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Osaka J. Math.
14 (1977), 173–177

K-GROUPS OF EIII AND FII

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(Received December 9, 1975)

1. Let G be a compact, connected, simply-connected Lie group and K a closed connected subgroup of G of maximal rank. As is well known [3], the complex K-group of G/K is isomorphic to $R(K)\otimes_{R(G)} Z$ and it is a free abelian group with rank equal to the quotient of the order of the Weyl group of G by the order of the Weyl group of K. Here $R(G)$ is the complex representation ring of G. The purpose of this paper is to determine an additive structure of the complex K-groups of symmetric spaces $EIII=E_6/Spin(10)\cdot SO(2)$ and $FII=F_4/Spin(9)$. To simplify the notation we write x for the element $x\otimes 1$ of $R(K)\otimes_{R(G)} Z$ in the following.

Let Δ^+ and Δ^- be the half-spin representations of $Spin(10)$, and let ρ and t be the canonical non-trivial 10- and 1-dimensional complex representations of $Spin(10)$ and $SO(2)$ respectively. Then

$$R(Spin(10)\times SO(2)) = Z[\lambda^1\rho, \lambda^2\rho, \lambda^3\rho, \Delta^+, \Delta^-, t, t^{-1}]$$

and $R(Spin(10)\cdot SO(2))$ is isomorphic to the subalgebra of $R(Spin(10)\times SO(2))$ generated by the representations of $Spin(10)\times SO(2)$ which are trivial on $Spin(10)\cap SO(2)=Z_4$ (See [5] and [2, I], Prop. 2.1). Furthermore then our result is stated as follows.

Theorem.

$$K^*(EIII)\cong Z\{x^i, x^iw, x^jw^2, x^kv \mid 0\leq i\leq 8,\ 0\leq j\leq 4,\ 0\leq k\leq 3\}$$

where

$$\begin{aligned} x &= t^4-1\,, \\ w &= (t^2\rho-10)-x^3+2x^2-5x\,, \\ v &= 45xw^3+26x^5w^2 \end{aligned}$$

and $Z\{a, b, c, \cdots\}$ *is the free abelian group generated by the set* $\{a, b, c,\cdots\}$.

Besides we have (2.1) of Section 2 concerning a ring structure of $K^*(EIII)$. Now, recently Steinberg [4] gave a general formula of a free basis over $R(G)$ for an $R(G)$-module $R(K)$ (by restriction).

2. Using the notation in Table V of [1] we denote by ρ_1 and ρ_2 the 27-dimensional representations of E_6 with the highest weights $\frac{1}{3}(4\alpha_1+3\alpha_2+5\alpha_3+6\alpha_4+4\alpha_5+2\alpha_6)$ and $\frac{1}{3}(2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6)$ respectively and by Ad the adjoint representation of E_6.

Lemma. *Let i^*: $R(E_6)\to R(Spin(10)\cdot SO(2))$ be the restriction induced by the natural inclusion i: $Spin(10)\cdot SO(2)\to E_6$. Then we have*

(i) $i^*(Ad)=\lambda^2\rho+t^3\Delta^++t^{-3}\Delta^-+1$
(ii) $i^*(\rho_1)=t^4+t\Delta^-+t^{-2}\rho$
(iii) $i^*(\rho_2)=t^{-4}+t^{-1}\Delta^++t^2\rho$
(iv) $i^*(\lambda^2\rho_1)=t^5\Delta^-+t^2\lambda^3\rho+t^2\rho+t^{-1}\Delta^-\rho+t^{-4}\lambda^2\rho$
(v) $i^*(\lambda^2\rho_2)=t^{-5}\Delta^++t^{-2}\lambda^3\rho+t^{-2}\rho+t\Delta^+\rho+t^4\lambda^2\rho$
(vi) $i^*(\lambda^3\rho_1)=i^*(\lambda^3\rho_2)=t^6\lambda^3\rho+t^{-6}\lambda^3\rho+t^3\Delta^+\lambda^2\rho+t^{-3}\Delta^-\lambda^2\rho+\rho\lambda^3\rho+\lambda^2\rho.$

Proof. (i)-(iii) are verified by observing the restriction of all weights of ρ_1, ρ_2 and Ad to $Spin(10)\cdot SO(2)$. Here this reduction is based on the formula given in Section 1 of [5], and the weights of ρ_1, ρ_2 and Ad are listed in Section 5.

Consider the exterior powers of the formulas (ii) and (iii) then we can aesily check (iv)-(vi) since $\lambda^2\Delta^\pm=\lambda^3\rho$, $\lambda^2(\lambda^2\rho)+\lambda^4\rho=\rho\lambda^3\rho$ and $\lambda^3\Delta^-+\rho\Delta^-=\Delta^+\lambda^2\rho$. q.e.d.

By Lemma we see that

(2.1) *$R(Spin(10)\cdot SO(2))\underset{R(E_6)}{\otimes} Z(\cong K^*(EIII))$ is multiplicatively generated by two elements x and w with relations*

$$(2.2)\quad (x^3+3x^2+3x)w^2+(x^{12}-x^{11}+x^{10}-x^9+x^8-x^7+3x^6)w +2(x^{16}-x^{15}+x^{14}-x^{13}+x^{12}-x^{11}+x^{10})-x^9=0$$

and

$$w^3-(2x^8-2x^7+2x^6-x^5+7x^4+5x^3+18x^2+15x)w^2 -\{8(x^{12}-x^{11}+x^{10}-x^9)+10x^8-2x^7+15x^6\}w-8(x^{16}-x^{15}+x^{14}-x^{13}+x^{12}) +7x^{11}-9x^{10}+5x^9=0\,.$$

When we calculate (2.2), note that

$$(2.3)\qquad x^{17}=0$$

since *EIII* is a differentiable manifold of dimension 32, and $x+1$ is invertible.

It follows from (2.2) that

$$(2.4)\quad \{x^9(x^5+15x^4+78x^3+182x^2+195x+78)w+13x^7+53x^6+84x^5+45x^4\}=0\,.$$

3. Proof of Theorem. By (2.3) and (2.4) we have inductively

$$624x^9w = 109x^{16}-154x^{15}+228x^{14}-360x^{13} \tag{3.1}$$

and so by this formula we have

$$x^9w^2 = 0\,. \tag{3.2}$$

Then we get

$$\begin{aligned} x^9 = &\{14(x^8+x^7+x^6+x^5+x^4)+13x^3+9x^2+3x\}w^2 \\ &+\{5(x^{12}+x^{11}+x^{10}+x^9+x^8+x^7)+3x^6\}w \end{aligned} \tag{3.3}$$

by (3.1), (3.2) and the first formula of (2.2), and moreover

$$xw^3 = (12x^8+16x^7+12x^6+15x^5)w^2+(x^{12}+9x^{11}+x^{10}+9x^9)w \tag{3.4}$$

by (3.1)-(3.3) and the secondary formula of (2.2).

It follows that $26x^{12}w=-15x^{16}$ and $x^{16}=3x^8w^2$ from (3.1) and (3.3) respectively. Therefore $26x^{12}w+45x^8w^2=0$ and so we see that

$$x^8w^2 = 26x^3v \quad \text{and} \quad x^{12}w = -45x^3v$$

using the equality $x^4w^3=15x^8w^2+9x^{12}w$ obtained by (3.4). The analogous arguments show inductively

$$x^5w^2 = -45\cdot 53944550x^3v+45\cdot 104903x^2v-45\cdot 246xv+26v \tag{3.5}$$

and

$$x^9w = 4196254501x^3v-5\cdot 1631629x^2v+19131xv-45v\,.$$

Consequently we have Theorem after a slight consideration because $x^4v=v^2=vw=0$ and the rank of $K^*(EIII)$ is equal to 27.

4. Denote by $j\colon F_4\to E_6$ the canonical imbedding of F_4 in E_6. Then $j^*\colon R(E_6)\to R(F_4)$ is surjective and particularly

$$j^*(\rho_1)=j^*(\rho_2)=\rho'+1 \quad \text{and} \quad j^*(Ad)=Ad'+\rho'$$

(See (6.7) and (6.8) of [2.I]) where ρ' is the irreducible representation of F_4 with the highest weight $\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$ using the notation in Table VIII of [1] and Ad' is the adjoint representation of F_4. Therefore Lemma implies the following

Corollary 1 (cf. [6], Theorem 15.1).

$$\begin{cases} k^*(\rho') = 1+\rho+\Delta \\ k^*(\lambda^2\rho') = \rho+2\lambda^2\rho+\lambda^3\rho+\Delta+\Delta\rho \\ k^*(\lambda^3\rho') = 2\lambda^2\rho+2\lambda^3\rho-\Delta+\rho\lambda^2\rho+\rho\lambda^3\rho+\Delta\rho+2\Delta\lambda^2\rho \\ k^*(Ad') = \lambda^2\rho+\Delta \end{cases}$$

where $k^*: R(F_4) \to R(Spin(9))$ *is the restriction induced by the natural inclusion* $k: Spin(9) \to F_4$, *and* Δ *is the spin representation of* $Spin(9)$ *and* ρ *is the canonical non-trivial* 9*-dimensional representaiton of* $Spin(9)$.

Let $l: FII \to EIII$ be the imbedding induced by j. Then we see that $l^*: K^*(EIII) \to K^*(FII)$ is surjective and so by the secondary formula of (2.2) or by the direct computation from Corollary 1 $K^*(FII)$ is generated by $l^*(w) = 16 - \Delta$ with relation $(l^*(w))^3 = 0$. Hence we have

Corollary 2 (cf. [2], Theorem 7.1).

$$K^*(FII) \cong Z[\Delta]/((\Delta - 16)^3)$$

where Δ *is as in Corollary* 1.

5. The following tables are obtained by acting the elements of the Weyl group of E_6 suitably on the highest weight of each irreducible representation.

Table 1

The weights of ρ_1:		
4 3 5 6 4 2	1 0 -1 0 -2 -1	1 0 2 3 1 2
1 3 5 6 4 2	-2 0 -1 0 -2 -1	1 0 2 0 1 2
1 3 2 6 4 2	-2 0 -1 -3 -2 -1	1 0 -1 0 1 2
1 3 2 3 4 2	-2 0 -4 -3 -2 -1	-2 0 -1 0 1 2
1 3 2 3 1 2	-2 -3 -4 -3 -2 -1	-2 0 -1 0 1 -1
1 3 2 3 1 -1	-2 -3 -4 -6 -2 -1	-2 -3 -1 -3 -2 -1
1 0 2 3 1 -1	-2 -3 -4 -6 -5 -1	1 0 -1 0 1 -1
1 0 2 0 1 -1	-2 -3 -4 -6 -5 -4	1 0 -1 -3 -2 -1
1 0 2 0 -2 -1	1 0 2 3 4 2	1 -3 -1 -3 -2 -1

Table 2

The weights of ρ_2:		
2 3 4 6 5 4	-1 0 -2 0 -1 1	-1 0 1 0 -1 1
2 3 4 6 5 1	-1 0 -2 0 -1 -2	-1 0 1 0 -1 -2
2 3 4 6 2 1	-1 -3 -2 -3 -1 -2	-1 0 -2 -3 -1 -2
2 3 4 3 2 1	-1 -3 -2 -3 -4 -2	-1 0 -2 -3 -4 -2
2 3 1 3 2 1	-1 -3 -2 -6 -4 -2	-4 -3 -5 -6 -4 -2
2 0 1 3 2 1	-1 -3 -5 -6 -4 -2	-1 3 1 3 2 1
-1 0 1 3 2 1	2 0 4 3 2 1	-1 0 -2 -3 -1 1
-1 0 1 0 2 1	2 0 1 0 2 1	-1 -3 -2 -3 -1 1
-1 0 -2 0 2 1	2 0 1 0 -1 1	2 0 1 0 -1 -2

Table 3

The positive roots of E_6:

1 2 2 3 2 1	1 1 1 2 1 1	0 1 0 1 0 0
1 1 2 3 2 1	1 1 1 1 1 1	0 1 0 0 0 0
1 1 2 2 2 1	1 0 1 1 1 1	0 1 1 1 1 1
1 1 2 2 1 1	1 0 1 1 1 0	0 1 0 1 1 1
1 1 2 2 1 0	0 0 1 1 1 0	0 1 0 1 1 0
1 1 1 2 1 0	0 0 1 1 0 0	0 0 0 1 1 0
1 1 1 1 1 0	0 0 1 0 0 0	0 0 0 1 0 0
1 1 1 1 0 0	0 1 1 2 2 1	0 0 1 1 1 1
1 0 1 1 0 0	0 1 1 2 1 1	0 0 0 1 1 1
1 0 1 0 0 0	0 1 1 2 1 0	0 0 0 0 1 1
1 0 0 0 0 0	0 1 1 1 1 0	0 0 0 0 1 0
1 1 1 2 2 1	0 1 1 1 0 0	0 0 0 0 0 1

where the sequence $m_1 \cdots m_6$ of integers indicates a weight $\frac{1}{3}(m_1\alpha_1 + \cdots + m_6\alpha_6)$ in Tables 1 and 2 and a root $m_1\alpha_1 + \cdots + m_6\alpha_6$ in Table 3 using the notation in page 261 of [1].

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