-q(t-r)} dr \int_E Q_{r-s}(i, x: dl, dz) p_{l,j} \tilde{T}_{t-r}^{j}(z, dy) . \end{cases}$$

The above equation is analogous to (5.5), as that the proof of Lemma 5.1 is applicable, therefore we can see that there exists $c > 0$ and $s_0 > 0$ such that

$$Q_{s_0}(i, x: j, dy) \geqq c dy$$

for all (i, x) in E and j in $\langle 1, n \rangle$. This proves (8.7).

For $t \geqq s_1$, we have

$$\begin{aligned} \phi(t+s_0) &= \int_E p_{0,t+s_0}(i_0, x_0: j, y) \mu(dj) dy \\ &\geqq \int_E p_{0,t}(i_0, x_0: k, z) \mu(dk) dz p_{t,t+s_0}(k, z: E) \\ &\quad + \int_E P_{0,t}^{(s)}(i_0, x_0: dk, dz) c \int_E \mu(dj) dy \\ &= \phi(t) + cn\omega(1 - \phi(t)) . \end{aligned}$$

Let $\phi(\infty) = \lim_{t \to \infty} \phi(t)$, then it follows that

$$\phi(\infty) \geqq \phi(\infty) + cn\omega(1 - \phi(\infty)) .$$

This means $\phi(\infty)=1$, and we have proved Lemma 8.2.

We now turn to the proof of Theorem 8.1. Let f be a measurable function such that $0\leqq f(i,x)\leqq 1$.

$\int_E p_{0,s}(i_0,x_0:k,z)\mu(dk)dzP_{s,s+t}(k,z:di,dx)=\Phi(t,di,dx)$ is a solution of the equation (7.1) with the initial value $p_{0,s}(i_0,x_0:i,x)\mu(di)dx$. Let $\Psi(t,i,x)$ be the solution of the equation (7.7) with the initial value $p_{0,s}(i_0,x_0:i,x)$. From (7.8) $\Phi(t,di,dx)=\Psi(t,i,x)\mu(di)dx$. Since $\int_E p_{0,s}(i_0,x_0:i,x)\mu(di)dx=\phi(s)$, (8.4) implies

$$\lim_{t\to\infty}\int_E f(i,x)\Psi(t,i,x)\mu(di)dx=\frac{\phi(s)}{\int_E u(i,x)\mu(di)dx}\int_E f(i,x)u(i,x)\mu(di)dx,\ i.e.,$$

$$\lim_{t\to\infty}\int_E p_{0,s}(i_0,x_0:i,x)P_{s,s+t}f(i,x)\mu(di)dx \tag{8.8}$$

$$=\frac{\phi(s)}{\int_E u(i,x)\mu(di)dx}\int_E f(i,x)u(i,x)\mu(di)dx\,.$$

From (8.6), it holds that

$$P_{0,t+s}f(i_0,x_0)=\int_E p_{0,s}(i_0,x_0:i,x)P_{s,s+t}f(i,x)\mu(di)dx$$
$$+\int_E P_{0,s}^{(s)}(i_0,x_0:di,dx)P_{s,s+t}f(i,x)\,.$$

Therefore, combining (8.8) with

$$\int_E p_{0,s}(i_0,x_0:i,x)P_{s,s+t}f(i,x)\mu(di)dx$$
$$\leqq\int_E P_{0,s+t}f(i_0,x_0)$$
$$\leqq\int_E p_{0,s}(i_0,x_0:i,x)P_{s,s+t}f(i,x)\mu(di)dx+\int_E P_{0,s}^{(s)}(i_0,x_0:di,dx)$$
$$=\int_E p_{0,s}(i_0,x_0:i,x)P_{s,s+1}f(i,x)\mu(di)dx+1-\phi(s)\,,$$

we obtain

$$\phi(s)\frac{\int_E u(i,x)f(i,x)\mu(di)dx}{\int_E u(i,x)\mu(di)dx}+1-\phi(s)$$
$$\geqq\overline{\lim_{t\to\infty}}P_{0,s+t}f(i_0,x_0)\geqq\underline{\lim_{t\to\infty}}P_{0,s+t}f(i_0,x_0)$$
$$\geqq\phi(s)\frac{\int_E u(i,x)f(i,x)\mu(di)dx}{\int_E u(i,x)\mu(di)dx}\,.$$

From Lemma 8.2,

$$\lim_{t\to\infty} P_{0,t}f(i_0, x_0) = \frac{\int_E u(i, x)f(i, x)\mu(di)dx}{\int_E u(i, x)\mu(di)dx}.$$

Therefore we have proved Theorem 8.1.

9. The case $n=2$. Examples and some degenerate cases

In this section, we always assume that $d_1=d_2=0$ and $v_1 \neq v_2$. We do not assume (1.3) unless otherwise stated.

9.1. Classification of equation (I)

Noting that $a_{j,k}^2 = -a_{j,k}^1$ by (1.2), equation (I) becomes

$$(E)\begin{cases} \dfrac{\partial}{\partial t}u(t, 1, x) = v_1\dfrac{\partial}{\partial x}u(t, 1, x) + a_{1,1}^1 u(t, 1, x)^2 \\ \qquad\qquad + (a_{1,2}^1 + a_{2,1}^1)u(t, 1, x)u(t, 2, x) + a_{2,2}^1 u(t, 2, x)^2, \\ \dfrac{\partial}{\partial t}u(t, 2, x) = v_2\dfrac{\partial}{\partial x}u(t, 2, x) - a_{1,1}^1 u(t, 1, x)^2 \\ \qquad\qquad - (a_{1,2}^1 + a_{2,1}^1)u(t, 1, x)u(t, 2, x) - a_{2,2}^1 u(t, 2, x)^2. \end{cases}$$

We first classify equation (E) into three cases.

Case 1. $a_{1,1}^1 \neq 0$.

Let $a=-a_{1,1}^1>0$, and we define α and β as the solution of equation

$$\begin{cases} \alpha+\beta = -\dfrac{a_{1,2}^1 + a_{2,1}^1}{a_{1,1}^1}, \\ \alpha\beta = \dfrac{a_{2,2}^1}{a_{1,1}^1}. \end{cases}$$

Baecause $a_{1,1}^1 a_{2,2}^1 \leqq 0$, we may assume that $\alpha \leqq 0 \leqq \beta$.

Case 2. $a_{1,1}^1 = a_{2,2}^1 = 0$.

Let $b=-(a_{1,2}^1 + a_{2,1}^1)$. If $b=0$, then equation (E) becomes a linear differential equation, and therefore we omit this case. We may assume $b>0$, for the case when $b<0$ is reduced to this case by interchanging u_1 and u_2.

Case 3. $a_{1,1}^1 = 0$, $a_{2,2}^1 \neq 0$.

This is reduced to case 1 again by interchanging u_1 and u_2. Therefore we may only consider equation (E) in two cases.

$$(E.1)\begin{cases}\dfrac{\partial}{\partial t}u(t,1,x)=v_1\dfrac{\partial}{\partial x}u(t,1,x)-a(u(t,1,x)-\alpha u(t,2,x))\,(u(t,1,x)\\ \qquad\qquad -\beta u(t,2,x))\,,\\ \dfrac{\partial}{\partial t}u(t,2,x)=v_2\dfrac{\partial}{\partial x}u(t,2,x)+a(u(t,1,x)-\alpha u(t,2,x))\,(u(t,1,x)\\ \qquad\qquad -\beta u(t,2,x))\,,\end{cases}$$

$$\alpha\leqq 0\leqq\beta,\ a>0\,.$$

$$(E.2)\begin{cases}\dfrac{\partial}{\partial t}u(t,1,x)=v_1\dfrac{\partial}{\partial x}u(t,1,x)-bu(t,1,x)u(t,2,x)\,,\\ \dfrac{\partial}{\partial t}u(t,2,x)=v_2\dfrac{\partial}{\partial x}u(t,2,x)+bu(t,1,x)u(t,2,x)\,,\end{cases}$$

$$b>0\,.$$

We discuss $(E.1)$ in four cases separately:

$(E.1.\mathrm{a})\quad \alpha<0<\beta\,,$

$(E.1.\mathrm{b})\quad \alpha=0<\beta\,,$

$(E.1.\mathrm{c})\quad \alpha<0=\beta\,,$

$(E.1.\mathrm{d})\quad \alpha=0=\beta\,.$

9.2. Behaviors of solutions of equation (E)

Hereafter, we shall assume that the initial values are non trivial, *i.e.*, $u(0,i,x)\not\equiv 0$, and we denote $m=\dfrac{1}{\omega}\displaystyle\int_{S^1}(u(0,1,x)+u(0,2,x))dx$. In cases $(E.1.\mathrm{a})$ and $(E.1.\mathrm{b})$, let $u_1^0=\dfrac{\beta}{\beta+1}$ and $u_2^0=\dfrac{1}{\beta+1}$. Then it follows that

$$\sum_{j,k=1}^{2}a_{j,k}^{i}u_j^0u_k^0=0\qquad i=1,2\,.$$

Let $b_{j,k}^{i}=(u_i^0)^{-1}a_{j,k}^{i}u_j^0u_k^0$ and $b_{i,j}=\sum_{k=1}^{2}(a_{j,k}^{i}+a_{k,j}^{i})u_k^0$ as in the Assumption 5.1. Then

$$(b_{i,j})=\frac{a(\beta-\alpha)}{1+\beta}\begin{pmatrix}-1, & \beta\\ 1, & -\beta\end{pmatrix}$$

is a nonnegative offdiagonal irreducible matrix. Therefore Assumption 5.1 is satisfied.

By $(E.1)$,

$$\sum_{j,k=1}^{2}a_{j,k}^{1}\eta_j\eta_k=-a(\eta_1-\alpha\eta_2)\,(\eta_1-\beta\eta_2)\,,$$

$$\sum_{j,k=1}^{2}a_{j,k}^{2}\eta_j\eta_k=a(\eta_1-\alpha\eta_2)\,(\eta_1-\beta\eta_2)\,.$$

Therefore,

$$\sum_{j,k=1}^{2} b_{j,k}^{1}\eta_j\eta_k = -\frac{a}{1+\beta}(\eta_1-\eta_2)(\beta\eta_1-\alpha\eta_2),$$

$$\sum_{j,k=1}^{2} b_{j,k}^{2}\eta_j\eta_k = \frac{a\beta}{1+\beta}(\eta_1-\eta_2)(\beta\eta_1-\alpha\eta_2).$$

This means for $c=\max\left(\frac{a(\beta-\alpha)}{1+\beta}, \frac{a\beta(\beta-\alpha)}{1+\beta}\right)$, and $0\leqq\eta_1, \eta_2\leqq 1$,

$$0\leqq \sum_{j,k=1}^{2} b_{j,k}^{1}\eta_j\eta_k+c\eta_1\leqq c,$$

$$0\leqq \sum_{j,k=1}^{2} b_{j,k}^{2}\eta_j\eta_k+c\eta_2\leqq c.$$

Therefore Assumption 6.1 is also satisfied. By Lemma 6.1, equation $(E.1)$ always has a bounded global solution.

9.2.1. Case (E.1.a)

We verify condition (a) in Lemma 6.2. Let $c\geqq\max\left(\frac{a\beta^2-2a\alpha\beta}{1+\beta}, \frac{a(\beta-\alpha)}{1+\beta}\right)$, then for $0\leqq\eta_1, \eta_2\leqq 1$,

$$\begin{aligned}\sum_{j,k=1}^{2} b_{j,k}^{1}\eta_j\eta_k+c\eta_1 &= \frac{-a\alpha}{1+\beta}\eta_2^2+\left(\frac{a\beta}{1+\beta}\eta_2-\frac{a\beta}{1+\beta}\eta_1+\frac{a\alpha}{1+\beta}\eta_2+c\right)\eta_1\\ &\geqq \frac{-a\alpha}{1+\beta}\eta_2^2+\left(c-\frac{a\beta}{1+\beta}+\frac{a\alpha}{1+\beta}\right)\eta_1^2,\\ \sum_{j,k=1}^{2} b_{j,k}^{2}\eta_j\eta_k+c\eta_2 &= \left(\frac{a\alpha\beta}{1+\beta}\eta_1-\frac{a\beta^2}{1+\beta}\eta_1+\frac{a\alpha\beta}{1+\beta}\eta_2+c\right)\eta_2+\frac{a\beta^2}{1+\beta}\eta_1^2\\ &\geqq \frac{a\beta^2}{1+\beta}\eta_1^2+\left(c+\frac{2a\alpha\beta}{1+\beta}-\frac{a\beta^2}{1+\beta}\right)\eta_2^2.\end{aligned}$$

Hence the irreducible matrix

$$\begin{pmatrix} c-\frac{a\beta}{1+\beta}+\frac{a\alpha}{1+\beta}, & \frac{-a\alpha}{1+\beta} \\ \frac{a\beta^2}{1+\beta} & , c+\frac{2a\alpha\beta}{1+\beta}-\frac{a\beta^2}{1+\beta} \end{pmatrix}$$

satisfies condition (6.8). Therefore by Theorem 6.2, the solution $(u(t, 1, x), u(t., 2, x))$ converges to $m\left(\frac{\beta}{1+\beta}, \frac{1}{1+\beta}\right)$ exponentially fast.

9.2.2. Case (E.1.b)

In this case, it is not difficult to see that condition (a) of Lemma 6.2 is not satisfied. However, condition (6.5) of Theorem 6.1 is replaced by

$$(9.1) \qquad \inf_{x\in S^1} u(0, 1, x) = \delta > 0 ,$$

by the following proposition.

Proposition 9.1. *If* (9.1) *is satisfied, then there exists* $t_1 > 0$ *and* $c > 0$ *such that*

$$\inf_{\substack{x\in S^1\\ i=1,2}} u(t_1, i, x) \geqq c .$$

Proof. Let $u(t, x) = \frac{1}{\beta} u(t, 1, x)$ and $v(t, x) = u(t, 2, x)$, then equation $(E.1)$ becomes

$$(9.2) \qquad \begin{cases} \dfrac{\partial}{\partial t} u(t, x) = v_1 \dfrac{\partial}{\partial x} u(t, x) - \beta a(u(t, x) - v(t, x))u(t, x) , \\ \dfrac{\partial}{\partial t} v(t, x) = v_2 \dfrac{\partial}{\partial x} v(t, x) + \beta^2 a(u(t, x) - v(t, x))u(t, x) . \end{cases}$$

We may assume that there exists $M > 0$ such that

$$0 \leqq u(t, x), v(t, x) \leqq M \qquad t \geqq 0, x \in S^1 .$$

Transform equation (9.2) as

$$(9.2)' \qquad \begin{cases} \dfrac{\partial}{\partial t} u(t, x) = v_1 \dfrac{\partial}{\partial x} u(t,x) - 2\beta a M u(t,x) + \beta a(v(t,x) - u(t,x) + 2M)u(t,x) , \\ \dfrac{\partial}{\partial t} v(t, x) = v_2 \dfrac{\partial}{\partial x} v(t,x) - 2\beta^2 a M v(t,x) + \beta^2 a(u(t,x) - v(t,x) + 2M)u(t,x) , \end{cases}$$

or equivalently,

$$\begin{cases} u(t, x) = e^{-2\beta a M t} u(0, x + v_1 t) \\ \qquad + \displaystyle\int_0^t e^{-2\beta a M(t-s)} \beta a(v(s, x + v_1(t-s)) - u(s, x + v_1(t-s)) + 2M)u(s, x + v_1(t-s))ds , \\ v(t, x) = e^{-2\beta^2 a M t} v(0, x + v_2 t) \\ \qquad + \displaystyle\int_0^t e^{-2\beta^2 a M(t-s)} \beta^2 a(u(s, x + v_2(t-s)) - v(s, x + v_2(t-s)) + 2M)u(s, x + v_2(t-s))ds . \end{cases}$$

Therefore, $u(t, x) \geqq e^{-2\beta a M t} u(0, x + v_1 t) \geqq e^{-2\beta a M t}\delta$, and therefore,

$$\begin{aligned} v(t, x) &\geqq \int_0^t e^{-2\beta^2 a M(t-s)} \beta^2 a u(s, x + v_2(t-s))^2 ds \\ &\geqq \int_0^t e^{-2\beta^2 a M(t-s) - 4\beta a M s} \delta^2 \beta^2 a ds \\ &\geqq t e^{-2\beta^2 a M t - 4\beta a M t} \delta^2 \beta^2 a . \end{aligned}$$

By this proposition, if $u(0, 1, x) \geqq \delta > 0$, then $(u(t, 1, x), u(t, 2, x))$ converges to $m\left(\frac{\beta}{1+\beta}, \frac{1}{1+\beta}\right)$ exponentially fast.

In the following three paragraphs, general theory in the preceding sections is not applicable, but we can show directly the asymptotic behavior of solutions.

9.2.3. Case (*E*.1.c)

Let $u(t,x)=-\frac{1}{\alpha}u(t,1,x)$ and $v(t,x)=u(t,2,x)$, then equation (*E*.1) becomes

$$(9.3)\qquad \begin{cases} \frac{\partial}{\partial t}u(t,x) = v_1\frac{\partial}{\partial x}u(t,x)+a\alpha u(t,x)\,(u(t,x)+v(t,x))\,, \\ \frac{\partial}{\partial t}v(t,x) = v_2\frac{\partial}{\partial x}v(t,x)+a\alpha^2u(t,x)\,(u(t,x)+v(t,x))\,, \end{cases}$$

or equivalently,

$$(9.3)'\qquad \begin{cases} u(t,x) = u(0,x+v_1t)\exp\left[a\alpha\int_0^t\{u(s,x+v_1(t-s))+v(s,x+v_1(t-s))\}ds\right]\,, \\ v(t,x) = v(0,x+v_2t)+\int_0^t a\alpha^2u(s,x+v_2(t-s))\,\{u(s,x+v_2(t-s)) \\ \qquad\qquad +v(s,x+v_2(t-s))\}ds\,. \end{cases}$$

Therefore, there exists $M>0$ such that

$$u(t,x)\leqq M \qquad \text{for any } t\geqq 0,\ x\in S^1\,,$$

and equation (9.3) has a global solution.
If $u(0,x)\equiv 0$, then $u(t,x)\equiv 0$, $v(t,x)=v(0,x+v_1t)$.

Proposition 9.2. *If $u(0,x)\not\equiv 0$, then there exists $N>0$ and $c>0$ such that*

$$(9.4)\qquad u(t,x)\leqq Ne^{-ct} \qquad t\geqq 0,\ x\in S^1\,,$$

and therefore, $v(t,x)$ is bounded in t and x.

Proof. By (9.3)′, for any $T<\infty$, there exists $c(T)>0$ such that

$$c(T)u(0,x+v_1t)\leqq u(t,x) \qquad 0\leqq t\leqq T,\ x\in S^1\,.$$

Therefore, for any $t\leqq T$,

$$v(t,x)\geqq\int_0^t a\alpha^2u(s,x+v_2(t-s))^2ds$$
$$\geqq a\alpha^2c(T)^2\int_0^t u(0,x+v_2t+(v_1-v_2)s)^2ds\,,$$

and there exists $T_1>0$ and $d>0$ such that

$$v(T_1,x)\geqq d \qquad x\in S^1\,.$$

Consider (9.3), $(u(T_1, x), v(T_{,1} x))$ as the initial value, we get

$$v(t, x) \geqq d \qquad t \geqq T_1,\ x \in S^1 .$$

By (9.3)$'$, for $t \geqq T_1$,

$$\begin{aligned} u(t, x) &\leqq u(0, x+v_2 t) \exp\left[a\alpha \int_0^t v(s, x+v_2(t-s))ds\right] \\ &\leqq u(0, x+v_2 t) \exp\left[a\alpha(t-T_1)d\right] . \end{aligned}$$

By (9.3) and (9.4), $v(t, x)$ remains bounded in t and x.
If $v_2=0$, then

$$\lim_{t\to\infty} v(t, x) = v(0, x)+\int_0^\infty a\alpha^2 u(s, x)\{u(s, x)+v(s, x)\}\, ds .$$

If $v_2 \neq 0$, then

$$\begin{aligned} \lim_{n\to\infty} v\left(\frac{n\omega}{|v_2|}+t, x\right) &= v(0, x+v_2 t) \\ &+\int_0^\infty a\alpha^2 u(s, x+v_2(t-s))\{u(s, x+v_2(t-s))+v(s, x+v_2(t-s))\}\, ds . \end{aligned}$$

9.2.4. Case (E.1.d)

Equation (E.1) becomes

$$(9.5) \qquad \begin{cases} \dfrac{\partial}{\partial t} u(t, 1, x) = v_1 \dfrac{\partial}{\partial x} u(t, 1, x) - a u(t, 1, x)^2 , \\ \dfrac{\partial}{\partial t} u(t, 2, x) = v_2 \dfrac{\partial}{\partial x} u(t, 2, x) + a u(t, 1, x)^2 , \end{cases}$$

or, equivalently,

$$\begin{cases} u(t, 1, x) = \dfrac{u(0, 1, x+v_1 t)}{1+tau(0, 1, x+v_1 t)} , \\ u(t, 2, x) = u(0, 2, x+v_2 t)+\displaystyle\int_0^t \left(\frac{u(0, 1, x+v_1 s+v_2(t-s))}{1+sau(0, 1, x+v_1 s+v_2(t-s))}\right)^2 ds . \end{cases}$$

By (9.5)$'$, $u(t, 1, x)$ converges to 0 as $t\to\infty$.
If $v_2 \neq 0$, then

$$\lim_{n\to\infty} u\left(\frac{n\omega}{|v_2|}+t, 2, x\right) = u(0, 2, x+v_2 t)+\int_0^\infty \left(\frac{u(0, 1, x+v_2 t+(v_1-v_2)s)}{1+sau(0,1,x+v_2 t+(v_1-v_2)s)}\right)^2 ds .$$

If $v_2=0$, then

$$\lim_{t\to\infty} u(t, 2, x) = u(0, 2, x)+\int_0^\infty \left(\frac{u(0, 1, x+v_1 s)}{1+sau(0, 1, x+v_1 s)}\right)^2 ds .$$

9.2.5. Equation ($E.2$)

Equation ($E.2$) is equivalent to

$$(E.2)' \quad \begin{cases} u(t,1,x)=u(0,1,x+v_1t)\exp\{-b\int_0^t u(s,2,x+v_1(t-s))ds\}\ , \\ u(t,2,x)=u(0,2,x+v_2t)\exp\{b\int_0^t u(s,1,x+v_2(t-s))ds\}\ . \end{cases}$$

By this expression,

$$(9.6) \quad \begin{cases} u(t,1,x)\leqq u(0,1,x+v_1t)\ , \\ u(t,2,x)\geqq u(0,2,x+v_2t)\ . \end{cases}$$

It then follows that

$$(9.7) \quad \begin{cases} u(t,1,x)\leqq u(0,1,x+v_1t)\exp\{-b\int_0^t u(0,2,x+v_1t+(v_2-v_1)s)ds\}\ , \\ u(t,2,x)\leqq u(0,2,x+v_2t)\exp\{b\int_0^t u(0,1,x+v_2t+(v_1-v_2)s)ds\}\ . \end{cases}$$

If $u(0,2,x)\equiv 0$, then $(E.2)'$ implies that $u(t,2,x)\equiv 0$ and therefore $u(t,1,x)=u(0,1,x+v_1t)$. Similarly if $u(0,1,x)\equiv 0$, then $u(t,1,x)\equiv 0$, $u(t,2,x)=u(0,2,x+v_2t)$.

In other case, since $u(0,2,x)\not\equiv 0$ and $v_1\neq v_2$, we have

$$\frac{1}{t}\int_0^t u(0,2,x+(v_1-v_2)s)ds=\frac{1}{t(v_1-v_2)}\int_0^{t(v_1-v_2)} u(0,2,x+s)ds$$

$$\to\frac{1}{\omega}\int_0^{\omega} u(0,2,y)dy>0$$

for any x in S^1. Therefore by (9.7),

$$u(t,1,x)\leqq Me^{-ct}$$

for some $c>0$ and $M>0$. By the second part of $(E.2)'$, $u(t,2,x)$ remains bounded.

If $v_2=0$, then by $(E.2)'$

$$\lim_{t\to\infty} u(t,2,x)=u(0,2,x)\exp\{a\int_0^\infty u(s,1,x)ds\}\ .$$

If $v_2\neq 0$, then

$$\lim_{n\to\infty} u\left(\frac{n\omega}{|v_2|}+t,2,x\right)=u(0,2,x+v_2t)\exp\{a\int_0^\infty u(s,1,x+v_2(t-s))ds\}\ .$$

Yoshikawa [13] has obtained more detailed results in this case.

9.3. Limit distributions of the associated Markov processes

We now assume (1.3). For a solution $(u(t,1,x),u(t,2,x))$ of (E), let (X_t)

be its associated Markov process defined in section 7.

9.3.1. Cases (*E*.1.a) and (*E*.1.b)

In case (*E*.1.a), $(u(t, 1, x), u(t, 2, x))$ converges to $m\left(\frac{\beta}{1+\beta}, \frac{1}{1+\beta}\right)$. By Theorem 8.1,

$$\lim_{t\to\infty} P_{0,(i,x)}(X_t\in(j, dy)) = \begin{cases} \frac{\beta}{1+\beta}\frac{dy}{\omega} & j=1, i=1, 2, x\in S^1, \\ \frac{1}{1+\beta}\frac{dy}{\omega} & j=2, i=1, 2, x\in S^1. \end{cases}$$

In case (*E*.1.b), we assume that the additional hypothesis (9.1) on $u(0, 1, x)$ is satisfied, then the same conclusion is valid.

9.3.2. Cases (*E*.1.c), (*E*.1.d) and equation (*E*.2)

We have seen in 9.2, that in general

$$\lim_{t\to\infty} u(t, 1, x) = 0,$$

and there exists a bounded nonnegative function $v(x)$ on S^1 such that if $v_2=0$, then $\lim_{t\to\infty} u(t, 2, x)=v(x)$, and if $v_2\neq 0$, then

$$\lim_{n\to\infty} u\left(\frac{n\omega}{|v_2|}+t, 2, x\right) = v(x+v_2t).$$

Therefore we can not apply Theorem 8.1 in these cases. However we can discuss on the limit distribution of (X_t) under the following additional assumption:

Assumption. There exists a bounded nonnegative function $v(x)$ which is not identically 0, such that if $v_2=0$, then

$$\begin{cases} u(t, 1, x) = 0, \\ u(t, 2, x) = v(x), \end{cases}$$

is a solution of equation (I), and if $v_2\neq 0$, then

$$\begin{cases} u(t, 1, x) = 0, \\ u(t, 2, x) = v(x+v_2t), \end{cases}$$

is a solution of equation (*E*).

In the present cases, since $a_{2,2}^{1}=a_{2,2}^{2}=0$, $a_{1,2}^{1}=-a_{1,2}^{2}<0$, we may only consider the following equation:

$$\text{(9.8)} \quad \begin{cases} \frac{\partial}{\partial t}\phi(t, 1, dx) = \phi(t, 1, dx)\left(-v_1\frac{\partial}{\partial x}+a_{1,2}^{1}u(t, 2, x)\right), \\ \frac{\partial}{\partial t}\phi(t, 2, dx) = \phi(t, 2, dx)\left(-v_2\frac{\partial}{\partial x}\right)+\phi(t, 1, dx)\left(a_{1,2}^{2}u(t, 2, x)\right), \end{cases}$$

or equivalently,

$$(9.8)' \quad \begin{cases} \phi(t,1,dx) = \int_{S^1}\phi(0,1,dy)e^{a_{1,2}^1\int_0^t u(s,2,y-v_1s)ds}\delta_{y-v_1t}(dx) \\ \qquad\qquad = \int_{S^1}\phi(0,1,dy)e^{a_{1,2}^1\int_0^t v(y+(v_2-v_1)s)ds}\delta_{y-v_1t}(dx)\,, \\ \phi(t,2,dx) \end{cases}$$

$$= \int_{S^1}\phi(0,2,dy)\delta_{y-v_2t}(dx)+\int_0^t dsa_{1,2}^2\int_{S^1}\phi(s,1,dy)u(s,2,y)\delta_{y-v_2(t-s)}(dx)$$

$$= \int_{S^1}\phi(0,2,dy)\delta_{y-v_2t}(dx)$$

$$+\int_0^t dsa_{1,2}^2\int_{S^1}\phi(0,1,dy)e^{a_{1,2}^1\int_0^s v(y+(v_1-v_2)r)dr}v(y+(v_2-v_1)s)\delta_{y-v_2t+(v_2-v_1)s}(dx)\,.$$

Since we have assumed that $v_2 \neq v_1$ and $v(x) \not\equiv 0$,

$$\lim_{t\to\infty}\int_0^t v(y+(v_2-v_1)s)ds = \infty \qquad \text{for any } y \text{ in } S^1\,,$$

and therefore,

$$\lim_{t\to\infty}\phi(t,1,S^1) = 0\,.$$

If $\phi(0,di,dy) = \delta_2(di)\delta_x(dy)$, then by (9.8)′

$$(9.9) \quad \begin{cases} \phi(t,2,dy) = \delta_{x-v_2t}(dy)\,, \\ \phi(t,1,dy) = 0\,. \end{cases}$$

If $\phi(0,di,dy) = \delta_1(di)\delta_x(dy)$, then by (9.8)′

$$\phi(t,2,dy) = \int_0^t dsa_{1,2}^2 e^{a_{1,2}^1\int_0^s v(x+(v_2-v_1)r)dr}v(x+(v_2-v_1)s)\delta_{x-v_2t+(v_2-v_1)s}(dy)\,.$$

Therefore if $v_2=0$, then

$$(9.10) \quad \lim_{t\to\infty}\phi(t,2,dy) = \int_0^\infty dsa_{1,2}^2 e^{a_{1,2}^1\int_0^s v(x-v_1r)dr}v(x-v_1s)\delta_{x-v_1s}(dy)\,,$$

and if $v_2 \neq 0$, then

$$(9.11) \quad \lim_{n\to\infty}\phi\left(\frac{n\omega}{|v_2|}+t,2,dy\right)$$

$$= \int_0^\infty dsa_{1,2}^2 e^{a_{1,2}^1\int_0^s v(x+(v_2-v_1)r)dr}v(x+(v_2-v_1)s)\delta_{x-v_2t+(v_2-v_1)s}(dy)\,.$$

Summing up (9.9), (9.10) and (9.11), we obtain:

$$\lim_{t\to\infty}P_{0,(i,x)}(X_t\in(1,S^1)) = 0 \qquad i=1,2,\ x\in S^1\,.$$

If $v_2=0$, then

$$\lim_{t\to\infty} P_{0,(2,x)}(X_t \in (2, dy)) = \delta_x(dy)\,,$$

$$\lim_{t\to\infty} P_{0,(1,x)}(X_t \in (2, dy)) = K(x, dy)\,,$$

where $K(x, dy)$ is the kernel defined by the right side of (9.10).
If $v_2 \neq 0$, then

$$\lim_{n\to\infty} P_{0,(2,x)}(X_{n\omega/|v_2|+t} \in (2, dy)) = \delta_{x-v_2t}(dy) \qquad 0 \leqq t \leqq \frac{\omega}{|v_2|}\,,$$

$$\lim_{n\to\infty} P_{0,(1,x)}(X_{n\omega/|v_2|+t} \in (2, dy)) = K_t(x, dy) \qquad 0 \leqq t \leqq \frac{\omega}{|v_2|}\,,$$

where $K_t(x, dy)$ is the kernel defined by the right side of (9.11).

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