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4-FOLD TRANSITIVE GROUPS IN WHICH ANY 2-ELEMENT AND ANY 3-ELEMENT OF A STABILIZER OF FOUR POINTS ARE COMMUTATIVE

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1. Introduction

The known 4-fold transitive groups in which the stabilizer of four points is nilpotent are S_4, S_5, S_6, A_6, A_7, M_{11} and M_{12}.

In this paper we shall prove the following

Theorem. *Let G be a 4-fold transitive group on Ω. If any 2-element and any 3-element of a stabilizer of four points in G are commutative, then G is S_4, S_5, S_6, A_6, A_7, M_{11} or M_{12}.*

As a corollary of this theorem, we have that a 4-fold transitive group in which the stabilizer of four points is nilpotent is one of the groups listed in our theorem.

Our notation is standard (cf. Wielandt [9]). For a subgroup X of G and a subset Δ of Ω, X_Δ is the point-wise stabilizer of Δ in X and if X fixes Δ as a set, the restriction of X on Δ will be denoted by X^Δ. $F(X)$ is the set of points in $\Omega=\{1, 2, \cdots, n\}$ fixed by every element of X.

2. Proof of the theorem

We proceed by way of contradiction and divide the proof in ten steps. From now on we assume that G^Ω is a counterexample to our theorem of the least possible degree. We set $D=G_{1234}$. Let P and R be a Sylow 2-subgroup and Sylow 3-subgroup of D, respectively.

By [2]–[8], we know the following.

(1) $F(D)=\{1, 2, 3, 4\}$, $|F(P)|=4$ *or* 5, $R\neq 1$ *and* $P^{\Omega-F(P)}$ *is not semi-regular.*

(2) $|F(P)|=4$.

Proof. From (1), we have only to show that $|F(P)|\neq 5$. Suppose $|F(P)|=5$. We set $F(P)=\{1, 2, 3, 4, i\}$ and $L=\langle R^d \mid d\in D\rangle$. By assumption, P centralizes L and so $i\in F(L)$. Then it follows from $L\trianglelefteq D$ that $i^D\subseteq F(L)$.

Since $2 \not| |D: D_i|$, $3 \not| |D: D_i$ and $|D: D_i| \neq 1$ by (1), we have $|D: D_i| \geq 5$ and so $|F(L)| \geq 9$. By the Witt's theorem, $N_G(L)^{F(L)}$ is a 4-fold transitive group. Furthermore $|F(L)| < n = |\Omega|$ since $R \neq 1$. Hence by our minimal choice of Ω we have $N_G(L)^{F(L)} \simeq M_{11}$ or M_{12} because $|F(L)| \geq 9$. Since $|F(P)| = 5$ and $[P, L] = 1$, we have $|F(L)|$ is odd and so $N_G(L)^{F(L)} \simeq M_{11}$. Clearly $P^{F(L)}$ is a Sylow 2-subgroup of a stabilizer of four points in $N_G(L)^{F(L)}$, so $P^{F(L)} = 1$ by the structure of M_{11}, hence $F(L)$ is a subset of $F(P)$, which is contrary to $|F(P)| = 5$. Thus we get (2).

Now let m be the maximal number of $|P_i|$ with $i \in \Omega - F(P)$. There exists some $j \in \Omega - F(P)$ such that $|P_j| = m$. We set $P_j = Q$. Then we have

(3) $1 \neq Q \neq P$.

Proof. By (1), $P^{\Omega - F(L)}$ is not semi-regular, and so $Q \neq 1$. It is clear that $Q \neq P$.

(4) *If Q^* is a 2-subgroup of G containing Q properly, then we have $|F(Q^*)| \leq 4$.*

Proof. Suppose that $|F(Q^*)| > 4$. Then there exists an element $g \in G$ with $(Q^*)^g \leq D$. Since P is a Sylow 2-subgroup of D, $(Q^*)^{gd} \leq P$ holds for some $d \in D$. By assumption we can choose an element k in $F((Q^*)^{gd})$ with $k \notin F(P)$. Then we have $m = |Q| < |Q^*| = |(Q^*)^{gd}| \leq |P_k|$, which is contrary to the choice of Q.

(5) *$N_G(Q)^{F(Q)}$ is 4-fold transitive.*

Proof. In the lemma 6 of [1], we put $G^\Omega = N_G(Q)^{F(Q)}$, $p = 2$ and $k = 4$, then (5) follows immediately from (2) and (4).

(6) *$N_G(Q)^{F(Q)} \simeq S_6$ or M_{12}.*

Proof. By (5), $N_G(Q)^{F(Q)}$ satisfies the assumption of our theorem. Since n is even by (2) and $Q \neq 1$, $n > |F(Q)| \geq 6$. By the minimal choice of Ω, we have $N_G(Q)^{F(Q)} \simeq A_6$, S_6 or M_{12}. On the other hand, we have $N_P(Q) > Q$ by (3), and so it follows from (4) that $|F(N_P(Q))| = 4$. Thus $N_G(Q)^{F(Q)} \not\simeq A_6$, which shows (6).

(7) *Let S be a nontrivial 3-subgroup of D. Then $F(S) = F(Q)$. Furthermore $R^{\Omega - F(R)}$ is semi-regular and the set $F(Q)$ depends only on D but is independent of the choice of P and Q.*

Proof. Since S centralizes Q, S is contained in $N_D(Q)$. By (6), $S^{F(Q)} = 1$, that is, $F(Q)$ is a subset of $F(S)$, hence $|F(S)| \geq 6$.

In the lemma 6 of [1], we put $G^\Omega = N_G(S)^{F(S)}$, $p = 2$ and $k = 4$, then we get

$N_G(S)^{F(S)}$ is 4-fold transitive. On the other hand $S \neq 1$, hence $n > |F(S)| \geq 6$. By the minimality of Ω, $N_G(S)^{F(S)} \simeq A_6$, S_6, A_7, M_{11} or M_{12}.

Suppose $F(S) \neq F(Q)$. Then we have $|F(S)| \geq 7$. Since P centralizes S, P acts on $F(S)-F(P)$ and so $|F(S)|$ is even by (2). Hence $N_G(S)^{F(S)} \simeq M_{12}$. Therefore $Q^{F(S)}=1$ by the structure of M_{12}. Hence $F(S)$ is a subset of $F(Q)$, a contradiction. Thus we conclude $F(Q)=F(S)$.

From this, the latter half of (7) immediately follows.

(8) *Let T be a Sylow 2-subgroup of $N_G(Q)$ and R^* be an arbitrary Sylow 3-subgroup of D. Then $[T, R^*]=1$.*

Proof. By (6), there is a 2-element x in $N_G(Q)$ such that $|F(x) \cap F(Q)|=4$ and $\langle x^g | g \in N_G(Q) \rangle^{F(Q)} = N_G(Q)^{F(Q)}$. We set $\langle x^g | g \in N_G(Q) \rangle = M$, $(N_G(Q))_{F(Q)}=K$. Then $N_G(Q)=MK$. Let T^* be an arbitrary Sylow 2-subgroup of M, then since by (4) Q is a unique Sylow 2-subgroup of K, T^*Q is a Sylow 2-subgroup of $N_G(Q)$. Applying (7), for any $u \in N_G(Q)$ we get $F(Q)=F(R^*)=F((R^*)^u)$, hence $(R^*)^u \leq K$. Since $|F(x^g) \cap F((R^*)^u)| = |F(x^g) \cap F(Q)|=4$ for any $g \in N_G(Q)$, x^g centralizes $(R^*)^u$. Hence M centralizes $(R^*)^u$, so $[T^*Q, (R^*)^u]=1$. Since T^*Q is a Sylow 2-subgroup of $N_G(Q)$, there exists an element $v \in N_G(Q)$ such that $T=(T^*Q)^v$. Thus we get $[T, (R^*)^{uv}]=1$. Put $u=v^{-1}$. Then (8) holds.

(9) *There exists an involution in Q. Let t be an involution in Q, then $|F(t)| \equiv 0$ (mod 3).*

Proof. From (3), the first statement is clear. Since $F(R)=F(Q) \subseteq F(t)$, $|F(Q)|=6$ or 12 and $[R, t]=1$, we have $|F(t)| = |F(R)| + |F(t)-F(R)| \equiv 0$ (mod 3).

(10) *We have now a contradiction in the following way.*

Let t be an involution in Q. For $i \in \Omega - F(t)$, we set $i^t=j$. Then t normalizes G_{12ij}. There exists an element z in G such that $G_{12ij}=D^z$. By (7), G_{12ij} fixes $F(Q^z)$ as a set. We set $N=(G_{12ij})_{F(Q^z)}$, then by (4) Q^z is a Sylow 2-subgroup of N. Again by (7), t normalzes N, hence t normalizes at least one of the Sylow 2-subgroups of N, say Q^{zm} where m is an element of N. Now $t^{m^{-1}z^{-1}}$ normalizes Q, so $t^{m^{-1}z^{-1}}$ centralizes R by (8), hence t centralizes R^{zm}. Since R^z is a subgroup of G_{12ij}, $R^z \leq N$ by (7). Hence $R^{zm} \leq N$. By (6), $N_G(Q^{zm})^{F(Q^{zm})} \simeq S_6$ or M_{12}. Since the set $F(R^{zm}) \cap F(t)$ is not empty and $F(R^{zm}) \cap F(t) \neq F(R^{zm}) = F(Q^{zm})$, we have $|F(R^{zm}) \cap F(t)|=2$ or 4. Since R^{zm} centralizes t, it follows from (7) that $|(\Omega - F(R^{zm})) \cap F(t)| \equiv 0$ (mod 3). Hence $|F(t)| = |F(t) \cap F(R^{zm})| + |(\Omega - F(R^{zm})) \cap F(t)| \equiv 1$ or 2 (mod 3), contrary to (9).

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