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ON AN EXPONENTIAL CHARACTER OF THE SPECTRAL DISTRIBUFION FUNCTION OF A RANDOM DIFFERENCE OPERATOR

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1. Let H^0 be a second order difference operator

$$(H^0u)(a) = \frac{\sigma^2}{2}\{u(a-1)-2u(a)+u(a+1)\}\,, \quad a\in Z\,,$$

u being a function on the space Z of all integers. We then consider a random difference operator H^ω defined by

$$(H^\omega u)(a) = -(H^0u)(a)+q(a,\omega)u(a)\,, \quad a\in Z\,,$$

where $\{q(a,\omega)\}_{a\in Z}$ is a family of random variables defined on a probability space $(\Omega, \mathcal{B}, P)$.
We assume that $\{q(a,\omega)\}_{a\in Z}$ forms a non-negative valued stationary Markov process with one step transition function $P(x, A)$ and absolute probability $\mu(A)$:

$$\begin{aligned}&P(q(a_1)\in A_1,\ q(a_2)\in A_2,\ \cdots,\ q(a_n)\in A_n)\\ &\quad = \int_{A_1\cdots A_n}\mu(dx_1)P^{(a_2-a_1)}(x_1, dx_2)P^{(a_3-a_2)}(x_2, dx_3)\cdots P^{(a_n-a_{n-1})}(x_{n-1}, dx_n)\end{aligned}$$

for integers $a_1<a_2<\cdots<a_n$ and Borel set $A_1, A_2, \cdots, A_n$ of $[0,\infty)$. Here $P^{(k)}(x, A)$ denotes the k-th iterate of $P(x, A)$.

Denote by $L^2(Z)$ the Hilbert space consisting of all square summable functions with inner product $(u, v)=\sum_{a\in Z}u(a)v(a)$. For each $\omega\in\Omega$, H^ω determines a selfadjoint operator A^ω by

$$\begin{cases}\mathcal{D}(A^\omega) = \{u\in L^2(Z);\ H^\omega u\in L^2(Z)\}\\ A^\omega u = H^\omega u \quad u\in\mathcal{D}(A^\omega)\,.\end{cases}$$

Let $\{E_\lambda^\omega, \lambda\in R^1\}$ be the resolution of the identity associated with A^ω. Then $(E_\lambda^\omega I_0, I_0)$ is measurable in ω and we can define the spectral distribution function ρ of $\{H^\omega\}$ by

$$\rho(\lambda) = E((E_\lambda^\omega I_0, I_0))$$

where E is the expectation in ω with respect to P and $I_0(a)=\delta_{0a}$, $a\in Z$ ([1]). $\rho(\lambda)$ vanishes for $\lambda<0$. Our present aim is to prove the following theorem.

Theorem.

(i) *If* $P(0, \{0\})=b>0$ *and* $\mu(\{0\})>0$, *then* $\underline{\lim}_{x\downarrow 0}\sqrt{x}\log\rho(x)>-\infty$.

(ii) *If* $\int_0^\infty \frac{1}{1+y}P(x, dy)<c$ μ*-a.e.* x *for some* $c<1$, *then* $\overline{\lim}_{x\downarrow 0}\sqrt{x}\log\rho(x)<0$.

A similar result has been obtained by M. Fukushima ([1]) when $q(a)$, $a\in Z$, are non-negative valued independent identically distributed random variables. We further mention the works of L. A. Pastur ([2]) and S. Nakao ([3]) for related results on the one dimensional Schrödinger operators with random potentials. The present novelty is to make use of a Markovian character of the local time (cf. M. L. Silverstein [4]).

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2. At first we collect some lemmas for the proof of our theorem.

We introduce the continuous Markov process $M=(\dot{\Omega}, \dot{\mathcal{B}}, \dot{X}_t, \dot{P}_a)$ on Z with the generator H^0. Denoting by $\dot{E}_0$ the expectation with respect to $\dot{P}_0$, we have Kac representation as follows.

Lemma 1 ([1]).

$$\int_0^\infty e^{-t\lambda}d\rho(\lambda)=E\times\dot{E}_0\left[\exp\left(-\int_0^t q(\dot{X}_s, \omega)ds\right);\ \dot{X}_t=0\right].$$

The proof of our theorem reduces to finding how fast $E\times\dot{E}_0\left[\exp\left(-\int_0^t q(\dot{X}_s, \omega)ds\right);\ \dot{X}_t=0\right]$ *tends to zero as* $t\to\infty$ *because of Lemma* 1 *and the following Tauberian theorem.*

Lemma 2 ([1]). *Let* $\phi(\lambda)$ *be non-decreasing function on* $[0, \infty)$ *with* $\phi(0)=0$ *and* $\psi(t)$ *be its Laplace transform*:

$$\psi(t)=\int_0^\infty e^{-t\lambda}d\phi(\lambda)$$

(i) *If* $\underline{\lim}_{t\uparrow\infty}\frac{1}{t^\gamma}\log\psi(t)>-\infty$ *then* $\underline{\lim}_{x\downarrow 0}x^{\gamma/1-\gamma}\log\phi(x)>-\infty$.

(ii) *If* $\overline{\lim}_{t\uparrow\infty}\frac{1}{t^\gamma}\log\psi(t)<0$ *then* $\overline{\lim}_{x\downarrow 0}x^{\gamma/1-\gamma}\log\phi(x)<0$.

For the investigation of asymptotic behavior of $E\times\dot{E}_0\left[\exp\left(-\int_0^t q(\dot{X}_s, \omega)\right);\right.$

$\dot{X}_t=0\Big]$ as $t\to\infty$, the following two lemmas are of great use.

Lemma 3 ([4]). *Put* $\dot{L}(t, x)=\int_0^t I_x(\dot{X}_s)ds$, $t>0$, $x\in Z$, $\sigma_x(s)=\sup\{t;\ \dot{L}(t, x)\leqq s$, $\dot{T}_{x,y}(s)=\dot{L}(\sigma_x(s), y)$, *then it holds that*

(i) $\{\dot{T}_{y,z}(s), y, z\geqq x\}$ *and* $\{\dot{T}_{y,z}(s), y, z\leqq x\}$ *are multually independent for each* $x\in Z$ *and* $s\geqq 0$,

(ii) $\dot{E}_0[\exp(-\alpha\dot{T}_{x,x-y-z}(s))\,|\,\dot{T}_{x,x-y}(s)=l]=\frac{1}{\alpha z+1}\exp\left(\frac{-l\alpha}{\alpha z+1}\right)$ *for each* $\alpha>0$, $x>0(<0)$ *and* $y, z\geqq 0$ *such that* $y+z\leqq x(\leqq -x)$,

(iii) $\{\dot{T}_{x,x-y}(s)\}$ *is Markovian in* $y\geqq 0$ *for fixed* $s\geqq 0$ *and* $x>0(<0)$.

Corollary *Put* $\mathscr{B}^x_{x-u}(s)=\sigma[\dot{T}_{x,x-v}(s), v=0, 1, \cdots, u]$ $s\geqq 0$, *then we have*

$$\dot{E}_0\left[\exp\left(-\alpha\dot{T}_{x,x-y-z}(s)\right)|\,\mathscr{B}^x_{x-y}(s)\right]\leqq\frac{1}{\alpha z+1}$$

for each $x>0$ (<0) *and* $y, z\geqq 0$ *such that* $y+z\leqq x$ $(y+z\leqq -x)$.

Lemma 4 ([1]). *Let* $\dot{R}_t$ *be the number of states where* $\dot{X}_s$ *visits during the interval* $[0, t)$, *then we have*

(i) $\varliminf_{t\uparrow\infty} t^{-1/3}\log\dot{E}_0[e^{-\beta_1\dot{R}_t};\ \dot{X}_t=0]>-\infty$

for any posiivte constant $\beta_1>0$,

(ii) $\varlimsup_{t\uparrow\infty} t^{-1/3}\log\dot{E}_0[e^{-\beta_2 R_t}]<0$

for any positive constant $\beta_2>0$.

3. Now we give the proof of our theorem.

Put $k(t)=E\times\dot{E}_0\left[\exp\left(-\int_0^t q(\dot{X}_s, \omega)ds\right);\ \dot{X}_t=0\right]$, then

$$k(t)=\sum_{k=0}^{\infty}\sum_{\substack{m+n=k\\ m,n\geqq 0}}E\times\dot{E}_0\left[\exp\left(-\sum_{x=-n}^{m}\dot{L}(t, x)q(x, \omega)\right);\ \dot{M}_t=m,\ \dot{m}_t=-n,\ \dot{X}_t=0\right.$$

where $\dot{M}_t=\sup\{\dot{X}_s\colon 0\leqq s\leqq t\}$ $\dot{m}_t=\inf\{X_s\colon 0\leqq s\leqq t\}$. Taking the expectation of $\exp(-\sum_{x=-n}^{m}\dot{L}(t, x)q(x, \omega))$ with respect to P, we have by stationarity and Markov property of $(\Omega, \mathscr{B}, P, q)$

$$E\left[\exp\left(-\sum_{x=-n}^{m}L(t, x)q(x, \omega)\right)\right]\geqq E\left[\prod_{x=-n}^{m}I_0(q(x, \omega))\right]=\mu(\{0\})b^{m+n}.$$

therefore

$$k(t) \geqq \sum_{k=0}^{\infty} \sum_{\substack{m+n=k \\ m,n \geqq 0}} \dot{E}_0[\mu(\{0\})b^k;\ \dot{M}_t=\dot{m},\ m_t=-n,\ \dot{X}_t=0]$$

$$= \frac{\mu(\{0\})}{b} \sum_{k=1}^{\infty} b^k \dot{P}_0(\dot{R}_t=k,\ \dot{X}_t=0) = \frac{\mu(\{0\})}{b} \dot{E}_0[e^{-\beta_1 \dot{R}_t};\ \dot{X}_t=0]\,,$$

$$\beta_1 = -\log b\,.$$

Because of Lemma 4

$$\varliminf_{t\uparrow\infty} t^{-1/3} \log k(t) > -\infty\,.$$

We get the first assertion (i) of our theorem by Lemma 1 and Lemma 2.

Turning to the proof of the second assertion (ii), we put

$$k_1(t) = E\times\dot{E}_0\left[\exp\left(-\int_0^t q(\dot{X}_s,\ \omega)ds\right);\ \dot{X}_t=0,\ \dot{R}_t<t\right],$$

$$k_2(t) = E\times\dot{E}_0\left[\exp\left(-\int_0^t q(\dot{X}_s,\ \omega)ds\right);\ \dot{X}_t=0,\ \dot{R}_t\geqq t\right],$$

then we get

$$k_1(t)^2 = \{\sum_{k=0}^{[t]-1} \sum_{\substack{m+n=k \\ m,n\geqq 0}} E\times\dot{E}_0[\exp(-\sum_{k=-n}^{m} \dot{L}(t,\ x)q(x,\ \omega));\ \dot{M}_t=m,\ \dot{m}_t=-n,\ \dot{X}_t=0]\}^2$$

$$\leqq \frac{[t]([t]+1)}{2} \sum_{k=0}^{[t]-1} \sum_{\substack{m+n=k \\ m,n\geqq 0}} \{E\times\dot{E}_0[\exp(-\sum_{x=-n}^{m} \dot{L}(t,\ x)q(x,\ \omega));\ \dot{M}_t=m,\ \dot{m}_t=-n,\ \dot{X}_t=0]\}^2$$

$$\leqq \frac{[t]([t]+1)}{2} \sum_{k=0}^{[t]-1} \sum_{\substack{m+n=k \\ m,n\geqq 0}} E\times\dot{E}_0[\exp(-2\sum_{x=-n}^{m} \dot{L}(t,\ x)q(x,\ \omega))]\ \dot{P}_0[\dot{M}_t=m,\ \dot{m}_t=-n,\ \dot{X}_t=0]\,.$$

Putting $\tau_i=t\wedge\inf\{s;\ \dot{X}_s=i\}$, $i\in Z$, it is clear that

$$\dot{L}(\tau_m,\ x)\leqq\dot{L}(t,\ x)\,,\quad \dot{L}(\tau_{-n},\ x)\leqq\dot{L}(t,\ x)\quad \text{and}$$
$$\dot{L}(\tau_m,\ x) = \dot{T}_{m,x}(0)\,,\quad \dot{L}(\tau_{-n},\ x) = \dot{T}_{-n,x}(0)\,.$$

Therefore

$$\exp(-2\sum_{x=-n}^{m} \dot{L}(t,\ x)q(x,\ \omega))$$

$$\leqq \exp(-2\sum_{x=0}^{m} \dot{T}_{m,x}(0)q(x,\ \omega) - 2\sum_{x=-n}^{-1} \dot{T}_{-n,x}(0)q(x,\ \omega))\,.$$

Taking the expectation with respect to $P\times\dot{P}_0$, we get

$$E\times\dot{E}_0[\exp(-2\sum_{x=-n}^{m} \dot{L}(t,\ x)q(x,\ \omega))] \leqq E\left[\prod_{x=-n}^{m} \frac{1}{1+2q(x,\ \omega)}\right]$$

because of Lemma 3, (i) (iii) and its Corollary. From Markov property of $(\Omega, \mathscr{B}, P, q)$ and the assumption in our theorem, it follows that

$$E\left[\prod_{x=-n}^{m} \frac{1}{1+2q(x,\omega)}\right] = E\left[\prod_{x=-n}^{m-1} \frac{1}{1+2q(x,\omega)} E\left[\frac{1}{1+2q(1,\omega)}\middle| q(0)\right]\right]$$
$$< cE\left[\prod_{x=-n}^{m-1} \frac{1}{1+2q(x,\omega)}\right] < c^{m+n+1}.$$

Now we have

$$k_1(t)^2 \leqq \frac{[t]([t]+1)}{2} \sum_{k=0}^{[t]-1} \sum_{\substack{m+n=k \\ m,n\geqq 0}} c^{m+n+1} \dot{P}_0[\dot{X}_t=0,\ \dot{M}_t=m,\ \dot{m}_t=-n]$$
$$= \frac{[t]([t]+1)}{2} \sum_{k=1}^{[t]} c^k \dot{P}_0(\dot{X}_t=0,\ \dot{R}_t=k)$$
$$\leqq \frac{[t]([t]+1)}{2} \dot{E}_0(e^{-\beta_2 \dot{R}_t};\ \dot{X}_t=0),\quad \beta_2 = -\log c\,.$$

On the other hand

$$k_2(t) \leqq E\times\dot{E}_0\left[\exp\left(-\int_0^t q(\dot{X}_s,\omega)ds;\ \dot{X}_t=0,\ \dot{M}_t \geqq \frac{t}{2}\right)\right]$$
$$+E\times\dot{E}_0\left[\exp\left(-\int_0^t q(\dot{X}_s,\omega);\ \dot{X}_t=0,\ \dot{m}_t \leqq -\frac{t}{2}\right)\right]$$
$$\leqq E\times\dot{E}_0\left[\exp\left(-\sum_{x=0}^{[t/2]} \dot{L}(\tau_{[t/2]}, x) q(x,\omega)\right)\right]$$
$$+E\times\dot{E}_0\left[\exp\left(-\sum_{x=-1}^{-[t/2]} \dot{L}(\tau_{-[t/2]}, x) q(x,\omega)\right)\right]$$
$$\leqq E\left[\prod_{x=0}^{[t/2]} \frac{1}{1+q(x,\omega)}\right] + E\left[\prod_{x=-1}^{-[t/2]} \frac{1}{1+q(x,\omega)}\right] < 2c^{[t/2]}$$
$$= 2e^{-\beta_2[t/2]}\,.$$

As a result

$$k(t) = k_1(t)+k_2(t) \leqq \sqrt{\frac{[t]([t]+1)}{2} \dot{E}_0(e^{-\beta_2 \dot{R}_t})} + 2e^{-\beta_2[t/2]}\,,$$

which, combined with Lemma 4, leads us to

$$\overline{\lim_{t\uparrow\infty}}\, t^{-1/3} \log k(t) < 0\,.$$

Hence we arrive at the second assertion of our theorem.

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