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DIFFUSION PROCESSES AND A CLASS OF MARKOV CHAINS RELATED TO POPULATION GENETICS

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1. Introduction

We investigate convergence of sequences of Markov chains induced by direct product branching processes, which are defined by Karlin and McGregor [7] with the intention of unified treatment of Markov chains in population genetics. The induced Markov chains that we deal with in this paper have d types ($d\geq 2$) with equal fertility (that is, selection does not occur), and mutation and migration are allowed for. Let $\boldsymbol{R}^{d-1}$ be the $(d-1)$-dimensional Euclidean space and let $\boldsymbol{K}$ be the set of $x=(x_1, \cdots, x_{d-1})\in \boldsymbol{R}^{d-1}$ such that $x_1\geq 0, \cdots, x_{d-1}\geq 0,\ 1-\sum_{l=1}^{d-1} x_l\geq 0$. Under some conditions, we prove convergence of the Markov chains (suitably normalized and interpolated) to the diffusion process on $\boldsymbol{K}$ with diffusion coefficient $a(x)=(a_{pq}(x))_{p,q=1,\cdots,d-1}$ and drift coefficient $b(x)=(b_p(x))_{p=1,\cdots,d-1}$ of the form

$$(1.1)\qquad a_{pp}(x)=\sigma^2 x_p(1-x_p)\,,$$

$$(1.2)\qquad a_{pq}(x)=-\sigma^2 x_p x_q \qquad (p\neq q)\,,$$

$$(1.3)\qquad b_p(x)=\sum_{l=1}^{d-1} x_l\alpha_{lp}+(1-\sum_{l=1}^{d-1} x_l)\alpha_{dp}+(1-x_p)\mu_p-x_p\mu_p'\,.$$

Here σ^2, α_{pq}, μ_p, μ_p' are constants satisfying $\sigma^2>0$, $\alpha_{pq}\geq 0$ $(p\neq q)$, $\alpha_{pp}\leq 0$, $\sum_{q=1}^{d}\alpha_{pq}=0$, $\mu_p\geq 0$, $\mu_p'=(\sum_{l=1}^{d}\mu_l)-\mu_p$. In some sense, $\alpha_{pq}(p\neq q)$ is the intensity of mutation from type p to type q and μ_p is the intensity of immigration of type p. Our conditions consist of two sorts. The first is some regularity of the branching process with immigration, which induces the Markov chain, imposed on the distributions of the number of offspring and of the number of immigrants. The second is uniqueness, in the sense of martingale problem, of the diffusion process on $\boldsymbol{K}$ associated with $a(x)$ and $b(x)$. We conjecture that the uniqueness always holds, but we do not yet have the proof. We give a proof of the uniqueness in some special cases including the following:

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(i) general one-dimensional case ($d-1=1$);
(ii) no mutation and no migration ($b(x)=0$);
(iii) migration allowed, but no mutation ($\alpha_{pq}=0$);
(iv) α_{pq} ($p\neq q$) depends on q alone.

Consequently, the convergence is established in the above cases. Our method applies also to some Markov chain models involving selection, which will be treated in another paper.

Our results have connection with diffusion approximation to genetics model Markov chains, which is not given a rigorous justification but is a powerful tool in population genetics (see Kimura [8], Crow and Kimura [2]).

In the one-dimensional case, Feller [3] considers convergence of Wright's model. Karlin and McGregor [7] make the assertion, without proof, of the convergence of the induced Markov chains under the condition $d-1=1$, $\alpha_{12}>0$, $\alpha_{21}>0$, $\mu_1=\mu_2=0$. They prove in [6] convergence to the same diffusion for a certain birth-and-death process model. Kushner [9] gives an invariance principle related to Section 3 of this paper and mentions an application to a genetics model, but he deals with convergence in the space D. As for the convergence of the corresponding eigenvalues, we give a detailed analysis in [11].

In Section 2 we will formulate our assumptions and results. The proof will be given in the subsequent three sections. Namely, we establish in Section 3 a general invariance principle in a form convenient for us, prove in Section 4 that the moments of the transition probabilities of the induced Markov chains have the desired properties and, in Section 5, check the uniqueness of the solution of the martingale problem in some cases. The most painstaking part is the derivation of the asymptotic form of the moments of the transition probabilities in Section 4. Our main tool is the powerful saddle point method, which is used in [11] in finding asymptotic behavior of the eigenvalues.

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2. Assumptions and results

Let $\boldsymbol{Z}_+^d$ be the set of d-dimensional lattice points with nonnegative coordinates. For each positive integer N, let $\{Z^{(N)}(n)=(Z_1^{(N)}(n), \cdots, Z_d^{(N)}(n));$ $n=0, 1, 2, \cdots\}$ be a d type branching process with stationary immigration. That is, $\{Z^{(N)}(n)\}$ is a Markov chain taking values in $\boldsymbol{Z}_+^d$ and there exist generating functions $f_{N,p}(s_1, \cdots, s_d)$, $p=1, \cdots, d$, and $g_N(s_1, \cdots, s_d)$ of distributions in $\boldsymbol{Z}_+^d$ such that, for any $j=(j_1, \cdots, j_d)$ and $k=(k_1, \cdots, k_d)$ in $\boldsymbol{Z}_+^d$,

$$P(\boldsymbol{Z}^{(N)}(n+1) = k \mid Z^{(N)}(n) = j) = \text{coefficient of } s_1^{k_1}\cdots s_d^{k_d} \text{ in}$$
$$g_N(s_1, \cdots, s_d) \prod_{p=1}^{d} f_{N,p}(s_1, \cdots, s_d)^{j_p} .$$

We make the following assumptions.

Assumptions 2.1. (i) *$f_{N,p}$ is of the form*

$$f_{N,p}(s_1, \cdots, s_d) = \sum_{n=0}^{\infty} c_n \Big(\sum_{q=1}^{d} \alpha_{pq}^{(N)} s_q\Big)^n \qquad \text{for} \quad N \geq 1,\ 1 \leq p \leq d,$$

where c_n and $\alpha_{pq}^{(N)}$ satisfy the following conditions.

(ii) *$\{c_n\}$ is a probability distribution in $\boldsymbol{Z}_+$ independent of N and p with $c_0>0$ and with maximum span* 1 (*that is, there is no pair of $\gamma>1$ and δ such that $\sum_n c_{n\gamma+\delta}=1$*). *Let $a=\sum_{n=0}^{\infty} nc_n$ (mean), $f(w)=\sum_{n=0}^{\infty} c_n w^n$ (generating function), $M(w)=\sum_{n=0}^{\infty} c_n e^{nw}$ (moment generating function), $F(w)=M(w)e^{-w}$, $b=\sup\{w; M(w)<\infty\}$. Then one of the following holds*:

(a) $1<a\leq+\infty$;

(b) $a=1$ *and* $b>0$;

(c) $a<1$ *and* $\lim_{w\uparrow b} F'(w)>0$.

(iii) *$\{\alpha_{pq}^{(N)}\}$ is of the form $\alpha_{pq}^{(N)}=\alpha_{pq}/N$ $(p\neq q)$ and $\alpha_{pp}^{(N)}=1+(\alpha_{pp}/N)$ for all sufficiently large N, where $\{\alpha_{pq}\}$ is independent of N and satisfies $\alpha_{pq}\geq 0$ $(p\neq q)$, $\alpha_{pp}\leq 0$, $\sum_{q=1}^{d}\alpha_{pq}=0$.*

As is remarked in [11], b is positive in the case (c). If $a<1$, $b>0$ and $\lim_{w\uparrow b} M(w)=\infty$, then (c) holds. It is easy to prove that (ii) implies the existence of a unique $\beta\in(-\infty, b)$ such that $F'(\beta)=0$. β is negative, zero, positive in the cases (a), (b), (c), respectively. Let $K(w)=\log M(w)$ for $w<b$. Then $K'(\beta)=1$ and $K''(\beta)>0$. See [11], Lemma 2.1. Let $\sigma^2=K''(\beta)$. If we define an associated distribution $\{\hat{c}_n\}$ of $\{c_n\}$ by $\hat{c}_n=c_n e^{n\beta}/M(\beta)$, then $\{\hat{c}_n\}$ has mean 1 and variance σ^2.

Assumption 2.2. *g_N is independent of N, that is,*

$$g_N(s_1, \cdots, s_d) = g(s_1, \cdots, s_d) = \sum_{k\in \boldsymbol{Z}_+^d} b_k s_1^{k_1}\cdots s_d^{k_d}$$

for some distribution $\{b_k\}$ in $\boldsymbol{Z}_+^d$. Moreover, g satisfies $g(e^{\beta+\varepsilon}, \cdots, e^{\beta+\varepsilon})<\infty$ for some $\varepsilon>0$.

Note that the last condition on g is automatically satisfied in the case (a). Assumption 2.1 implies that reproduction of offspring by one individual of type p is made in two steps—first it produces independently a random number of children of the same type p according to the distribution $\{c_n\}$, and then, each child has a chance of mutation to type q $(p\neq q)$ with probability α_{pq}/N.

Assumption 2.2 implies that the immigration probability is independent of N. Hence dependence of the law of $\{Z^{(N)}(n)\}$ on N lies only in $\{\alpha_{pq}^{(N)}\}$.

Let $\{\hat{b}_k\}$ be an associated distribution of $\{b_k\}$ defined by

$$\hat{b}_k = b_k e^{(k_1+\cdots+k_d)\beta}/g(e^\beta, \cdots, e^\beta) \qquad \text{for} \quad k=(k_1, \cdots, k_d)\in \boldsymbol{Z}_+^d .$$

Let $(\mu_1, \cdots, \mu_d)$ be the mean of $\{\hat{b}_k\}$, $\mu_p = \sum_{k\in \boldsymbol{Z}_+^d} k_p \hat{b}_k$, and let $\mu_p' = (\sum_{q=1}^d \mu_q) - \mu_p$. These are important characterstics in our discussion.

Let us define the induced Markov chain. Let $\boldsymbol{J}(N)$ be the set of points $j=(j_1, \cdots, j_d)\in \boldsymbol{Z}_+^d$ such that $\sum_{p=1}^d j_p = N$. For $j, k \in \boldsymbol{J}(N)$ let

$$P_{jk}^{(N)} = P(Z^{(N)}(n+1) = k \mid Z^{(N)}(n) = j,\ Z^{(N)}(n+1)\in \boldsymbol{J}(N)) . \tag{2.1}$$

Clearly,

$$P_{jk}^{(N)} = \frac{1}{A_1(N)} (\text{coefficients of } s_1^{k_1}\cdots s_d^{k_d} \text{ in } g(s_1, \cdots, s_d) \prod_{p=1}^d f(\sum_{q=1}^d \alpha_{pq}^{(N)} s_q)^{j_p}) \tag{2.2}$$

where

$$\begin{aligned} A_1(N) &= P(Z^{(N)}(n+1)\in \boldsymbol{J}(N) \mid Z^{(N)}(n) = j) \\ &= \text{coefficient of } w^N \quad \text{in} \quad g(w, \cdots, w) f(w)^N . \end{aligned} \tag{2.3}$$

Assumption 2.1 guarantees $A_1(N) > 0$ for farge N. See (4.25) in a later section. Hence we can define $P_{jk}^{(N)}$ for all $j, k\in \boldsymbol{J}(N)$ if N is large. Let $\{X^{(N)}(n) = (X_1^{(N)}(n), \cdots, X_d^{(N)}(n));\ n=0, 1, \cdots\}$ be a Markov chain defined on a probability space $(\Omega^{(N)}, Q^{(N)})$, taking values in $\boldsymbol{J}(N)$ with one-step transition probability $P_{jk}^{(N)}$. The initial distribution is given arbitrarily. $X^{(N)}(n)$ is the *induced Markov chain* of Karlin and McGregor. The fact that this includes various genetics models as special cases is shown in Karlin [5].

Since the sum of components is N, we can consider the induced Markov chain as a Markov chain on a $(d-1)$-dimensional state space. We normalize and interpolate this chain as follows:

$$Y^{(N)}(t) = \left(\frac{1}{N} X_1^{(N)}(n), \cdots, \frac{1}{N} X_{d-1}^{(N)}(n)\right) \qquad \text{for} \quad t = \frac{n}{N} , \tag{2.4}$$

$$Y^{(N)}(t) = (n+1-Nt) Y^{(N)}\left(\frac{n}{N}\right) + (Nt-n) Y^{(N)}\left(\frac{n+1}{N}\right) \qquad \text{for} \quad \frac{n}{N} \le t \le \frac{n+1}{N} . \tag{2.5}$$

Then $\{Y^{(N)}(t);\ 0\le t<\infty\}$ is a continuous process taking values in $\boldsymbol{K}$ ($\boldsymbol{K}$ is defined in Section 1). Let Ω be the space of continuous paths ω: $[0, \infty)\to \boldsymbol{K}$, endowed with the topology of uniform convergence on compact subsets of $[0, \infty)$.

There is a complete separable metric compatible with this topology. Let $x(t, \omega)=\omega(t)$. Let $\mathcal{M}$ be the topological σ-algebra of Ω and $\mathcal{M}_t$ be the σ-algebra generatd by $x(s)$, $s\leq t$. Let $P^{(N)}$ be the probability measure on $(\Omega, \mathcal{M})$ that $(\Omega^{(N)}, Q^{(N)}, Y^{(N)}(t); 0\leq t<\infty)$ induces. That is, $P^{(N)}(B)=Q^{(N)}(\varphi^{-1}(B))$, $B\in\mathcal{M}$, where φ is the measurable mapping from $\Omega^{(N)}$ into Ω defined by $\varphi(\omega^{(N)})=\omega$, $\omega(t)=Y^{(N)}(t, \omega^{(N)})$. Obviously, $P^{(N)}$ is determined by the initial distribution of $(X^{(N)}(n))$ and its transition probability $(P_{jk}^{(N)})$. What we would like to prove is the convergence of the sequence of the probability measures $(P^{(N)})$.

Define $a(x)=(a_{pq}(x))_{p,q=1,\dots,d-1}$ and $b(x)=(b_p(x))_{p=1,\dots,d-1}$ by (1.1)—(1.3) and consider the following martingale problem: given $x\in\boldsymbol{K}$, to find a probability measure P_x on $(\Omega, \mathcal{M})$ such that $P_x(x(0)=x)=1$ and, for each $\theta\in\boldsymbol{R}^{d-1}$, $(M_\theta(t), \mathcal{M}_t, P_x; 0\leq t<\infty)$ is a martingale, where $M_\theta(t)$ is defined by

$$M_\theta(t)=\exp\{\langle\theta, x(t)-x(0)\rangle-\int_0^t\langle\theta, b(x(u))\rangle du -\frac{1}{2}\int_0^t\langle\theta, a(x(u))\theta\rangle du\}\,. \tag{2.6}$$

Here $\langle\,,\rangle$ denotes the inner product. We call this problem *the martingale problem on* $\boldsymbol{K}$ *for* a, b *starting from* x, or, for short, *the martingale problem* $(\boldsymbol{K}, a, b, x)$. Such a problem was originated by Stroock and Varadhan [12], but the above problem is different from theirs on the point of the restriction of the state space.

We will prove the following results.

Theorem 2.1. *For any set of* σ^2, α_{pq}, μ_p, *the martingale problem* $(\boldsymbol{K}, a, b, x)$ *has a solution for each* $x\in\boldsymbol{K}$.

Theorem 2.2. *Suppose that the solution* P_x *of the martingale problem* $(\boldsymbol{K}, a, b, x)$ *is unique. Let* $Y^{(N)}(0)=x^{(N)}$, *which is non-random. If* $x^{(N)}\to x$, *then the sequence of the measures* $\{P^{(N)}\}$ *weakly converges to* P_x, *that is, for each bounded, continuous, real function* $\xi(\omega)$ *on* Ω, *we have*

$$\int_\Omega\xi(\omega)P^{(N)}(d\omega)\to\int_\Omega\xi(\omega)P_x(d\omega)\,,\quad N\to\infty\,. \tag{2.7}$$

It is well known that the weak convergence of $P^{(N)}$ to P_x implies (2.7) for any bounded, measurable, real function $\xi(\omega)$ on Ω whose discontinuity is a set of P_x measure 0. Sojourn times and hitting times are included in applications in many cases. See Billingsley [1].

The above two theorems reduce the point to the uniqueness of the solution of the martingale problem $(\boldsymbol{K}, a, b, x)$. We conjecture that the uniqueness holds, and the following theorems give the proof in some cases.

Theorem 2.3. *Let $d-1=1$. Then, for each $x\in\boldsymbol{K}$, the solution of the martingale problem $(\boldsymbol{K}, a, b, x)$ is unique, and hence the conclusion of Theorem 2.2 holds.*

Theorem 2.4. *Suppose that if $p>q$ and $p'>q$ then $\alpha_{pq}=\alpha_{p'q}$. Then, for each $x\in\boldsymbol{K}$, the solution of the martingale problem $(\boldsymbol{K}, a, b, x)$ is unique, and the conclusion of Theorem 2.2 holds.*

Note that Theorem 2.4 covers the cases (ii), (iii), (iv) in Section 1.

If the martingale problem $(\boldsymbol{K}, a, b, x)$ has a unique solution P_x for every $x\in\boldsymbol{K}$, then $(x(t), \mathcal{M}_t, P_x; x\in\boldsymbol{K})$ is a strong Markov process, to which the process $\{Y^{(N)}(t); 0\leq t<\infty\}$ converges as $N\to\infty$. In case of $d-1=1$, it is a diffusion process on the interval $[0, 1]$ with backward Kolmogorov equation

$$\frac{\partial u}{\partial t}=\frac{\sigma^2}{2}x(1-x)\frac{\partial^2 u}{\partial x^2}+\{-x(\alpha_{12}+\mu_2)+(1-x)(\alpha_{21}+\mu_1)\}\frac{\partial u}{\partial x}. \tag{2.8}$$

If $d-1=2$, then it is a diffusion process on the triangular region $\boldsymbol{K}$ with backward Kolmogorov equation

$$\begin{aligned}\frac{\partial u}{\partial t}=&\frac{\sigma^2}{2}\left\{x_1(1-x_1)\frac{\partial^2 u}{\partial x_1^2}-2x_1x_2\frac{\partial^2 u}{\partial x_1\partial x_2}+x_2(1-x_2)\frac{\partial^2 u}{\partial x_2^2}\right\}\\&+\{-x_1(\alpha_{12}+\alpha_{13}+\mu_2+\mu_3)+x_2\alpha_{21}+(1-x_1-x_2)\alpha_{31}+(1-x_1)\mu_1\}\frac{\partial u}{\partial x_1}\\&+\{-x_2(\alpha_{21}+\alpha_{23}+\mu_1+\mu_3)+x_1\alpha_{12}+(1-x_1-x_2)\alpha_{32}+(1-x_2)\mu_2\}\frac{\partial u}{\partial x_2}.\end{aligned} \tag{2.9}$$

Properties of the boundaries for these diffusion processes will be discussed at the end of Section 5.

3. An invariance principle

In this section $\boldsymbol{K}$ denotes an arbitrary compact set in $\boldsymbol{R}^l$ and Ω denotes the space of continuous paths ω: $[0, \infty)\to\boldsymbol{K}$. The topology of Ω and the σ-algebras $\mathcal{M}$, $\mathcal{M}_t$ are defined as in Secion 2. Let $a(x)=(a_{pq}(x))_{p,q=1,\dots,l}$ be a continuous, symmetric, nonnegative-definite $l\times l$-matrix defined on $\boldsymbol{K}$ and $b(x)=(b_p(x))_{p=1,\dots,l}$ be a continuous l-vector defined on $\boldsymbol{K}$. By the martingale problem $(\boldsymbol{K}, a, b, x)$ we mean the problem to find a probability measure P_x on $(\Omega, \mathcal{M})$ such that $P_x(x(0)=x)=1$ and, for each $\theta\in\boldsymbol{R}^l$, $(M_\theta(t), \mathcal{M}_t\, P_x; 0\leq t<\infty)$ is a martingale, where $M_\theta(t)$ is defined by (2.6). Suppose that, on a probability space $(\Omega^{(N)}, Q^{(N)})$, time homogeneous Markov chains $\{\bar{Y}^{(N)}(n); n=0, 1, \cdots\}$ taking values in $\boldsymbol{K}$ are given. Let $\Pi^{(N)}(x, dy)$ be the one-step transition probability of $\{\bar{Y}^{(N)}(n)\}$. Let $\bar{Y}^{(N)}(0)=x_0^{(N)}$ (non-random). Let $\boldsymbol{K}(N)$ be a Borel subset of $\boldsymbol{K}$ such that $Q^{(N)}(\bar{Y}^{(N)}(n)\in\boldsymbol{K}(N))=1$, and suppose that $\Pi^{(N)}(x, dy)$ is defined for $x \in \boldsymbol{K}(N)$ and $dy \subset \boldsymbol{K}(N)$. Define $Y^{(N)}(t)$ by

$$Y^{(N)}(t) = \bar{Y}^{(N)}(n) \qquad \text{for} \quad t = n/N\,,$$
$$Y^{(N)}(t) = (n+1-Nt)\bar{Y}^{(N)}(n) + (Nt-n)\bar{Y}^{(N)}(n+1) \qquad \text{for} \quad n/N \leq t \leq (n+1)/N\,.$$

Let $P^{(N)}$ be the probability measure induced on $(\Omega, \mathcal{M})$ by the process $\{Y^{(N)}(t);\ 0 \leq t < \infty\}$.

We will prove the following theorems. By convergence of measures we mean weak convergence.

Theorem 3.1. *Suppose that the following conditions are satisfied:*

(i) $\lim_{N\to\infty} \sup_{x\in K(N)} N\,\Pi^{(N)}(x, \{y; |y-x| > \varepsilon\}) = 0$ *for every* $\varepsilon > 0$.

(ii) $\lim_{N\to\infty} \sup_{x\in K(N)} |N \int_{K(N)} (y_p - x_p)\,\Pi^{(N)}(x, dy) - b_p(x)| = 0$ *for* $p = 1, \cdots, l$.

(iii) $\lim_{N\to\infty} \sup_{x\in K(N)} |N \int_{K(N)} (y_p - x_p)(y_q - x_q)\,\Pi^{(N)}(x, dy) - a_{pq}(x)| = 0$ *for* $p, q = 1, \cdots, l$.

(iv) $\lim_{N\to\infty} x_0^{(N)} = x_0$.

Then, the sequence $\{P^{(N)}\}$ *is relatively compact and the limit of any convergent subsequence is a solution of the martingale problem* $(\boldsymbol{K}, a, b, x_0)$.

REMARK. An obvious sufficient condition for (i) is

(i)′ $\lim_{N\to\infty} \sup_{x\in K(N)} N \int_{K(N)} |y-x|^{2+\delta}\,\Pi^{(N)}(x, dy) = 0$ for some $\delta > 0$.

Theorem 3.2. *If* (i)—(iv) *hold and if, moreover, uniqueness of the solution* P_{x_0} *of the martingale problem* $(\boldsymbol{K}, a, b, x_0)$ *holds, then* $P^{(N)}$ *converges to* P_{x_0}.

These theorems are close to results of Stroock and Varadhan [12]. The difference lies in the following three points. First, the state space is a set $\boldsymbol{K}$, not $\boldsymbol{R}^l$. Secondly, $a(x)$ may be degenerate. Thirdly, $\Pi^{(N)}(x, dy)$ need not be defined on the whole of $\boldsymbol{K}$. The compactness of the state space makes the proof of the invariance principle easier. But the restriction to $\boldsymbol{K}$ of the state space gives to Theorem 3.1 applicability to the proof of the existence of the solution of the martingale problem $(\boldsymbol{K}, a, b, x)$. Note that, while the martingale problem $(\boldsymbol{R}^l, a, b, x)$ is proved to have a solution for general bounded continuous a, b, the solution of the problem $(\boldsymbol{K}, a, b, x)$ exists only in some special degenerate cases.

Theorem 3.2 is an obvious consequence of Theorem 3.1. Theorem 3.1 can be proved by an appropriate modification of [12]. We give the proof here for completeness.

Lemma 3.1. *$\{P^{(N)}\}$ is relatively compact if* (i) *and the following two conditions are satisfied*:

$$\text{(ii)}' \quad \sup_N \sup_{x \in K(N)} |N\int_{K(N)} (y_p - x_p)\, \Pi^{(N)}(x, dy)| < \infty \qquad \textit{for} \quad p = 1, \cdots, l,$$

$$\text{(iii)}' \quad \sup_N \sup_{x \in K(N)} |N\int_{K(N)} (y_p - x_p)(y_q - x_q)\, \Pi^{(N)}(x, dy)| < \infty$$
$$\textit{for} \quad p, q = 1, \cdots, l.$$

Proof. For each finite T, the restriction of $P^{(N)}$ to $\mathcal{M}_T$ can be viewed as a probability measure on $C([0, T], \boldsymbol{K})$, the space of continuous paths $[0, T] \rightarrow \boldsymbol{K}$. For T fixed, this restriction of $\{P^{(N)}\}$ is relatively compact by [12], Lemma 10.2. Let $\{P^{(n_N)}\}$ be a subsequence of $\{P^{(N)}\}$. Using the diagonal procedure, one can choose a subsequence $\{P^{(m_N)}\}$ of $\{P^{(n_N)}\}$ such that if, for some T, $\xi_T(\omega)$ is a bounded, continuous, real function on Ω determined by the value $x(t, \omega)$, $t \leq T$, then $\int_\Omega \xi_T(\omega) P^{(m_N)}(d\omega)$ is convergent. Let $\rho(\omega, \omega') = \sum_{n=1}^{\infty} 2^{-n}(1 \wedge \max_{0 \leq t \leq n} |\omega(t) - \omega'(t)|)$. Then ρ is a metric compatible with the topology of Ω. For each $\omega \in \Omega$, let $\omega_T(t) = \omega(t \wedge T)$. Then we see that $\sup_{\omega \in \Omega} \rho(\omega, \omega_T) \rightarrow 0$ as $T \rightarrow \infty$. Hence $\int_\Omega \xi(\omega) P^{(m_N)}(d\omega)$ is convergent for every bounded, ρ-uniformly continuous, real function $\xi(\omega)$. This suffices for the convergence of $P^{(m_N)}$.

Proof of Theorem 3.1. Relative compactness of $\{P^{(N)}\}$ follows from Lemma 3.1. Let P be the limit of a subsequence of $\{P^{(N)}\}$. What we have to prove is that P is a solution of the martingale problem $(\boldsymbol{K}, a, b, x_0)$. For simplicity of notations, we assume that $P^{(N)}$ converges to P. Since $\{\omega; |x(0) - x_0| \leq \varepsilon\}$ is a closed set, we have

$$P(|x(0) - x_0| \leq \varepsilon) \geq \limsup_{N \rightarrow \infty} P^{(N)}(|x(0) - x_0| \leq \varepsilon) = 1,$$

and $P(x(0) = x_0) = 1$ follows. Fix $\theta \in \boldsymbol{R}^l$ and write $M(t)$ for $M_\theta(t)$. Let

$$\varphi^{(N)}(x) = \log \{\int_{K(N)} e^{\langle \theta, y - x \rangle}\, \Pi^{(N)}(x, dy)\},$$
$$\bar{M}^{(N)}(n) = \exp \{\langle \theta, \bar{Y}^{(N)}(n) - \bar{Y}^{(N)}(0) \rangle - \sum_{m=0}^{n-1} \varphi^{(N)}(\bar{Y}^{(N)}(m))\},$$
$$M^{(N)}(t) = \exp \{\langle \theta, x(t) - x(0) \rangle - N\int_0^t \varphi^{(N)}\Big(x\Big(\frac{[Nu]}{N}\Big)\Big) du\},$$

where $[Nu]$ is the greatest integer not exceeding Nu, and let $\bar{\mathcal{M}}_n^{(N)}$ be the σ-algebra generated by $\{\bar{Y}^{(N)}(m); m\leq n\}$. It follows from the Markov property of $\{\bar{Y}^{(N)}(n)\}$ that $(\bar{M}^{(N)}(n), \bar{\mathcal{M}}_n^{(N)}, Q^{(N)}; n=0, 1, \cdots)$ is a martingale. Hence $(M^{(N)}(n/N), \mathcal{M}_{n/N}, P^{(N)}; n=0, 1, \cdots)$ is a martingale, because

$$M^{(N)}\Big(\frac{n}{N}\Big) = \exp\Big\{\Big\langle\theta, x\Big(\frac{n}{N}\Big)-x(0)\Big\rangle-\sum_{m=0}^{n-1}\varphi^{(N)}\Big(x\Big(\frac{m}{N}\Big)\Big)\Big\}.$$

We will prove that

$$\lim_{N\to\infty} E^{P^{(N)}}\Big(\xi\cdot M^{(N)}\Big(\frac{[Nt]+1}{N}\Big)\Big) = E^P(\xi\cdot M(t)) \tag{3.1}$$

for every bounded, continuous, $\mathcal{M}_t$-measurable function $\xi(\omega)$. Here E^P and $E^{P^{(N)}}$ denote expectation with respect to P and $P^{(N)}$, respectively. Suppose, for a moment, that (3.1) is proved. If $s<t$ and ξ is bounded, continuous, $\mathcal{M}_s$-measurable, then

$$E^{P^{(N)}}\Big(\xi\cdot M^{(N)}\Big(\frac{[Ns]+1}{N}\Big)\Big) = E^{P^{(N)}}\Big(\xi\cdot M^{(N)}\Big(\frac{[Nt]+1}{N}\Big)\Big)$$

by the martingale property, and, letting $N\to\infty$, we get

$$E^P(\xi\cdot M(s)) = E^P(\xi\cdot M(t)).$$

This is just the martingale property of $(M(t), \mathcal{M}_t, P)$ and the proof would be complete.

In order to prove (3.1), we will first show

$$\lim_{N\to\infty}\sup_{x\in K(N)}\Big|N\Big(\int_{K(N)} e^{\langle\theta, y-x\rangle}\,\Pi^{(N)}(x, dy)-1\Big)-\langle\theta, b(x)\rangle -\frac{1}{2}\langle\theta, a(x)\theta\rangle\Big| = 0. \tag{3.2}$$

We have

$$\begin{aligned}\int e^{\langle\theta, y-x\rangle}\,\Pi^{(N)}(x, dy)-1 &= \int\langle\theta, y-x\rangle\,\Pi^{(N)}(x, dy)\\ &+\frac{1}{2}\int\langle\theta, y-x\rangle^2\,\Pi^{(N)}(x, dy)\\ &+\int_{|y-x|\leq\varepsilon}\langle\theta, y-x\rangle^2\delta(y-x)\,\Pi^{(N)}(x, dy)+B\cdot\Pi^{(N)}(x, V_\varepsilon(x))\end{aligned}$$

where $V_\varepsilon(x)=\{y;\ |y-x|>\varepsilon\}$, $\delta(y-x)\to0$ as $|y-x|\to0$, and B is a bounded function of N, x, ε. Let $f_\varepsilon(N)=\sup_{x\in K(N)} N\,\Pi^{(N)}(x, V_\varepsilon(x))$. Then $f_\varepsilon(N)\to0$ as $N\to\infty$ for each fixed $\varepsilon>0$ by the assumption (i). Using the assumptions (ii) and (iii) we see that the absolute value in (3.2) is bounded by

$$g(N)+C\sup_{|y-x|\leq\varepsilon}|\delta(y-x)|+Cf_\varepsilon(N)$$

for every $\varepsilon>0$ with $g(N)\to 0$ and a constant C. Choosing ε_N which decreases to zero sufficiently slowly, we get $f_{\varepsilon_N}(N)\to 0$, and hence (3.2). Noting that

$$\varphi^{(N)}(x)=\int e^{\langle\theta,y-x\rangle}\Pi^{(N)}(x,dy)-1+o\Big(\int e^{\langle\theta,y-x\rangle}\Pi^{(N)}(x,dy)-1\Big),$$

we get

$$\lim_{N\to\infty}\sup_{x\in K(N)}|N\varphi^{(N)}(x)-\langle\theta,b(x)\rangle-\frac{1}{2}\langle\theta,a(x)\theta\rangle|=0 \tag{3.3}$$

from (3.2).

Let $\Gamma^{(N)}=\{\omega;\,x(n/N,\omega)\in\boldsymbol{K}(N)\text{ for }n=0,1,\cdots\}$. If $\Lambda\in\mathcal{M}_T$ and if Λ is compact when it is viewed as a subset of $C([0,T],\boldsymbol{K})$, then, for any $t<T$

$$\lim_{N\to\infty}\sup_{\omega\in\Gamma^{(N)}\cap\Lambda}\Big|M^{(N)}\Big(\frac{[Nt]+1}{N},\omega\Big)-M(t,\omega)\Big|=0\,. \tag{3.4}$$

In fact, let us write $([Nt]+1)/N=t_N$ and notice that

$$M^{(N)}(t_N,\omega)=\exp\Big\{\langle\theta,x(t_N)-x(0)\rangle-N\int_0^{t_N}\varphi^{(N)}\Big(x\Big(\frac{[Nu]}{N}\Big)\Big)du\Big\}$$

and

$$\begin{aligned}&N\int_0^{t_N}\varphi^{(N)}\Big(x\Big(\frac{[Nu]}{N}\Big)\Big)du-\int_0^t\langle\theta,b(x(u))\rangle du-\frac{1}{2}\int_0^t\langle\theta,a(x(u))\theta\rangle du\\&=\int_0^{t_N}\Big\{N\varphi^{(N)}\Big(x\Big(\frac{[Nu]}{N}\Big)\Big)-\Big\langle\theta,b\Big(x\Big(\frac{[Nu]}{N}\Big)\Big)\Big\rangle-\frac{1}{2}\Big\langle\theta,a\Big(x\Big(\frac{[Nu]}{N}\Big)\Big)\theta\Big\rangle\Big\}du\\&\quad+\int_0^{t_N}\Big\{\Big\langle\theta,b\Big(x\Big(\frac{[Nu]}{N}\Big)\Big)-b(x(u))\Big\rangle+\frac{1}{2}\Big\langle\theta,\Big(a\Big(x\Big(\frac{[Nu]}{N}\Big)\Big)-a(x(u))\Big)\theta\Big\rangle\Big\}du\\&\quad+\int_t^{t_N}\Big\{\langle\theta,b(x(u))\rangle+\frac{1}{2}\langle\theta,a(x(u))\theta\rangle\Big\}du\,.\end{aligned}$$

Uniformly in $\omega\in\Lambda\cap\Gamma^{(N)}$, the first integral in the right-hand side tends to zero by (3.3), and so does the second integral by the uniform continuity of a, b, and also the third integral tends to zero by the boundedness of a, b. Note that $|x([Nu]/N,\omega)-x(u,\omega)|\to 0$ uniformly in $u<T$ and $\omega\in\Lambda$.

Now let us prove (3.1). Let

$$\Delta=\sup_N\sup_{\omega\in\Gamma^{(N)}}\Big|\xi(\omega)\Big(M^{(N)}\Big(\frac{[Nt]+1}{N},\omega\Big)-M(t,\omega)\Big)\Big|,$$

which is finite by (3.3). Given $\varepsilon>0$, choose $\Lambda\in\mathcal{M}_T$ compact as a subset of $C([0,T],\boldsymbol{K})$ in such a way that $\inf_N P^{(N)}(\Lambda)>1-(\varepsilon/\Delta)$. This can be done since $\{P^{(N)}\}$ is tight by Lemma 3.1. Then,

$$|E^{P^{(N)}}\left(\xi\cdot M^{(N)}\left(\frac{[Nt]+1}{N}\right)\right)-E^{P}(\xi\cdot M(t))|\leq\Delta(1-P^{(N)}(\Lambda))$$
$$+\|\xi\|\int_{\Lambda\cap\Gamma^{(N)}}|M^{(N)}\left(\frac{[Nt]+1}{N}\right)-M(t)|\,dP^{(N)}$$
$$+|E^{P^{(N)}}(\xi\cdot M(t))-E^{P}(\xi\cdot M(t))|$$
$$<3\varepsilon$$

if N is large. Here we have used $P^{(N)}\to P$ in the third term and (3.4) in the second term. The proof is complete.

4. Asymptotic estimate of the moments of the induced Markov chains and proof of Theorems 2.1 and 2.2

All notations in this section are the same as in Section 2. In particular, $\{X^{(N)}(n)\}$ is the induced Markov chain on $\boldsymbol{J}(N)$. In order to prove Theorems 2.1 and 2.2, we apply Theorems 3.1 and 3.2 with $l=d-1$ and

$$Y^{(N)}(n)=\left(\frac{1}{N}X_1^{(N)}(n),\cdots,\frac{1}{N}X_{d-1}^{(N)}(n)\right).$$

Let $\boldsymbol{J}^0(N)$ be the set of points $j=(j_1,\cdots,j_{d-1})\in\boldsymbol{Z}_+^{d-1}$ such that $\sum_{p=1}^{d-1}j_p\leq N$ and let $\boldsymbol{K}(N)=\left\{\frac{j}{N};\,j\in\boldsymbol{J}^0(N)\right\}$. For $j\in\boldsymbol{J}^0(N)$, let

$$b_p^{(N)}\left(\frac{j}{N}\right)=N\sum_{k\in J(N)}\left(\frac{k_p}{N}-\frac{j_p}{N}\right)P_{jk}^{(N)}, \tag{4.1}$$

$$a_{pq}^{(N)}\left(\frac{j}{N}\right)=N\sum_{k\in J(N)}\left(\frac{k_p}{N}-\frac{j_p}{N}\right)\left(\frac{k_q}{N}-\frac{j_q}{N}\right)P_{jk}^{(N)}, \tag{4.2}$$

$$e_p^{(N)}\left(\frac{j}{N}\right)=N\sum_{k\in J(N)}\left(\frac{k_p}{N}-\frac{j_p}{N}\right)^4P_{jk}^{(N)}. \tag{4.3}$$

Here we have identified $j=(j_1,\cdots,j_{d-1})\in\boldsymbol{J}^0(N)$ with $(j_1,\cdots,j_{d-1},1-\sum_{p=1}^{N-1}j_p)\in\boldsymbol{J}(N)$. $b_p(x)$ and $a_{pq}(x)$ are difined by (1.1)—(1.3).

Lemma 4.1. *Suppose that for $p,q=1,\cdots,d-1$*

$$\lim_{N\to\infty}\sup_{j\in J^0(N)}|b_p^{(N)}\left(\frac{j}{N}\right)-b_p\left(\frac{j}{N}\right)|=0, \tag{4.4}$$

$$\lim_{N\to\infty}\sup_{j\in J^0(N)}|a_{pq}^{(N)}\left(\frac{j}{N}\right)-a_{pq}\left(\frac{j}{N}\right)|=0, \tag{4.5}$$

$$\lim_{N\to\infty}\sup_{j\in J^0(N)}e_p^{(N)}\left(\frac{j}{N}\right)=0. \tag{4.6}$$

Then Theorems 2.1 *and* 2.2 *follow.*

Proof. Clearly (4.4) and (4.5) imply Conditions (ii) and (iii) of Theorem 3.1. Since

$$\left(\sum_{p=1}^{d-1}(k_p-j_p)^2\right)^2\leq(d-1)\sum_{p=1}^{d-1}(k_p-j_p)^4,$$

(4.6) implies (i)′ for $\delta=2$. Hence, if $Y^{(N)}(0)=x_0^{(N)}\to x_0$, then $\{P^{(N)}\}$ is relatively compact and the limit of any convergent subsequence of $\{P^{(N)}\}$ is a solution of the martingale problem $(\boldsymbol{K}, a, b, x_0)$. Now, using any branching process with immigration satisfying Assumptions 2.1 and 2.2 (given any set of σ^2, α_{pq}, μ_p, we can find such a process), we see that the martingale problem $(\boldsymbol{K}, a, b, x)$ has at least one solution. That Theorem 2.2 follows is a consequence of Theorem 3.2.

Our task in this section is to prove the estimate (4.4)—(4.6). We will prove the following stronger asymptotic formulas :

(4.7) $$b_p^{(N)}\left(\frac{j}{N}\right)=b_p\left(\frac{j}{N}\right)+O\left(\frac{1}{N}\right)\quad\text{uniformly in } j\in\boldsymbol{J}^0(N),$$

(4.8) $$a_{pq}^{(N)}\left(\frac{j}{N}\right)=a_{pq}\left(\frac{j}{N}\right)+O\left(\frac{1}{N}\right)\quad\text{uniformly in } j\in\boldsymbol{J}^0(N),$$

(4.9) $$e_p^{(N)}\left(\frac{j}{N}\right)=O\left(\frac{1}{N}\right)\quad\text{uniformly in } j\in\boldsymbol{J}^0(N).$$

What we mean by (4.7) is

$$\sup_N\sup_{j\in\boldsymbol{J}^0(N)}N\left|b_p^{(N)}\left(\frac{j}{N}\right)-b_p\left(\frac{j}{N}\right)\right|<\infty .$$

We use the phrase "uniformly in $j\in\boldsymbol{J}^0(N)$" in this meaning. In order to prove (4.7)—(4.9) we have to make many estimations of coefficients of power series. Our tool for this is Lemma 4.3 below which is proved by the saddle point method. Naturally the proof of (4.9) is the most complicated. This condition guarantees that the limit process is a diffusion process (that is, sample functions are continuous).

Given $N\geq1$ and $j\in\boldsymbol{J}(N)$, let

(4.10) $$G(s_1,\cdots,s_d)=\sum_{k\in\boldsymbol{J}(N)}P_{jk}^{(N)}s_1^{k_1}\cdots s_d^{k_d},$$

(4.11) $$\Phi(w,s_1,\cdots,s_d)=\prod_{p=1}^{d}f(ws_p)^{j_p},$$

(4.12) $$\Psi(w,s_1,\cdots,s_d)=\Phi\left(w,\sum_{l=1}^{d}\alpha_{1l}^{(N)}s_l,\cdots,\sum_{l=1}^{d}\alpha_{dl}^{(N)}s_l\right),$$

(4.13) $$\Theta(w,s_1,\cdots,s_d)=\Psi(w,s_1,\cdots,s_d)g(ws_1,\cdots,ws_d).$$

Since we have

$$P_{jk}^{(N)} = A_1(N)^{-1} \text{ (coefficient of } w^N s_1^{k_1} \cdots s_d^{k_d} \text{ in } \Theta(w, s_1, \cdots, s_d))$$

by (2.2), we see that

$$G(s_1, \cdots, s_d) = A_1(N)^{-1} \text{ (coefficient of } w^N \text{ in } \Theta(w, s_1, \cdots, s_d)) . \tag{4.14}$$

Let us denote

$$D_{p_1 \cdots p_m} = \frac{\partial^m}{\partial s_{p_1} \cdots \partial s_{p_m}} ,$$

which operates on functions of $(s_1, \cdots, s_d)$ or of $(w, s_1, \cdots, s_d)$.
Let

$$C_{p_1 \cdots p_m} = D_{p_1 \cdots p_m} G(1, \cdots, 1), \; C^*_{p_1 \cdots p_m} = \sum_{k \in J(N)} k_{p_1} \cdots k_{p_m} P_{jk}^{(N)} .$$

Lemma 4.2.

$$C^*_p = C_p, \; C^*_{pq} = C_{pq} \; (p \neq q), \; C^*_{pp} = C_{pp} + C_p ,$$
$$C^*_{ppp} = C_{ppp} + 3C_{pp} + C_p, \quad C^*_{pppp} = C_{pppp} + 6C_{ppp} + 7C_{pp} + C_p .$$

Proof is easy and omitted.

Functions $M(w)$ and $K(w)$ are defined in Section 2. $M(w)$ extends to an analytic function $M(z)$ of complex z with $\operatorname{Re} z < b$. $K(w)$ extends to $K(z)$ analytic in a neighborhood of β. Define κ_n by

$$K(z) = \sum_{n=0}^{\infty} \kappa_n (z - \beta)^n . \tag{4.15}$$

Note that $\kappa_0 = K(\beta)$, $\kappa_1 = 1$, $\kappa_2 = \sigma^2/2$. Let

$$L(s_1, \cdots, s_d) = g(e^{s_1}, \cdots, e^{s_d}) . \tag{4.16}$$

By Assumption 2.2, $L(s_1, \cdots, s_d)$ can be extended to $L(z_1, \cdots, z_d)$ for complex $z_1, \cdots, z_d$ with $\operatorname{Re} z_l < \beta + \varepsilon$ $(l = 1, \cdots, d)$. Similarly $D_{p_1 \cdots p_m} L(s_1, \cdots, s_d)$ can be extended to $D_{p_1 \cdots p_m} L(z_1, \cdots, z_d)$.

Lemma 4.3. *Let*

$$\tilde{A}(N) = \frac{1}{2\pi i} \int_{\beta - i\pi}^{\beta + i\pi} M(z)^{N-r} \tilde{M}(z) e^{-Nz} dz , \quad N \geq r , \tag{4.17}$$

where the integral is along the line segment from $\beta - i\pi$ to $\beta + i\pi$, r is a fixed integer, and $\tilde{M}(z)$ is a bounded continuous function on the segment. Suppose $\tilde{M}(z)$ is analytic in a neighborhood of $z = \beta$ and let

$$M(z)^{-r} \tilde{M}(z) = \sum_{n=0}^{\infty} \tilde{\rho}_n (z - \beta)^n$$

there. Then,

$$\tilde{A}(N) = \Delta_N(\tilde{\rho}_0 + N^{-1}\tilde{a} + O(N^{-2})) \qquad as \quad N \to \infty ,$$

where

$$\Delta_N = \frac{e^{N(K(\beta)-\beta)}}{\sigma\sqrt{2\pi}\sqrt{N}} ,$$

$$\tilde{a} = \tilde{\rho}_0\left(\frac{3\kappa_4}{\sigma^4} - \frac{15\kappa_3^2}{2\sigma^6}\right) + \tilde{\rho}_1\frac{3\kappa_3}{\sigma^4} - \tilde{\rho}_2\frac{1}{\sigma^2} .$$

Proof is omitted, since this is part of Lemma 3.1 of Sato [11].

Let

$A_\nu(N)$=coefficient of w^N in a power series $h_\nu(w)g(w, \cdots, w)$, where we define $h_\nu(w)$ as follows:

$$\begin{aligned}
h_1(w) &= f(w)^N , \\
h_2(w) &= f(w)^{N-1}f'(w)w , \\
h_3(w) &= f(w)^{N-2}f'(w)^2w^2 , \\
h_4(w) &= f(w)^{N-1}f''(w)w^2 , \\
h_5(w) &= f(w)^{N-3}f'(w)^3w^3 , \\
h_6(w) &= f(w)^{N-2}f'(w)f''(w)w^3 , \\
h_7(w) &= f(w)^{N-1}f'''(w)w^3 , \\
h_8(w) &= f(w)^{N-4}f'(w)^4w^4 , \\
h_9(w) &= f(w)^{N-3}f'(w)^2f''(w)w^4 , \\
h_{10}(w) &= f(w)^{N-2}f''(w)^2w^4 , \\
h_{11}(w) &= f(w)^{N-2}f'(w)f'''(w)w^4 , \\
h_{12}(w) &= f(w)^{N-1}f''''(w)w^4 .
\end{aligned}$$

Note that $A_1(N)$ is the same that is defined in (2.3). Further, let

$$A_{\nu p_1 \cdots p_m}(N) = \text{coefficient of } w^N \text{ in } h_\nu(w)D_{p_1\cdots p_m}g(w, \cdots, w)w^m .$$

All functions which appear above are power series of w with non-negative coefficients and the convergence radii are bigger than e^β. Hence they are convergent at e^z with $\mathrm{Re}\, z=\beta$.

Lemma 4.4. *For each ν, m, $p_1, \cdots, p_m$, we have*

$$A_\nu(N) = \frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi} h_\nu(e^z)g(e^z, \cdots, e^z)e^{-Nz}dz , \tag{4.18}$$

$$A_{\nu p_1\cdots p_m}(N) = \frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi} h_\nu(e^z)D_{p_1\cdots p_m}g(e^z, \cdots, e^z)e^{mz}e^{-Nz}dz , \tag{4.19}$$

and Lemma 4.3 *applies.*

Proof. We need only use Fubini's theorem to get the integral representation (4.18), (4.19). It is easy to see that they are of the form (4.17) satisfying the conditions in Lemma 4.3.

Given $N \geq 1$ and $j \in \boldsymbol{J}(N)$, we write $x = N^{-1}j$. Hence $x_p = N^{-1}j_p$ for $1 \leq p \leq d$ and $\sum_{p=1}^{d} x_p = 1$. Sometimes we write $y_p = \sum_{l=1}^{d} x_l \alpha_{lp}$. We use the notation $(m)_n = m(m-1)\cdots(m-n+1)$.

Lemma 4.5.

$$C_p = j_p + \sum_{l=1}^{d} x_l \alpha_{lp} + \mu_p - x_p \sum_{l=1}^{d} \mu_l + O(N^{-1}) \text{ uniformly in } j \in \boldsymbol{J}(N)\,, \tag{4.20}$$

and

$$b_p^{(N)}(x) = \sum_{l=1}^{d} x_l \alpha_{lp} + \mu_p - x_p \sum_{l=1}^{d} \mu_l + O(N^{-1}) \text{ uniformly in } j \in \boldsymbol{J}(N)\,, \tag{4.21}$$

that is, (4.7) *holds.*

Proof. It follows from (4.14) that

$$C_p = D_p G(1, \cdots, 1) = A_1(N)^{-1} \text{ (coefficient of } w^N \text{ in } D_p \Theta(w, 1, \cdots, 1))\,.$$

From (4.11)—(4.13) we have

$$\begin{aligned} D_p\Theta(w, s_1, \cdots, s_d) &= D_p\Psi(w, s_1, \cdots, s_d) g(ws_1, \cdots, ws_d) \\ &\quad + \Psi(w, s_1, \cdots, s_d) D_p g(ws_1, \cdots, ws_d) w\,, \\ D_p\Psi(w, s_1, \cdots, s_d) &= \sum_l D_l\Phi(w, \sum_m \alpha_{1m}^{(N)} s_m, \cdots, \sum_m \alpha_{dm}^{(N)} s_m) \alpha_{lp}^{(N)}\,, \\ D_l\Phi(w, s_1, \cdots, s_d) &= f(ws_1)^{j_1} \cdots \{j_l f(ws_l)^{j_l - 1} f'(ws_l) w\} \cdots f(ws_d)^{j_d}\,. \end{aligned}$$

Noting that

$$\Psi(w, 1, \cdots, 1) = \Phi(w, 1, \cdots, 1) = f(w)^N\,,$$
$$\sum_l j_l \alpha_{lp}^{(N)} = j_p + y_p\,,$$

we have

$$D_l\Phi(w, 1, \cdots, 1) = j_l h_2(w)\,, \quad D_p\Psi(w, 1, \cdots, 1) = (j_p + y_p) h_2(w)\,,$$
$$D_p\Theta(w, 1, \cdots, 1) = (j_p + y_p) h_2(w) g(w, \cdots, w) + h_1(w) D_p g(w, \cdots, w) w\,.$$

Hence

$$C_p = (j_p + y_p) \frac{A_2(N)}{A_1(N)} + \frac{A_{1p}(N)}{A_1(N)}\,. \tag{4.22}$$

Let us estimate $A_1(N)$, $A_2(N)$, $A_{1p}(N)$. By Lemma 4.4,

$$A_1(N) = \frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi} M(z)^N L(z, \cdots, z) e^{-Nz}\, dz\,.$$

Define ρ_n by the expansion

(4.23) $\quad L(z, \cdots, z) = \sum_{n=0}^{\infty} \rho_n(z-\beta)^n$ in a neighborhood of β.

We have $\rho_0 = g(e^\beta, \cdots, e^\beta) > 0$ and

(4.24) $$\rho_1 = \rho_0 \sum_{l=1}^{d} \mu_l\,.$$

Thus

(4.25) $$A_1(N) = \Delta_N(\rho_0 + N^{-1}a_1 + O(N^{-2}))\,,$$

(4.26) $$a_1 = \rho_0\left(\frac{3\kappa_4}{\sigma^4} - \frac{15\kappa_3^2}{2\sigma^6}\right) + \rho_1\frac{3\kappa_3}{\sigma^4} - \rho_2\frac{1}{\sigma^2}$$

by Lemma 4.3. Similarly,

$$A_2(N) = \frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi} M(z)^{N-1}M'(z) L(z, \cdots, z) e^{-Nz}\, dz$$

since $f'(e^z)e^z = M'(z)$. We have

$$\begin{aligned} M(z)^{-1}M'(z)L(z, \cdots, z) &= K'(z)L(z, \cdots, z) \\ = \rho_0 + (\sigma^2\rho_0 + \rho_1)(z-\beta) &+ (3\kappa_3\rho_0 + \sigma^2\rho_1 + \rho_2)(z-\beta)^2 + \cdots \end{aligned}$$

by (4.15) and (4.23), and hence

(4.27) $$A_2(N) = \Delta_N(\rho_0 + N^{-1}a_2 + O(N^{-2})), \quad a_2 = a_1 - \rho_1$$

by Lemma 4.3 and (4.26). Further,

$$A_{1p}(N) = \frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi} M(z)^N D_p L(z, \cdots, z) e^{-Nz}\, dz$$

since $D_pL(s_1, \cdots, s_d) = D_pg(e^{s_1}, \cdots, e^{s_d})e^{s_p}$. Noting that $D_pL(\beta, \cdots, \beta) = \rho_0\mu_p$, we get

(4.28) $$A_{1p}(N) = \Delta_N(\rho_0\mu_p + O(N^{-1}))\,.$$

Using (4.25), (4.27), (4.28) in (4.22), we obtain

$$C_p = (j_p + y_p)(1 + N^{-1}\rho_0^{-1}(a_2 - a_1) + O(N^{-2})) + \mu_p + O(N^{-1}) \text{ uniformly,}$$

from which (4.20) follows since $\rho_0^{-1}(a_2-a_1)=-\sum_{l=1}^{d}\mu_l$. Note that j_p is at most N and y_p is bounded. Since $b_p^{(N)}(x)=C_p-j_p$, we get (4.21) from (4.20),.

Lemma 4.6.

$$C_{pq}=j_pj_q-N^{-1}j_pj_q(\sigma^2+2\sum_{l=1}^{d}\mu_l)+j_p\sum_{l=1}^{d}x_l\alpha_{lq}+j_q\sum_{l=1}^{d}x_l\alpha_{lp} +j_p\mu_q+j_q\mu_p+O(1)\qquad for\quad p\neq q\,, \tag{4.29}$$

$$C_{pp}=j_p^2+j_p(\sigma^2-1+2\mu_p+2\sum_{l=1}^{d}x_l\alpha_{lp}-x_p(\sigma^2+2\sum_{l=1}^{d}\mu_l))+O(1)\,, \tag{4.30}$$

$$a_{pq}^{(N)}(x)=-\sigma^2x_px_q+O(N^{-1})\qquad for\quad p\neq q\,, \tag{4.31}$$

$$a_{pp}^{(N)}(x)=\sigma^2x_p(1-x_p)+O(N^{-1})\,. \tag{4.32}$$

All O signs here are uniform in $j\in\boldsymbol{J}(N)$.

Proof. We can prove

$$C_{pq}=\{(j_p+y_p)(j_q+y_q)-\sum_{l=1}^{d}j_l\alpha_{lp}^{(N)}\alpha_{lq}^{(N)}\}\frac{A_3(N)}{A_1(N)}+\sum_{l=1}^{d}j_l\alpha_{lp}^{(N)}\alpha_{lq}^{(N)}\frac{A_4(N)}{A_1(N)} +(j_p+y_p)\frac{A_{2q}(N)}{A_1(N)}+(j_q+y_q)\frac{A_{2p}(N)}{A_1(N)}+\frac{A_{1pq}(N)}{A_1(N)}\,. \tag{4.33}$$

where $p=q$ is not excluded. In fact,

$$C_{pq}=A_1(N)^{-1}\text{ (coefficient of } w^N \text{ in } D_{pq}\Theta(w,1,\cdots,1))\,,$$

and $D_{pq}\Theta(w,1,\cdots,1)$ is obtained as follows:

$$\begin{aligned}
D_{pq}\Psi(w,s_1,\cdots,s_d)&=\sum_{l,m}D_{lm}\Phi(w,\sum_n\alpha_{1n}^{(N)}s_n,\cdots,\sum_n\alpha_{dn}^{(N)}s_n)\alpha_{lp}^{(N)}\alpha_{mq}^{(N)}\,,\\
D_{lm}\Phi(w,s_1,\cdots,s_d)&=f(ws_1)^{j_1}\cdots\{j_lf(ws_l)^{j_l-1}f'(ws_l)w\}\cdots\\
&\qquad\{j_mf(ws_m)^{j_m-1}f'(ws_m)w\}\cdots f(ws_d)^{j_d}\qquad\text{if}\quad l<m\,,\\
D_{ll}\Phi(w,s_1,\cdots,s_d)&=f(ws_1)^{j_1}\cdots\{(j_l)_2f(ws_l)^{j_l-2}f'(ws_l)^2w^2\\
&\qquad+j_lf(ws_l)^{j_l-1}f''(ws_l)w^2\}\cdots f(ws_d)^{j_d}\,,
\end{aligned}$$

and hence

$$\begin{aligned}
D_{lm}\Phi(w,1,\cdots,1)&=j_lj_mh_3(w)\qquad\text{if}\quad l\neq m\\
D_{ll}\Phi(w,1,\cdots,1)&=(j_l)_2h_3(w)+j_lh_4(w)\,.
\end{aligned}$$

Thus,

$$D_{pq}\Psi(w,1,\cdots,1)=\{\sum_{l,m}j_lj_m\alpha_{lp}^{(N)}\alpha_{mq}^{(N)}-\sum_lj_l\alpha_{lp}^{(N)}\alpha_{lq}^{(N)}\}h_3(w) +\sum_lj_l\alpha_{lp}^{(N)}\alpha_{lq}^{(N)}h_4(w)\,.$$

Since

$$D_{pq}\Theta(w, 1, \cdots, 1) = D_{pq}\Psi(w, 1, \cdots, 1)g(w, \cdots, w) \\ +D_p\Psi(w, 1, \cdots, 1)D_q g(w, \cdots, w)w+D_q\Psi(w, 1, \cdots, 1)D_p g(w, \cdots, w)w \\ +\Psi(w, 1, \cdots, 1)D_{pq}g(w, \cdots, w)w^2$$

and the terms except the first in the right-hand side are treated in the proof of Lemma 4.5, (4.33) follows.

Having

$$(4.34) \qquad \sum_l j_l \alpha_{lp}^{(N)}\alpha_{lq}^{(N)} = \begin{cases} O(1) & \text{if } p \neq q \\ j_p+O(1) & \text{if } p = q \end{cases}$$

in mind, let us estimate the right-hand side of (4.33). We have

$$A_3(N) = \frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi} M(z)^{N-2}M'(z)^2 L(z, \cdots, z)e^{-Nz}dz$$

and, in a neighborhood of β,

$$M(z)^{-2}M'(z)^2L(z, \cdots, z) = K'(z)^2L(z, \cdots, z) \\ = \rho_0+(2\sigma^2\rho_0+\rho_1)(z-\beta)+(6\kappa_3\rho_0+\sigma^4\rho_0+2\sigma^2\rho_1+\rho_2)(z-\beta)^2+\cdots .$$

Hence

$$(4.35) \qquad A_3(N) = \Delta_N(\rho_0+N^{-1}a_3+O(N^{-2})), \quad a_3 = a_1-\sigma^2\rho_0-2\rho_1$$

by Lemma 4.3 and (4.26). Next,

$$A_4(N) = \frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi} M(z)^{N-1}(M''(z)-M'(z))L(z, \cdots, z)e^{-Nz}dz$$

by $f''(e^z)e^{2z} = M''(z)-M'(z)$, and

$$M(z)^{-1}(M''(z)-M'(z))L(z, \cdots, z) \\ = (K'(z)^2+K''(z)-K'(z))L(z, \cdots, z) = \sigma^2\rho_0$$

at $z=\beta$. Hence

$$(4.36) \qquad A_4(N) = \Delta_N(\sigma^2\rho_0+O(N^{-1})) .$$

Also $A_{2p}(N)$ and $A_{1pq}(N)$ have analogous expression by Lemma 4.4. Thus

$$(4.37) \qquad A_{2p}(N) = \frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi} M(z)^{N-1}M'(z)D_pL(z, \cdots, z)e^{-Nz}dz \\ = \Delta_N(\rho_0\mu_p+O(N^{-1}))$$

since $M(z)^{-1}M'(z)D_pL(z, \cdots, z)=K'(z)D_pL(z, \cdots, z)=\rho_0\mu_p$ at $z=\beta$, and

(4.38) $\quad A_{1pq}(N)=\Delta_N O(1)$.

It follows from (4.25), (4.33)—(4.38) that

$$C_{pq}=(j_pj_q+j_py_q+j_qy_p)(1-N^{-1}(\sigma^2+2\rho_0^{-1}\rho_1))+j_p\mu_q+j_q\mu_p+O(1) \quad \text{for} \quad p\neq q\,,$$
$$C_{pp}=(j_p^2+2j_py_p-j_p)(1-N^{-1}(\sigma^2+2\rho_0^{-1}\rho_1))+j_p\sigma^2+2j_p\mu_p+O(1)\,.$$

Hence we get (4.29), (4.30) by (4.24). Since

$$a_{pq}^{(N)}(x)=N^{-1}(C_{pq}-j_pC_q-j_qC_p+j_pj_q) \qquad \text{for} \quad p\neq q\,,$$
$$a_{pp}^{(N)}(x)=N^{-1}(C_{pp}+C_p-2j_pC_p+j_p^2)\,,$$

(4.31) and (4.32) follow from (4.20), (4.29), (4.30).

Lemma 4.7.

(4.39) $$C_{ppp}=j_p^3+3j_p^2(\sigma^2-1+\mu_p+\sum_{l=1}^d x_l\alpha_{lp}-x_p(\sigma^2+\sum_{l=1}^d \mu_l))+O(N)$$

uniformly in $J(N)$.

Proof. Clearly

(4.40) $\quad C_{ppp}=U_1+3U_2+3U_3+U_4$

where U_1, U_2, U_3 and U_4 are $A_1(N)^{-1}$ times coefficients of w^N in $D_{ppp}\,\Psi(w,1,\cdots,1)g(w,\cdots,w)$, $D_{pp}\Psi(w,1,\cdots,1)D_pg(w,\cdots,w)w$, $D_p\Psi(w,1,\cdots,1)D_{pp}g(w,\cdots,w)w^2$, and $\Psi(w,1,\cdots,1)D_{ppp}g(w,\cdots,w)w^3$, respectively. We have

$$D_{ppp}\Psi(w,s_1,\cdots,s_d)=\sum_{l,m,n}D_{lmn}\Phi(w,\sum_r\alpha_{1r}^{(N)}s_r,\cdots,\sum_r\alpha_{dr}^{(N)}s_r)\,\alpha_{lp}^{(N)}\alpha_{mp}^{(N)}\alpha_{np}^{(N)}\,,$$

$$D_{lmn}\Phi(w,1,\cdots,1)=j_lj_mj_nh_5(w) \qquad \text{if } l,m,n \text{ are all different},$$
$$D_{lln}\Phi(w,1,\cdots,1)=(j_l^2j_n-j_lj_n)h_5(w)+j_lj_nh_6(w) \qquad \text{if} \quad l\neq n$$

and

$$D_{lll}\Phi(w,1,\cdots,1)=(j_l^3-3j_l^2+2j_l)h_5(w)+3(j_l^2-j_l)h_6(w)+j_lh_7(w)\,,$$

because

$$D_{lll}\Phi(w,s_1,\cdots,s_d)=f(ws_1)^{j_1}\cdots\{(j_l)_3f(ws_l)^{j_l-3}f'(ws_l)^3w^3+3(j_l)_2f(ws_l)^{j_l-2}f'(ws_l)f''(ws_l)w^3+j_lf(ws_l)^{j_l-1}f'''(ws_l)w^3\}\cdots f(ws_d)^{j_d}\,.$$

Hence,

$$U_1=A_1(N)^{-1}\,(\text{coefficient of } w^N \text{ in } \sum_{l,m,n}D_{lmn}\Phi(w,1,\cdots,1)\,g(w,\cdots,w)\alpha_{lp}^{(N)}\alpha_{mp}^{(N)}\alpha_{np}^{(N)})$$
$$=(S_1-3S_2+2S_3)\frac{A_5(N)}{A_1(N)}+(3S_2-3S_3)\frac{A_6(N)}{A_1(N)}+S_3\frac{A_7(N)}{A_1(N)}\,,$$

where

$$S_1=\sum_{l,m,n} j_l j_m j_n \alpha_{lp}^{(N)}\alpha_{mp}^{(N)}\alpha_{np}^{(N)}=(j_p+y_p)^3=j_p^3+3j_p^2y_p+O(N),$$
$$S_2=\sum_{l,n} j_l j_n(\alpha_{lp}^{(N)})^2\alpha_{np}^{(N)}=j_p^2+O(N),$$
$$S_3=\sum_{l} j_l(\alpha_{lp}^{(N)})^3=O(N).$$

As in the proof of Lemmas 4.5 and 4.6, U_2, U_3 and U_4 are expressed as follows:

$$U_2=\{(j_p+y_p)^2-\sum_l j_l(\alpha_{lp}^{(N)})^2\}\frac{A_{3p}(N)}{A_1(N)}+\sum_l j_l(\alpha_{lp}^{(N)})^2\frac{A_{4p}(N)}{A_1(N)},$$
$$U_3=(j_p+y_p)\frac{A_{2pp}(N)}{A_1(N)},\quad U_4=\frac{A_{1ppp}(N)}{A_1(N)}.$$

Hence

$$C_{ppp}=(j_p^3+3j_p^2y_p-3j_p^2)\frac{A_5(N)}{A_1(N)}+3j_p^2\frac{A_6(N)}{A_1(N)}+3j_p^2\frac{A_{3p}(N)}{A_1(N)}+O(N) \tag{4.41}$$

by (4.40) and Lemma 4.4. We have

$$\begin{aligned} A_5(N)&=\frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi}M(z)^{N-3}M'(z)^3L(z,\cdots,z)e^{-Nz}dz \\ &=\Delta_N(\rho_0+N^{-1}a_5+O(N^{-2})),\quad a_5=a_1-3\sigma^2\rho_0-3\rho_1 \end{aligned} \tag{4.42}$$

by Lemma 4.3 and (4.26), since

$$M^{-3}M'^3L=K'^3L=\rho_0+(3\sigma^2\rho_0+\rho_1)(z-\beta)+(3\sigma^4\rho_0+9\kappa_3\rho_0+3\sigma^2\rho_1+\rho_2)(z-\beta)^2+\cdots$$

in a neighborhood of β. Similarly

$$\begin{aligned} A_6(N)&=\frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi}M(z)^{N-2}M'(z)(M''(z)-M'(z))L(z,\cdots,z)e^{-Nz}dz \\ &=\Delta_N(\sigma^2\rho_0+O(N^{-1})) \end{aligned} \tag{4.43}$$

since $M^{-2}M'(M''-M')L=K'(K'^2+K''-K')L=\sigma^2\rho_0$ at $z=\beta$, and

$$\begin{aligned} A_{3p}(N)&=\frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi}M(z)^{N-2}M'(z)^2D_pL(z,\cdots,z)e^{-Nz}dz \\ &=\Delta_N(\rho_0\mu_p+O(N^{-1})) \end{aligned} \tag{4.44}$$

as in (4.37). It follows from (4.25), (4.41)—(4.44) that

$$C_{ppp}=(j_p^3+3j_p^2y_p-3j_p^2)(1-3N^{-1}(\sigma^2+\rho_0^{-1}\rho_1))+3j_p^2\sigma^2+3j_p^2\mu_p+O(N),$$

which is (4.39).

Lemma 4.8.

$$(4.45)\qquad C_{pppp}=j_p^4+2j_p^3(3\sigma^2-3+2\mu_p+2\sum_{l=1}^{d}x_l\alpha_{lp}-x_p(3\sigma^2+2\sum_{l=1}^{d}\mu_l))+O(N^2)$$

uniformly tn $j\in I(N)$.

Proof. This time we have

$$C_{pppp}=V_1+4V_2+6V_3+4V_4+V_5\,,$$

where $V_1,\cdots,V_5$ are $A_1(N)^{-1}$ times coefficients of w^N in $D_{pppp}\Psi(w,1,\cdots,1)$ $g(w,\cdots,w)$, $D_{ppp}\Psi(w,1,\cdots,1)D_pg(w,\cdots,w)w$, $D_{pp}\Psi(w,1,\cdots,1)D_{pp}g(w,\cdots,w)w^2$, $D_p\Psi(w,1,\cdots,1)D_{ppp}g(w,\cdots,w)w^3$, $\Psi(w,1,\cdots,1)D_{pppp}g(w,\cdots,w)w^4$, respectively. The most involved part is

$$(4.46)\qquad D_{pppp}\Psi(w,1,\cdots,1)=\sum_{l,m,n,r}D_{lmnr}\Phi(w,1,\cdots,1)\alpha_{lp}^{(N)}\alpha_{mp}^{(N)}\alpha_{np}^{(N)}\alpha_{rp}^{(N)}$$

and we have

$$D_{lmnr}\Phi(w,1,\cdots,1)=j_lj_mj_nj_rh_8(w)\qquad \text{if}\quad l,m,n,r,\text{ are all different,}$$
$$D_{llnr}\Phi(w,1,\cdots,1)=(j_l^2j_nj_r-j_lj_nj_r)h_8(w)+j_lj_nj_rh_9(w)\qquad \text{if}\quad l\neq n\neq r\neq l\,,$$
$$D_{llnn}\Phi(w,1,\cdots,1)=(j_l^2j_n^2-j_l^2j_n-j_lj_n^2+j_lj_n)h_8(w)+(j_l^2j_n+j_lj_n^2-2j_lj_n)h_9(w)+j_lj_nh_{10}(w)\qquad \text{if}\quad l\neq n\,,$$
$$D_{lllr}\Phi(w,1,\cdots,1)=(j_l^3j_r-3j_l^2j_r+2j_lj_r)h_8(w)+3(j_l^2j_r-j_lj_r)h_9(w)+j_lj_rh_{11}(w)\qquad \text{if}\quad l\neq r\,,$$
$$D_{llll}\Phi(w,1,\cdots,1)=(j_l^4-6j_l^3+11j_l^2-6j_l)h_8(w)+6(j_l^3-3j_l^2+2j_l)h_9(w)+3(j_l^2-j_l)h_{10}(w)+4(j_l^2-j_l)h_{11}(w)+j_lh_{12}(w)\,.$$

The last one cames from the expression

$$D_{llll}\Phi(w,s_1,\cdots,s_d)=f(ws_1)^{j_1}\cdots\{(j_l)_4f(ws_l)^{j_l-4}f'(ws_l)^4w^4+6(j_l)_3f(ws_l)^{j_l-3}f'(ws_l)^2f''(ws_l)w^4+3(j_l)_2f(ws_l)^{j_l-2}f''(ws_l)^2w^4+4(j_l)_2f(ws_l)^{j_l-2}f'(ws_l)f'''(ws_l)w^4+j_lf(ws_l)^{j_l-1}f''''(ws_l)w^4\}\cdots f(ws_d)^{j_d}$$

Let $\sum^{(1)}$, $\sum^{(2)}_{(l=m)}$, $\sum^{(3)}_{(l=m,n=r)}$, $\sum^{(4)}_{(l=m=n)}$, $\sum^{(5)}$ denote the summations over $\{(l,m,n,r);\,l,m,n,r$ are different from each other$\}$, $\{(l,m,n,r);\,l=m\neq n\neq r\neq l\}$, $\{(l,m,n,r);\,l=m\neq n=r\}$, $\{(l,m,n,r);\,l=m=n\neq r\}$, $\{(l,m,n,r);\,l=m=n=r\}$, respectively. Decompose the right-hand side of (4.46) so that

$$\sum_{l,m,n,r}=\sum^{(1)}+(\sum^{(2)}_{(l=m)}+\sum^{(2)}_{(l=n)}+\sum^{(2)}_{(l=r)}+\sum^{(2)}_{(m=n)}+\sum^{(2)}_{(m=r)}+\sum^{(2)}_{(n=r)})+(\sum^{(3)}_{(l=m,n=r)}+\sum^{(3)}_{(l=n,m=r)}+\sum^{(3)}_{(l=r,m=n)})+(\sum^{(4)}_{(l=m=n)}+\sum^{(4)}_{(l=m=r)}+\sum^{(4)}_{(l=n=r)}+\sum^{(4)}_{(m=n=r)})+\sum^{(5)}\,,$$

and use the above formulas. Then, by Lemma 4.4, we get

$$V_1 = (\sum_l j_l \alpha_{lp}^{(N)})^4 \frac{A_8(N)}{A_1(N)} + 6(\sum_l j_l (\alpha_{lp}^{(N)})^2)(\sum_l j_l \alpha_{lp}^{(N)})^2 \frac{A_9(N)-A_8(N)}{A_1(N)} + O(N^2)$$
$$= (j_p^4 + 4j_p^3 y_p - 6j_p^3) \frac{A_8(N)}{A_1(N)} + 6j_p^3 \frac{A_9(N)}{A_1(N)} + O(N^2)$$

after some reflection. By argument quite similar to the proof of Lemmas 4.5, 4.6, 4.7, we get

$$V_2 = (S_1 - 3S_2 + 2S_3)\frac{A_{5p}(N)}{A_1(N)} + (3S_2 - 3S_3)\frac{A_{6p}(N)}{A_1(N)} + S_3 \frac{A_{7p}(N)}{A_1(N)}$$
$$= j_p^3 \frac{A_{5p}(N)}{A_1(N)} + O(N^2),$$
$$V_3 = O(N^2), \quad V_4 = O(N), \quad V_5 = O(1).$$

Hence,

$$C_{pppp} = (j_p^4 + 4j_p^3 y_p - 6j_p^3)\frac{A_8(N)}{A_1(N)} + 6j_p^3 \frac{A_9(N)}{A_1(N)} + 4j_p^3 \frac{A_{5p}(N)}{A_1(N)} + O(N^2). \tag{4.47}$$

We have

$$A_8(N) = \frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi} M(z)^{N-4} M'(z)^4 L(z, \cdots, z) e^{-Nz} dz \tag{4.48}$$
$$= \Delta_N(\rho_0 + N^{-1} a_8 + O(N^{-2})), \quad a_8 = a_1 - 6\sigma^2 \rho_0 - 4\rho_1$$

since

$$M^{-4} M'^4 L = K'^4 L = \rho_0 + (4\sigma^2\rho_0 + \rho_1)(z-\beta) + (12\kappa_3\rho_0 + 6\sigma^4\rho_0 + 4\sigma^2\rho_1 + \rho_2)(z-\beta)^2 + \cdots$$

in a neighborhood of β,

$$A_9(N) = \frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi} M(z)^{N-3} M'(z)^2 (M''(z) - M'(z)) L(z, \cdots, z) e^{-Nz} dz \tag{4.49}$$
$$= \Delta_N(\sigma^2\rho_0 + O(N^{-1}))$$

since $M^{-3}M'^2(M''-M')L = K'^2(K'^2+K''-K')L = \sigma^2\rho_0$ at β, and

$$A_{5p}(N) = \frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi} M(z)^{N-3} M'(z)^3 D_p L(z, \cdots, z) e^{-Nz} dz \tag{4.50}$$
$$= \Delta_N(\rho_0 \mu_p + O(N^{-1}))$$

as in (4.37). By (4.25), (4.47)—(4.50) we have

$$C_{pppp} = (j_p^4 + 4j_p^3 y_p - 6j_p^3)(1 - N^{-1}(6\sigma^2 + 4\rho_0^{-1}\rho_1)) + 6j_p^3\sigma^2 + 4j_p^3\mu_p + O(N^2),$$

and the proof of (4.45) is complete.

Lemma 4.9.

(4.51) $\quad e_p^{(N)}(x) = O(N^{-1}) \qquad$ *uniformly in* $j \in \boldsymbol{J}(N)$.

Proof. We have

$$\begin{aligned} e_p^{(N)}(x) &= N^{-3}(C^*_{pppp} - 4j_p C^*_{ppp} + 6j_p^2 C^*_{pp} - 4j_p^3 C^*_p + j_p^4) \\ &= N^{-3}\{C_{pppp} + (-4j_p + 6)C_{ppp} + (6j_p^2 - 12j_p + 7)C_{pp} \\ &\qquad + (-4j_p^3 + 6j_p^2 - 4j_p + 1)C_p + j_p^4\} \end{aligned}$$

by Lemma 4.2. Use Lemmas 4.5—4.8 and substitute (4.20), (4.30), (4.39), (4.45) for C_p, C_{pp}, C_{ppp}, C_{pppp}. Then, all terms cancel except terms of magnitude $O(N^{-1})$, and (4.51) follows.

Now Lemmas 4.5, 4.6 and 4.9 say that (4.7), (4.8) and (4.9) hold. Hence the proof of Theorems 2.1 and 2.2 is complete.

5. Uniqueness of the solution of the martingale problem and proof of Theorems 2.3 and 2.4

All notations in this section are the same as in Section 2. By Theorems 2.1 and 2.2, convergence of the interpolated normalization of the induced Markov chains is proved if the solution of the martingale problem ($\boldsymbol{K}$, a, b, x) is unique. This uniqueness problem is a problem on some stochastic differential equations, as the following lemma says.

Lemma 5.1. *Let* $c(x) = (c_{pq}(x))$ *be a bounded, Borel measurable* $(d-1)\times(d-1)$*-matrix on* $\boldsymbol{K}$ *satisfying*

(5.1) $\quad c(x)c(x)' = a(x)\,,$

where $c(x)'$ *is the transpose of* $c(x)$. *Consider a stochastic differential equation*

(5.2) $$\begin{cases} dX_p(t) = \sum_{q=1}^{d-1} c_{pq}(X(t))\,dB_q(t) + b_p(X(t))\,dt, & p = 1, \cdots, d-1 \\ X(0) = x \end{cases}$$

where $X(t) = (X_1(t), \cdots, X_{d-1}(t))$ *is a process taking values in* $\boldsymbol{K}$ *and* $B(t) = (B_1(t), \cdots, B_{d-1}(t))$ *is a* $(d-1)$*-dimensional Brownian motion. Then, uniqueness of the solution of the martingale problem* ($\boldsymbol{K}$, a, b, x) *is equivalent to uniqueness, in the sense of probability measure induced in* $(\Omega, \mathcal{M})$, *of the solution of* (5.2).

Proof. This is proved essentially by Stroock and Varadhan [13]. Priouret [10] contains a detailed account.

We choose $c(x)=(c_{pq}(x))_{p,q=1\cdots,d-1}$ as a lower triangular matrix. Let $\boldsymbol{K}^0$ and $\partial\boldsymbol{K}$ be the interior and the boundary of $\boldsymbol{K}$ and let, for $x\in\boldsymbol{K}^0$,

$$
\begin{aligned}
&c_{pq}(x)=0 \qquad \text{for} \quad q>p \\
&c_{11}(x)=\sigma x_1^{1/2}(1-x_1)^{1/2}, \\
&c_{21}(x)=-\sigma x_2 x_1^{1/2}(1-x_1)^{-1/2}, \quad c_{22}(x)=\sigma x_2^{1/2}(1-x_1-x_2)^{1/2}(1-x_1)^{-1/2}, \\
&c_{31}(x)=-\sigma x_3 x_1^{1/2}(1-x_1)^{-1/2}, \quad c_{32}(x)=-\sigma x_3 x_2^{1/2}(1-x_1)^{-1/2}(1-x_1-x_2)^{-1/2}, \\
&c_{33}(x)=\sigma x_3^{1/2}(1-x_1-x_2-x_3)^{1/2}(1-x_1-x_2)^{-1/2}, \\
&c_{41}(x)=-\sigma x_4 x_1^{1/2}(1-x_1)^{-1/2}, \quad c_{42}(x)=-\sigma x_4 x_2^{1/2}(1-x_1)^{-1/2}(1-x_1-x_2)^{-1/2}, \\
&c_{43}(x)=-\sigma x_4 x_3^{1/2}(1-x_1-x_2)^{-1/2}(1-x_1-x_2-x_3)^{-1/2}, \\
&c_{44}(x)=\sigma x_4^{1/2}(1-x_1-x_2-x_3-x_4)^{1/2}(1-x_1-x_2-x_3)^{-1/2}, \\
&\cdots\cdots\cdots
\end{aligned}
\tag{5.3}
$$

Lemma 5.2. *$c(x)$ defined above on $\boldsymbol{K}^0$ extends continuously to $\boldsymbol{K}$. Denote the extension by the same $c(x)$. Then*

$$a_{pq}(x)=\sum_{r=1}^{p\wedge q} c_{pr}(x)c_{qr}(x)\,, \tag{5.4}$$

that is, (5.1) *holds.*

Proof. Let $x\in\partial\boldsymbol{K}$. We define $c_{pq}(x)$ by the formulas in (5.3) when negative powers of 0 do not appear, and define $c_{pq}(x)=0$ when negative powers of 0 appear. Then all c_{pq} are continuous on $\boldsymbol{K}$, since we have

$$
\begin{aligned}
&0\geq c_{21}(x)\geq-\sigma x_1^{1/2}(1-x_1)^{1/2}, \quad 0\leq c_{22}(x)\leq\sigma(1-x_1)^{1/2}, \\
&0\geq c_{31}(x)\geq-\sigma x_1^{1/2}(1-x_1)^{1/2}, \quad 0\geq c_{32}(x)\geq-\sigma(1-x_1-x_2)^{1/2}, \\
&0\leq c_{33}(x)\leq\sigma(1-x_1-x_2)^{1/2}, \\
&\cdots\cdots\cdots
\end{aligned}
\tag{5.5}
$$

for $x\in\boldsymbol{K}^0$. In order to show (5.4), we may assume $x\in\boldsymbol{K}^0$. Let

$$1-x_1=y_1, \quad 1-x_1-x_2=y_2, \quad 1-x_1-x_2-x_3=y_3, \ \cdots.$$

Then,

$$
\begin{aligned}
&c_{11}=\sigma(x_1y_1)^{1/2}, \\
&c_{21}=-\sigma x_2\left(\frac{x_1}{y_1}\right)^{1/2}=-\sigma x_2\left(\frac{1}{y_1}-1\right)^{1/2}, \quad c_{22}=\sigma\left(\frac{x_2y_2}{y_1}\right)^{1/2}, \\
&c_{31}=-\sigma x_3\left(\frac{x_1}{y_1}\right)^{1/2}=-\sigma x_3\left(\frac{1}{y_1}-1\right)^{1/2}, \\
&c_{32}=-\sigma x_3\left(\frac{x_2}{y_1y_2}\right)^{1/2}=-\sigma x_3\left(\frac{1}{y_2}-\frac{1}{y_1}\right)^{1/2}, \quad c_{33}=\sigma\left(\frac{x_3y_3}{y_2}\right)^{1/2}, \\
&\cdots\cdots\cdots
\end{aligned}
$$

and (5.4) is obtained by direct calculation.

Henceforth $c(x)$ denotes the one defined by (5.3) and Lemma 5.2.

Lemma 5.3. *Let $d-1=1$. Then the solution of the stochastic differential equation* (5.2) *is pathwise unique. That is, if $X(t)$ and $\tilde{X}(t)$ are solutions of* (5.2) *with a common one-dimensional Brownian motion $B(t)$, then $X(t)=\tilde{X}(t)$ almost surely.*

Proof. In this case (5.2) is a single stochastic differential equation

$$dX(t) = c_{11}(X(t))\,dB(t)+b_1(X(t))\,dt, \quad X(0) = x\,.$$

c_{11} is extended on $\boldsymbol{R}^1$ to a Hölder continuous function with exponent 1/2 and b_1 is extended to a Lipschitz continuous function on $\boldsymbol{R}^1$. Hence the theorem of Yamada and S. Watanabe [15] applies.

Lemma 5.4. *If $x, y\in \boldsymbol{K}$ and $x_l=y_l$ for $l=1, \cdots, p-1$, then*

$$(5.6) \qquad |c_{pq}(x)-c_{pq}(y)|\leq 2\sigma|x_p-y_p|^{1/2}\,.$$

Proof. Let $q=1<p$. If $1-\varepsilon\leq x_1\leq 1$, then $0\geq c_{p1}(x)\geq -\sigma\varepsilon^{1/2}$ and $0\geq c_{p1}(y)\geq -\sigma\varepsilon^{1/2}$ by (5.5). If $1-\varepsilon>x_1\geq 0$, then

$$|c_{p1}(x)-c_{p1}(y)| = \sigma|x_p-y_p|x_1^{1/2}(1-x_1)^{-1/2}\leq\sigma\varepsilon^{-1/2}|x_p-y_p|\,.$$

Hence

$$|c_{p1}(x)-c_{p1}(y)|\leq\sigma(\varepsilon^{-1/2}|x_p-y_p|+\varepsilon^{1/2})$$

in any case. Since ε is arbitrary we may choose $\varepsilon=|x_p-y_p|^{1/2}$ and get (5.6) for $q=1<p$. The proof for $1<q<p$ is similar. In case $p=q$, we let $1-x_1-\cdots-x_{p-1}=1-y_1-\cdots-y_{p-1}=\xi$ and have

$$\begin{aligned}|c_{pp}(x)-c_{pp}(y)| &\leq\sigma\xi^{-1/2}(|x_p^{1/2}(\xi-x_p)^{1/2}-y_p^{1/2}(\xi-x_p)^{1/2}| \\ &\qquad +|y_p^{1/2}(\xi-x_p)^{1/2}-y_p^{1/2}(\xi-y_p)^{1/2}|) \\ &\leq\sigma\xi^{1/2}((\xi-x_p)^{1/2}|x_p-y_p|^{1/2}+y_p^{1/2}|x_p-y_p|^{1/2})\leq 2\sigma|x_p-y_p|^{1/2}\,,\end{aligned}$$

since

$$|\eta^{1/2}-\zeta^{1/2}|\leq|\eta-\zeta|^{1/2}\,.$$

Lemma 5.5. *Let $d-1>1$ and suppose that $b(x)$ satisfies the condition in Theorem* 2.4. *Then the solution of the stochastic differential equation* (5.2) *is pathwise unique.*

Proof. By the assumption $b_p(x)$ is a linear function only of $x_1, \cdots, x_p$. As is seen from (5.3), $c_{pq}(x)$ is also a function of $x_1, \cdots, x_p$. Suppose that $X(t)=(X_1(t), \cdots, X_{d-1}(t))$ and $\tilde{X}(t)=(\tilde{X}_1(t), \cdots, \tilde{X}_{d-1}(t))$ are solutions of (5.2) taking values in $\boldsymbol{K}$ with a common Brownian motion $B(t)=(B_1(t), \cdots, B_{d-1}(t))$. First, we get $X_1(t)=\tilde{X}_1(t)$ *a.s.* as in Lemma 5.3, since the equation for $X_1(t)$ does not involve $X_2(t), \cdots, X_{d-1}(t)$. Let us prove $X_p(t)=\tilde{X}_p(t)$ *a.s.* for all p by induction. Suppose $X_l(t)=\tilde{X}_l(t)$ *a.s.* for $l=1, \cdots, p-1$. The following proof is essentially the same as part of Yamada and S. Watanabe [15]. Choose nonnegative C^2 functions $\varphi_n(\xi)$ on $\boldsymbol{R}^1$ such that $\varphi_n(\xi)$ increases to $|\xi|$ as $n\to\infty$, $\varphi_n(\xi)=0$ in a neighborhood of 0, $\varphi_n'(\xi)$ is bounded, and $0\leq\varphi_n''(\xi)\leq 2n^{-1}|\xi|^{-1}$. Since

$$X_p(t)-\tilde{X}_p(t) = \sum_{q=1}^{p}\int_0^t \{c_{pq}(X(s))-c_{pq}(\tilde{X}(s))\}\, dB_q(s) + \int_0^t \{b_p(X(s))-b_p(\tilde{X}(s))\}\, ds\,,$$

we get

$$\varphi_n(X_p(t)-\tilde{X}_p(t)) = \text{stochastic integral}+\frac{1}{2}\int_0^t \varphi_n''(X_p(s)-\tilde{X}_p(s)) \sum_{q=1}^{p} \{c_{pq}(X(s))-c_{pq}(\tilde{X}(s))\}^2 ds+\int_0^t \varphi_n'(X_p(s)-\tilde{X}_p(s))\{b_p(X(s))-b_p(\tilde{X}(s))\}\, ds\,.$$

by the Ito formula. We have $\{c_{pq}(X(s))-c_{pq}(\tilde{X}(s))\}^2\leq 4\sigma^2|X_p(s)-\tilde{X}_p(s)|$ by Lemma 5.4 and hence

$$E[\varphi_n(X_p(t)-\tilde{X}_p(t))]\leq\frac{1}{2}\int_0^t \frac{2}{n} 4p\sigma^2 ds + CE[\int_0^t |X_p(s)-\tilde{X}_p(s)|\, ds]\,, \quad C = \text{const.}$$

Going to the limit as $n\to\infty$ we get

$$E|X_p(t)-\tilde{X}_p(t)|\leq C\int_0^t E|X_p(s)-\tilde{X}_p(s)|\, ds\,.$$

By iteration it follows that $E|X_p(t)-\tilde{X}_p(t)|=0$. Hence $X_p(t)=\tilde{X}_p(t)$, *a.s.*

Proof of Theorems 2.3 and 2.4. Yamada and S. Watanabe [15] prove that pathwise uniqueness of the solution of a stochastic differential equation implies uniqueness in the sense of probability measure induced in $(\Omega, \mathcal{M})$. Hence Theorems 2.3 and 2.4 follow from Lemmas 5.3 and 5.5, respectively.

Properties of the limiting diffusions

Let $d-1=1$. We have proved that the martingale problem $(\boldsymbol{K}, a, b, x)$ has a unique solution (Theorem 2.3). It is a one-dimensional diffusion process on [0, 1] with backward Kolmogorov equation (2.8). Let $\lambda_1=\alpha_{21}+\mu_1$ and $\lambda_2=\alpha_{12}+\mu_2$. In Feller's boundary classification into four types (regular, pure exit, pure entrance, natural), the boundary 0 is pure exit, regular, pure entrance according as $\lambda_1=0$, $0<\lambda_1<\sigma^2/2$, $\lambda_1\geq\sigma^2/2$, respectively. Similarly, the boundary 1 is pure exit, regular, pure entrance according as $\lambda_2=0$, $0<\lambda_2<\sigma^2/2$, $\lambda_2\geq\sigma^2/2$, respectively. The proof is a standard argument. In case the boundary is pure exist, it is a trap. This is a consequence of the general theory, but this is clear also from the uniqueness of the solution since in case $\lambda_1=0$ the process standing still at 0 is a solution. If $\lambda_1>0$, it is shown by the method of S. Watanabe [14], p. 459, that $E_x[\int_0^\infty \chi_{\{0\}}(x(t))\,dt]=0$ for every $x\in[0, 1]$, where $\chi_{\{0\}}$ is the indicator function of the set $\{0\}$. Therefore, in case the boundary is regular, the limiting diffusion has reflecting boundary condition there, while, in case of pure entrance boundary, the limiting diffusion starting from there immediately enters the interior and never returns.

Let $d-1=2$ and suppose that the uniqueness holds for the martingale problem $(\boldsymbol{K}, a, b, x)$ for every x. Thus $(x(t), \mathcal{M}_t, P_x; x\in\boldsymbol{K})$ is a diffusion process on the triangular region $\boldsymbol{K}=\{(x_1, x_2);\, x_1\geq 0,\, x_2\geq 0,\, 1-x_1-x_2\geq 0\}$ and its backward equation is (2.9). Let us examine its boundary properties. Since all the three sides of $\boldsymbol{K}$ are similar, we examine $\Gamma=\{(x_1, 0);\, 0<x_1<1\}$. Following Hasminsky [4] and S. Watanabe [14], we define regular and repulsive boundary points and unattainable and pure entrance boundary segments as follows. Let $x\in\Gamma$ and let U be a neighborhood in $\boldsymbol{K}$ of x having positive distance with $\partial\boldsymbol{K}-\Gamma$. For $\eta>0$, let $U_\eta=U\cap\{(x_1, x_2);\, x_2>\eta\}$, $T_\eta(\omega)$=first leaving time of U_η, and $T(\omega)=\lim_{\eta\downarrow 0} T_\eta(\omega)$. Let $\Gamma_U^T(\omega)$ be the set of all limit points of $x(t, \omega)$ when $t\uparrow T(\omega)$. x is called *regular* if, for every U and for every neighborhood V in $\boldsymbol{K}$ of x, we have

$$\lim_{y\in\boldsymbol{K}^0,\, y\to x} P_y(\Gamma_U^T(\omega)\subset V\cap\Gamma) = 1\,.$$

x is called *repulsive*, if, for some U and for some $\eta>0$, we have

$$\liminf_{y\in\boldsymbol{K}^0,\, y\to x} P_y(\Gamma_U^T(\omega)\subset(\partial U)_\eta)<1\,.$$

Here $(\partial U)_\eta=\partial U\cap\{y=(y_1, y_2);\, y_2<\eta\}$, ∂U being the boundary in $\boldsymbol{R}^2$ of U. Let $\sum$ be an open interval in Γ. $\sum$ is called *unattainable* if for every $x\in\sum$ there exists a neighborhood U in $\boldsymbol{K}$ of x such that

$$P_y(\Gamma_U^T(\omega)\cap\Gamma \text{ is empty}) = 1 \qquad \text{for every } y\in U\cap\boldsymbol{K}^0\,.$$

$\sum$ is called a *pure entrance boundary segment* if $\sum$ is unattainable and if every $x\in\sum$ has a neighborhood U such that $P_y(T_U<\infty)=1$ for every $y\in U$. Here T_U is the first leaving time of U.

Define $\sum_1$ and $\sum_2$ as follows:

$$\textstyle\sum_1 = \{(x_1, 0)\in\Gamma;\, x_1(\alpha_{12}-\alpha_{32})+\alpha_{32}+\mu_2<\sigma^2/2\}\,,$$
$$\textstyle\sum_2 = \{(x_1, 0)\in\Gamma;\, x_1(\alpha_{12}-\alpha_{32})+\alpha_{32}+\mu_2>\sigma^2/2\}\,.$$

$\sum_1$ and $\sum_2$ are open intervals or empty.

Theorem 5.1. (i) *Every point in* Σ_1 *is regular. If* $\alpha_{12}=\alpha_{32}=\mu_2=0$, *then* $\Sigma_1=\Gamma$ *and, after hitting* Γ, *the process moves in* $\{(x_1, 0); 0\leq x_1\leq 1\}$ *following the backward equation*

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} x_1(1-x_1)\frac{\partial^2 u}{\partial x_1^2}+\{-x_1(\alpha_{13}+\mu_3)+(1-x_1)(\alpha_{31}+\mu_1)\} \frac{\partial u}{\partial x_1}$$

with the behavior at (0, 0) *and* (1, 0) *described in the case* $d-1=1$. *If at least one of* α_{12}, α_{32} *and* μ_2 *is positive, then*

$$\int_0^\infty \chi_{\Sigma_1}(x(t))\,dt = 0 \qquad a.s. \tag{5.7}$$

and, for any open subinterval Σ *of* Σ_1, *the process starting from* $\boldsymbol{K}^0$ *hits* Σ *in a finite time with positive probability.*

(ii) Σ_2 *is a pure entrance boundary segment and every point in* Σ_2 *is repulsive.*

Proof. Regularity of the points in Σ_1, repulsiveness of the points in Σ_2 and unattainability of Σ_2 are applications of Hasminsky's tests [4]. The property described in (i) in case $\alpha_{12}=\alpha_{32}=\mu_2=0$ is a consequence of the assumed uniqueness of the solution of the martingale problem. The proof of the other properties is similar to the discussion by S. Watanabe [14] of two-dimensional diffusion processes with branching property. In proving (5.7) we use the expression

$$\begin{cases} dX_1(t) = c_{11}^*(X(t))\,dB_1^*(t)+c_{12}^*(X(t))\,dB_2^*(t)+b_1(X(t))\,dt \\ dX_2(t) = c_{22}^*(X(t))\,dB_2^*(t)+b_2(X(t))\,dt \end{cases}$$

where $B^*(t) = (B_1^*(t), B_2^*(t))$ is a two-dimensional Brownian motion and

$$c_{11}^*(x) = \sigma x_1^{1/2}(1-x_1-x_2)^{1/2}(1-x_2)^{-1/2}, \quad c_{12}^*(x) = -\sigma x_1 x_2^{1/2}(1-x_2)^{-1/2},$$
$$c_{22}^*(x) = \sigma x_2^{1/2}(1-x_2)^{1/2}.$$

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Added in proof. S. N. Ethier (*A class of degenerate diffusion processes occurring in population genetics*, Comm. Pure Appl. Math. **29** (1976), 483–493) gives a proof to our conjecture on the uniqueness of the solution of the martingale problem (K, a, b, x) in general dimensions. Thus the assumption in our Theorem 2.2 is always satisfied.