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A CERATAIN DEGENERATE EVOLUTION EQUATION OF THE SUBDIFFERENTIAL TYPE

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0. Introduction

In this paper we are concerned with the weak solution of the following nonlinear evolution equation

$$\frac{du}{dt}(t)+\partial(c(t)\varphi^t+I_{\bar{D}})u(t)\ni f(t) \tag{0.1}$$

in a real Hilbert space H. Here φ^t, $0\leqq t\leqq T$, is a family of convex, $\not\equiv+\infty$, lower-semi-continuous functions from H into $]-\infty, +\infty]$. Putting $D(\varphi^t)=\{u\in H;\ \varphi^t(u)<+\infty\}$, we assume that $D(\varphi^t)\equiv D$ is independent of t. The operator $\partial(c(t)\varphi^t+I_{\bar{D}})$ is the subdifferential of $c(t)\varphi^t+I_{\bar{D}}$ where $c(t)$ is nonnegative continuous function, $\bar{D}$ is the closure of D and $I_{\bar{D}}$ is the indicator function of $\bar{D}$ (see [1]). The function $c(t)$ may vanish somewhere, hence (0.1) is an equation of degenerate type. We denote the inner product and norm in H by $(\ ,\ )$ and $|\ |$, respectively. Moreover we assume that there exist a continuous function $h(t)$ and two positive constants L_r and l_r depending on $r>0$ such that

$$|\varphi^t u-\varphi^s u|\leqq L_r|h(t)-h(s)|\{\varphi^s u+l_r\} \tag{0.2}$$

holds for $0\leqq s\leqq t\leqq T$, $u\in D$ and $|u|\leqq r$.
Next, for $c(t)$ we assume

$$\text{measure}\,(\{t\in]0, T[;\ c(t)=0\}-int\,\{t\in]0, T[;\ c(t)=0\})=0\,. \tag{0.3}$$

DEFINITION 0.1. We say that $u\in C([0, T];\bar{D})$ is a weak solution of (0.1) if and only if u satisfies $(c(t)\varphi^t+I_{\bar{D}})u(t)=c(t)\varphi^t u(t)\in L^1(0, T)$ and

$$\begin{aligned}&\int_{t_1}^{t_2}c(s)\varphi^s v(s)\,ds-\int_{t_1}^{t_2}c(s)\ \varphi^s u(s)\,ds\\ \geqq&\int_{t_1}^{t_2}(f(s)-\frac{dv}{ds}(s),\ v(s)-u(s))\,ds\\ &\quad+\frac{1}{2}|v(t_2)-u(t_2)|^2-\frac{1}{2}|v(t_1)-v(t_1)|^2\end{aligned}$$

for any $v \in W^{1,1}(t_1, t_2 : \bar{D})$ where t_1 and t_2 are nonnegative numbers such that $0 \leqq t_1 \leqq t_2 \leqq T$.

Now we state our main theorem.

Theorem. *Under the above assumptions the weak solution of* (0.1) *satisfying the initial condition* $u(0)=u_0$ *exists and is unique for any* $f \in L^1(0, T: H)$ *and* $u_0 \in \bar{D}$.

Abstract equations of this form have been considered by H. Brezis [1], H. Attouch et A. Damlamian [2], N. Kenmochi and T. Nagai [3], K. Maruo [4], J. Watanabe [5], e.t.c. This paper has been motivated by [5], in which the strong solution was considered under the condition of nondegenerate type. For the linear case, Friedman and Shuss [6] considered a degenerate equation of this form. In this paper, we have attempted to extend the results of [5] and [6] in a certain sence.

First, in section 1, we introduce some definitions and lemmas which will be used throghout this paper. Next, in section 2, we will prove the theorem when φ^t is independent of t. Finally, in section 3, we will prove the existence and the uniqueness of weak solutions of (0.1) in the general case.

1. Definitions and fundamental lemmas

Definition 1.1. We say that $u \in C([0, T]: H)$ is a strong solution of (0.1) if u is absolutely continuous on any compact subset of $]0,T[$ and satisfies (0,1) almost everyshere on$]0, T[$.

DEFINITION 1.2. We say that $u \in C([0, T]: H)$ is a piecewise strong solution of (0.1) if there exists a partition of $[0, T]$: $0=t_0<t_1<\cdots<t_{i-1}<t_i=T$, and u is strong solution of (0.1) on every $[t_k, t_{k+1}]_{k=0,1,\dots,i-1}$.

DEFINITION 1.3. We say that $u \in C([0, T]: H)$ is a s-weak solution of (0,1) if there exists $\{u^j\}_{j\geqq 1} \subset C([0, T]: H)$ and $\{f^j\}_{j\geq 1} \subset L^1(0, T: H)$ such that u^j is a strong solution of

$$\frac{du^j}{dt}(t)+\partial(c(t)\varphi^t+I_{\bar{D}})u^j(t) \ni f^j(t)$$

and $f^j \to f$ in $L^1(0, T: H)$ and $u^j \to u$ in $C([0, T]: H)$ as $j \to \infty$.

DEFINITION 1.4. We say that $u \in C([0, T]: H)$ is a piecewise s-weak solution of (0.1) if there exists a partition of $[0, T]$: $0=t_0<t_1<\cdots<t_{i-1}<t_i=T$, and u is a s-weak solution of (0.1) on every $[t_k, t_{k+1}]_{k=0,1,\dots,i-1}$.

Lemma 1.1. *The following diagram hold:*

1) *strong solution* $\Rightarrow$ *s-weak solution* $\Rightarrow$ *piecewise s-weak solution* $\Rightarrow$ *waek solution.*

2) *strong solution* $\Rightarrow$ *piecewise strong solution* $\Rightarrow$ *piecewise s-weak solution.*

Proof. If u be a strong solution, then for any $v \in D$

$$c(t)\varphi^t v - c(t)\varphi^t u(t) \geqq \left(f(t) - \frac{du}{dt}(t),\ v - u(t)\right)$$

$$= (f(t),\ v - u(t)) + \frac{1}{2}\frac{d}{dt}|u(t) - v|^2 .$$

Hence $c(t)\varphi^t u(t)$ is absolutely integrable on $[0, T]$. The remaining part of the proof is simple and is omitted.

Lemma 1.2. *Let* u *and* v *be piecewise s-weak solution of*

$$\frac{du}{dt}(t) + \partial(c(t)\varphi^t + I_{\bar{D}})u(t) \ni f(t) \tag{1.1}$$

$$\frac{dv}{dt}(t) + \partial(c(t)\varphi^t + I_{\bar{D}})v(t) \ni g(t) \tag{1.2}$$

respectively where f *and* $g \in L^1(0, T: H)$. *Then we have*

$$\frac{1}{2}|u(t) - v(t)|^2 \leqq \frac{1}{2}|u(s) - v(s)|^2 + \int_s^t (f(x) - g(x),\ u(x) - v(x))\,dx , \tag{1.3}$$

$$|u(t) - v(t)| \leqq |u(s) - v(s)| + \int_s^t |f(x) - g(x)|\,dx , \tag{1.4}$$

for any $0 \leqq s \leqq t \leqq T$.

Proof. If u and v are s-weak solutions, we have known these results from [1], p. 64, Lemme 3.1. If u and v are piecewise s-weak solutions, we obtain the above inequalities for respective partitions. Next, if we add these inequalities, we have our results.

2. Some lemmas

In this section, we will prove Theorem when φ^t is independent of t in Lemma 2.2 and Lemma 2.3.

Lemma 2.1. *For any* $h(t) \in L^1(0, T: H)$ *and* $u_0 \in \bar{D}$, *there exists a s-weak solution* $u(t)$ *of*

$$\frac{du}{dt}(t) + \partial(c(t)\varphi + I_{\bar{D}})u(t) \ni \sqrt{c(t)} \cdot h(t), \quad u(0) = u_0 \tag{2.1}$$

and it is the unique piecewise s-weak solution of this problem. In particular, if

$h(t)\in L^2(0, T: H)$ and $u_0\in\bar{D}$, *then u is a unique piecewise strong solution.*

Proof. The uniqueness follows from Lemma 1.2. We will prove the existence of a solution. We can always reduce to the case min $\varphi=0$ since $D(\partial\varphi)\neq\phi$ in view of [1]. We put for any $t\in[0, T]$ and $\varepsilon\in]0, 1]$,

$$c_\varepsilon(t) = c(t)+\varepsilon,\ y = \sigma_\varepsilon(t) = \int_0^t c_\varepsilon(s)\,ds\,,$$

$$\sigma(t) = \int_0^t c(s)\,ds,\ g_\varepsilon(y) = c_\varepsilon(t)^{-1}\cdot\sqrt{c(t)}\cdot h(t)\,.$$

Case (1): $h(t)\in L^2(0, T: H)$ and $u_0\in D$.
Using [1] p. 72, Theorem 3.6, we get a strong solution of

$$\frac{du}{dt}v_\varepsilon(y)+\partial\varphi v_\varepsilon(y)\ni g_\varepsilon(y),\ v_\varepsilon(0) = u_0\,, \tag{2.2}$$

and have the next equality almost everywhere on $[0, T_\varepsilon]$ where $T_\varepsilon=\sigma_\varepsilon(T)$,

$$\left|\frac{d}{dy}v_\varepsilon(y)\right|^2+\frac{d}{dy}\varphi v_\varepsilon(y) = (g_\varepsilon(y), \frac{d}{dy}v_\varepsilon(y))\,.$$

Hence we get for any $y\in[0, T_\varepsilon]$,

$$\frac{1}{2}\int_0^y\left|\frac{d}{dx}v_\varepsilon(x)\right|^2dx+\varphi v_\varepsilon(y)-\varphi u_0\leqq\frac{1}{2}\int_0^y|g_\varepsilon(x)|^2dx\leqq\frac{1}{2}\int_0^T|h(x)|^2dx\,.$$

We have for any $y\in[0, T_\varepsilon]$

$$\int_0^y\left|\frac{d}{dx}v_\varepsilon(x)\right|^2dx\ \leqq 2\varphi u_0+\int_0^T|h(t)|^2dt\,, \tag{2.3}$$

$$0\leqq\varphi v_\varepsilon(y)\leqq\varphi u_0+\frac{1}{2}\int_0^T|h(t)|^2dt\,. \tag{2.4}$$

Next we put $u_\varepsilon(t)=v_\varepsilon(\sigma_\varepsilon(t))$; $t\in[0, T]$, then $u_\varepsilon(t)$ is a unique strong solution of

$$\frac{d}{dt}u_\varepsilon(t)+c_\varepsilon(t)\partial\varphi u_\varepsilon(t)\ni\sqrt{c(t)}\,h(t),\ u_\varepsilon(0) = u_0,\ t\in[0, T]\,. \tag{2.5}$$

By (2.3) and (2.4), we get

$$\int_0^T\left|\frac{d}{dt}u_\varepsilon(t)\right|^2dt\leqq C_1 \tag{2.6}$$

$$0\leqq\varphi u_\varepsilon(t)\leqq C_1 \tag{2.7}$$

where C_1 is a positive constant independent of ε and t.
On the other hand, from (2.5), we have

$$\begin{aligned}&c_\varepsilon(t)\varphi u_{\varepsilon'}(t)-c_\varepsilon(t)\varphi u_\varepsilon(t)\\ &\geqq(\sqrt{c(t)}h(t)-\frac{d}{dt}u_\varepsilon(t),\ u_{\varepsilon'}(t)-u_\varepsilon(t))\end{aligned} \tag{2.8}$$

$$\begin{aligned}&c_{\varepsilon'}(t)\varphi u_\varepsilon(t)-c_{\varepsilon'}(t)\varphi u_{\varepsilon'}(t)\\ &\geqq(\sqrt{c(t)}h(t)-\frac{d}{dt}u_{\varepsilon'}(t),\ u_\varepsilon(t)-u_{\varepsilon'}(t))\end{aligned} \tag{2.9}$$

By adding these two inequalities (2.8) and (2.9), we have

$$\begin{aligned}&(\varepsilon-\varepsilon')\varphi u_{\varepsilon'}(t)+(\varepsilon'-\varepsilon)\varphi u_\varepsilon(t)\\ &\geqq\frac{1}{2}\frac{d}{dt}|u_\varepsilon(t)-u_{\varepsilon'}(t)|^2 .\end{aligned}$$

Integrate above inequality on $[0,t]$. Then using (2.7), we get

$$\frac{1}{2}|u_\varepsilon(t)-u_{\varepsilon'}(t)|^2\leqq 4TC_1\cdot\varepsilon \qquad \text{for any } t\in[0,\ T],\ 0<\varepsilon'<\varepsilon .$$

So, there exists a $u\in C([0,\ T]:\ H)$ such that

$$u_\varepsilon\to u \quad \text{as} \quad \varepsilon\to 0 \quad \text{in} \quad C([0,\ T]:\ H) .$$

By (2.6), there exists a sunsequence $\varepsilon_j\to 0$ and $w\in L^2\ (0,\ T:\ H)$ such that

$$\frac{d}{dt}u_{\varepsilon_j}\to w \quad \text{as} \quad j\to\infty \quad \text{in} \quad \text{w}-L^2(0,\ T:\ H) .$$

Hence we get

$$u\in W^{1,1}(0,\ T:\ H),\ w=\frac{du}{dt}\in L^2(0,\ T:\ H)$$

By (2.5), we have

$$\begin{aligned}&\int_s^t c_\varepsilon(x)\varphi(v)\,dx-\int_s^t c_\varepsilon(x)\varphi u_\varepsilon(x)\,dx\\ &\geqq\int_s^t(\sqrt{c(x)}h(x)-\frac{du_\varepsilon}{dx}(x),\ v-u_\varepsilon(x))\,dx\end{aligned}$$

$$\text{for any} \quad v\in D(\varphi) \quad \text{and} \quad [s,\ t]\subset[0,\ T] .$$

Letting $\varepsilon=\varepsilon_j\to 0$, we get $c(t)\varphi u(t)\in L^1(0,\ T)$ and

$$\begin{aligned}&\int_s^t c(x)\varphi(v)\,dx-\int_s^t c(x)\varphi u(x)\,dx\\ &\geqq\int_s^t(\sqrt{c(x)}h(x)-\frac{d}{dx}u(x),\ v-u(x))\,dx .\end{aligned}$$

For any Lebesque point of $c(t)\varphi u(t)$, $\sqrt{c(t)}\ h(t)$ and $\frac{du}{dt}(t)$, we have

$$c(t)\varphi(v)-c(t)\varphi u(t)\geqq(\sqrt{c(t)}h(t)-\frac{du}{dt}(t),\ v-u(t))$$
$$\text{for any} \quad v\in D(\varphi)\,.$$

Hence we get that $u(t)$ is a strong solution of (2.1), and in view of (2.7) we know that $u(t)\in D$ for any $t\in[0, T]$.
Cace (2): $h(t)\in L^1(0, T: H)$, $u_0\in\bar{D}$.
There exists $\{u_0^j\}_{j\geq1}\subset D$ such that $u_0^j\to u_0$ as $j\to\infty$ in H and $\{h^j(t)\}_{j\geq1}\subset L^2(0, T: H)$ such that $h^j\to h$ as $j\to\infty$ in $L^1(0, T: H)$. Let $u^j\in C([0, T]: H)$ be a strong solution of (2.1) with the initial deta u_0^j. By Lemma 1.2, we get

$$|u^j(t)-u^k(t)|\leqq|u_0^j-u_0^k|+\int_0^T\sqrt{c(s)}\,|h^j(s)-h^k(s)|\,ds\,.$$

So, $\{u^j\}_{j\geq1}$ is a Cauchy sequence in $C([0, T]: H)$.
There exists a $u\in C([0, T]; H)$ such that

$$u^j\to u \qquad \text{in} \quad C([0, T]: H)\,.$$

By the definition, u is a s-weak solution of (2.1).
Cace (3): $h(t)\in L^2(0, T: H)$, $u_0\in\bar{D}$.
There exists $\{u_0^j\}_{j\geq1}\subset D$ such that $u_0^j\to u_0$ as as $j\to\infty$ in H. If $\int_0^t c(s)\,ds>0$, for any $t\in]0, T]$, we denote by $v_\varepsilon^j(y)$ a unique strong solution of (2.2) with a initial deta u_0^j. By [1] Theorem 3.6, we get, for any $\delta\in]0, T]$,

$$\int_{\sigma_\varepsilon(\delta)}^{\sigma_\varepsilon(T)}\left|\frac{d}{dy}v_\varepsilon^j(y)\right|^2dy\leqq(\sqrt{2\sigma(\delta)})^{-1}\cdot C_2+C_3 \tag{2.10}$$

where C_2 and C_3 are positive constants independent of ε, j and δ.

Let $u_\varepsilon^j(t)=v_\varepsilon^j(\sigma_\varepsilon(t))$ $t\in[0, T]$, then we know that $u_\varepsilon^j(t)$ is a unique strong solution of (2.1) with the initial deta u_0^j. By (2.10), we get

$$\int_\delta^T\left|\frac{d}{dt}u_\varepsilon^j(t)\right|^2dt\leqq(\sqrt{2\sigma(\delta)})^{-1}\cdot C_4+C_5$$

where C_4 and C_5 are positive constants.
As in Case (1) we get

$$u_\varepsilon^j\to u^j \quad \text{as} \quad \varepsilon\to0 \quad \text{in} \quad C([0, T]: H)$$

where u^j is a unique strong solution of (2.1) with the initial deta u_0^j.
Using the method of Case (1) we have

$$\int_{\delta}^{T}\left|\frac{d}{dt}u^j(t)\right|^2 dt \leqq (\sqrt{2\sigma(\delta)})^{-1}\cdot C_4+C_5\,. \tag{2.11}$$

Now, recalling the proof of Case (2), we know that u is a unique s-weak solution of (2.1) with the initial deta u_0 where $u^j\to u$ as $j\to\infty$ in $C([0, T]: H)$ and moreover u is a strong solution of (2.1) by (2.11).
Finally, if there exists positive constant δ_0 such that

$$\delta_0 = inf\,\{\delta\in[0, T];\ \int_0^{\delta} c(s)\,ds>0\}\ ,$$

we put

$$u(t) = \begin{cases} u_0 & 0\leqq t\leqq\delta_0 \\ \text{the solution of (2.1) with the initial deta } u_0 & \\ \qquad\qquad \delta_0\leqq t\leqq T\,. \end{cases}$$

Using the above method and Lemma 1.2, then $u(t)$ is a picewise strong solution of (2.1). We complete the proof of this Lemma.

Now, for simplicity we denote by $E(f, u_0)$ the initial value problem

$$\frac{du}{dt}+\partial(c(t)\varphi+I_{\bar{D}})u\ni f\,,\ u(0) = u_0\,.$$

Lemma 2.2. *For any $f(t)\in L^1(0, T: H)$ and $u_0\in\bar{D}$ there exists a weak solution of $E(f, u_0)$.*

Proof. We put

$$N = \{t\in[0, T];\ c(t) = 0\},\ P = \{t\in[0, T];\ c(t)>0\} \quad \text{and}$$
$$R = P\cup int\,N\,.$$

By the assumption (0.3), we have measure $([0, T]-R)=0$. Since R is an open set, there exists an at most countable set of disjoint open intervals $\{]a_i, b_i[\}_{i\geqq 1}$ such that $R=\sum_{i\geqq 1}\,]a_i, b_i[$. If $\{i\}$ is a finite set, the proof is easier. Thus we assume $\{i\}$ is an infinite set. Let $f^i\in L^1(0, T: H)$ be the function such that

$$\begin{cases} f^i(t) = f(t);\ a_j+2^{-i-1}(b_j-a_j)<t<b_j-2^{-i-1}(b_j-a_j) \\ \qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \cdots, i\,. \\ f^i(t) = 0;\ \text{otherwise.} \end{cases} \tag{2.12}$$

Clearly

$$f^i\to f \quad \text{as} \quad i\to\infty \quad \text{in} \quad L^1(0, T: H)\,. \tag{2.13}$$

Let us consider the equation

$$\frac{du}{dt}(t)+\partial(c(t)\varphi+I_{\bar{D}})u(t)\ni f^i(t)\,. \tag{2.14}$$

If $]a_j, b_j[\subset P$, we can apply Lemma 2.1 in this interval since $f^i(t)$ vanishes near a_j and b_j. If $]a_j, b_j[\subset int\, N$, (2.14) reduces to

$$\frac{du}{dt}(t)+\partial I_{\bar{D}}u(t)\ni f^i(t)\,,$$

there. Hence there exists a unique piecewise s-weak solution u^i of $E(f^i, u_0)$. By Lemma 1.2, we have

$$|u^j(t)-u^k(t)|\leqq\int_0^T|f^j(x)-f^k(x)|\,dx \qquad \text{for any} \quad t\in[0, T]\,.$$

Thus there exists a function such that

$$u^i\to u \quad \text{as} \quad i\to\infty \quad \text{in} \quad C([0, T]: H)\,. \tag{2.15}$$

By Lemma 1.1 we get the next iniquality for any $0\leqq t_1\leqq t_2\leqq T$.

$$\begin{aligned}&\int_{t_1}^{t_2}(c(t)\varphi+I_{\bar{D}})v(t)\,dt-\int_{t1}^{t_2}(c(t)\varphi+I_{\bar{D}})u^i(t)\,dt\\ &\geqq\int_{t_1}^{t_2}(f^i(t)-\frac{dv}{dt}(t),\, v(t)-u^i(t))\,dt\\ &+\frac{1}{2}|v(t_2)-u^i(t_2)|^2-\frac{1}{2}|v(t_1)-u^i(t_1)|^2\end{aligned}$$

where $v\in W^{1,1}(t_1, t_2: H)$ such that $v(t)\in\bar{D}$ on $[t_1, t_2]$ and $(c(t)\varphi+I_{\bar{D}})v(t)\in L^1(0, T)$.

Using (2.13) and (2.15), we can complete the proof.

Next we will get the uniqueness of weak solutions by establishing the following Lemma.

Lemma 2.3. *Let u and $v\in C([0, T]: H)$ be two weak solution of $E(f, u(0))$ and $E(g, v(0))$ respectively. Then we get the same inequalities as* (1.3) *and* (1.4).

Proof. Fix $z\in\bar{D}$, we define the operator K_z from $L^1(0, T: H)$ to $C([0, T]: H)$ such that K_zf is the weak solution of $E(f, z)$ which is actually constructed in Lemma 2.2. Then we know that K_z is a singlevalued operator and $D(K_z)=L^1(0, T: H)$. Let $u_1=K_zf_1$, $u_2=K_zf_2$. By the proof of Lemma 2.2, there exist $\{u_1^i\}_{i\geqq1}$ and $\{u_2^i\}_{i\geqq1}$ such that u_p^i is a piecewises-weak solution of $E(f_p^i, z)$ $p=1, 2$. By Lemma 1.2 we get

$$|u_1^i(t)-u_2^i(t)|\leqq\int_u^T|f_1^i(x)-f_2^i(x)|\,dx\,. \tag{2.16}$$

and so

$$|u_1(t)-u_2(t)| \leqq \int_0^T |f_1(t)-f_2(t)|\,dt$$

for any $t\in[0, T]$. Hence K_z is a continuous operator from $L^1(0, T: H)$ to $C([0, T]: H)$. For $f\in L^1(0, T; H)$, we denote by $M_z f$ the set of the weak solutions of $E(f, z)$. Then we get

$$D(M_z) = L^1(0, T: H)\,, \; K_z \subset M_z\,.$$

Next, let $[f_1, u_1]\in M_z$ and $[f, u]\in K_z$. Since $f\in L^1(0, T: H)$, there exists $\{f_k\}_{k\geqq 1}\subset L^2(0, T: H)$ such that

$$f_k \to f \quad \text{as} \quad k\to\infty \quad \text{in} \quad L^1(0, T: H)\,.$$

Let f^i be the function defined by (2.12) and f_k^i be the function defined similaly with f_k in place of f. There exists $\{u^i\}_{i\geqq 1}$ such that u^i is a piecewise s-weak solution of $E(f^i, z)$ and

$$u^i \to u \quad \text{as} \quad i\to\infty \quad \text{in} \quad C([0, T]: H)$$

by the proof of Lemma 2.2. And there exists $\{u_k^i\}_{i\geqq 1,\, k\geqq 1}$ such that u_k^i is a piecewise strong solution of $E(f_k^i, z)$ on $0=t_0^{i,k}<t_1^{i,k}<\cdots<t_{N_{i,k}}^{i,k}=T$. Consequently by Lemma 1.2, we have for any $t\in[0, T]$

$$|u_k^i(t)-u^i(t)| \leqq \int_0^T |f^i(t)-f_k^i(t)|\,dt\,.$$

So, $u_k^i\to u^i$ as $k\to\infty$ in $C([0, T]: H)$.

For fixed i,k, we take a positive constant p such that

$$0<p<2^{-1}\cdot\min\,\{t_j^{i,k}-t_{j-1}^{i,k};\, j=1, 2, \cdots, N_{i,k}\}\,.$$

For simplicity, we write t_j instead of $t_j^{i,k}$. Thus we know

$$\int_{t_j+p}^{t_{j+1}-p}(c(t)\varphi+I_{\bar{D}})u_1(t)\,dt-\int_{t_j+p}^{t_{j+1}-p}(c(t)\varphi+I_{\bar{D}})u_k^i(t)\,dt$$
$$\geqq \int_{t_j+p}^{t_{j+1}-p}\Big(f_k^i-\frac{d}{dt}u_k^i,\, u_1-u_k^i\Big)dt\,, \quad j=0, 1, \cdots, N_{i,k}-1\,. \tag{2.17}$$

Since $[f_1, u_1]\in M_z$, we have

$$\int_{t_j+p}^{t_{j+1}-p}(c(x)\varphi+I_{\bar{D}})u_k^i(t)\,dt-\int_{t_j+p}^{t_{j+1}-p}(c(t)\varphi+I_{\bar{D}})u_1(t)\,dt$$
$$\geqq \int_{t_j+p}^{t_{j+1}-p}\Big(f_1-\frac{d}{dt}u_k^i,\, u_k^i-u_1\Big)dx$$

$$+\frac{1}{2}|u_k^i(t_{j+1}-p)-u_1(t_{j+1}-p)|^2-\frac{1}{2}|u_k^i(t_j+p)-u_1(t_j+p)|^2$$
$$\text{for} \quad j=0,1,\cdots,N_{i,k}-1\,. \tag{2.18}$$

Adding (2.17), (2.18) and letting $p\to 0$, we get

$$0\geqq\int_0^T(f_1-f_k^i,\, u_k^i-u_1)dt+\frac{1}{2}|u_k^i(T)-u_1(T)|^2-\frac{1}{2}|u_k^i(0)-u_1(0)|\,.$$

Then we have

$$\int_0^T(f-f_1,\, u-u_1)dt\geqq 0\,, \qquad \text{for any} \quad [f,u]\in K_z,\ [f_1,u_1]\in M\,. \tag{2.19}$$

Let h be an arbitrary element of $L^1(0,T:H)$ and $0<q<1$. Substituting $f=(1-q)f_1+qh$, in (2.19) and using the continuity of K_z, we easily get

$$\int_0^T(h-f_1,\, K_zf_1-u_1)dt\geqq 0\,.$$

So we have $K_zf_1=u_1$. Thus we obtain $K_z=M_z$. Recalling the proof of Lemma 2.2 we can easily show that (1.3) and (1.4) are valid for $[f,u]\in K_{u(0)}$, $[g,v]\in K_{v(0)}$. Thus the proof of the lemma is complete.

3. Proof of Theorem

Lemma 3.1. *For any $f\in L^1(0,T:H)$ and $u_0\in\bar{D}$, there exists a weak solution of* (0,1) *with the initial deta u_0.*

Proof. There exists a sequence $\{f_k\}_{k\geqq 1}\subset L^2(0,T:H)$ such that $f_k\to f$ as $k\to\infty$ in $L^1(0,T:H)$. Let $\{f_k^i\}_{k\geqq 1,\, i\geqq 1}$ be the family of functions defined in the proof of Lemma 2.3. Setting $t_p^n=2^{-n}\cdot p\cdot T$, define

$$\varphi_n^t(u)=\varphi^{t_p^n}(u);\ \text{for}\ t_p^n\leqq t<t_{p+1}^n,\ p=0,1,\cdots,2^n-1\,.$$

Let $\{u_n\}_{n\geqq 1}\subset C([0,T]:H)$ be the functions such that $u_n(t)$; $t_p\leqq t\leqq t_{p+1}$, $p=0,1,\cdots,2^n-1$, is a unique weak solution of

$$\frac{d}{dt}u_n(t)+\partial(c(t)\varphi_n^t+I_{\bar{D}})u_n(t)\ni f(t)$$

$$u_n(t_p^n)=\begin{cases} u_0\,, & \text{if}\quad p=0\,.\\ \text{the value of } u_n(t)\ (t_{p-1}^n\leqq t\leqq t_p^n) & \text{at}\quad t=t_p^n\,,\\ \text{if}\quad p=1,2,\cdots,2^n-1\,. & \end{cases}$$

These functions $\{u_n\}_{n\geqq 1}$ exist by virtue of Lemma 2.2. For simplicity we write $u_n=S_n(f,u_0)$ and similarly define $u_{n,k}=S_n(f_k,u_0)$ and $u_{n,k}^i=S_n(f_k^i,u_0)$. By

Lemma 2.1 $u^i_{n,k}(t)$; $t^n_p \leqq t \leqq t^n_{p+1}$, is a piecewise strong solution of

$$\frac{d}{dt} u^i_{n,k}(t) + \partial(c(t)\varphi^t_n + I_{\bar{D}}) u^i_{n,k} \ni f^i_k(t) .$$

Using Lemma 2.3, we get

$$|u^i_{n,k}(t) - u_{n,k}(t)| \leqq \int_0^T |f^i_k - f_k| dx , \qquad \text{for any} \quad t \in [0, T] .$$

Consequently we find $u^i_{n,k} \to u_{n,k}$ as $i \to \infty$ in $C([0, T]: H)$. We also get $u_{n,k} \to u_n$ as $k \to \infty$ in $C([0, T]: H)$. Taking $v \in D$, we have

$$\begin{aligned} &(c(t)\varphi^t_n + I_{\bar{D}})v - (c(t)\varphi^t_n + I_{\bar{D}}) u^i_{n,k} \\ &\geqq (f^i_k - \frac{d}{dt} u^i_{n,k}, v - u^i_{n,k}) \\ &= (f^i_k, v - u^i_{n,k}) + \frac{1}{2} \frac{d}{dt} |v - u^i_{n,k}|^2, \quad a.e.\ [0, T] , \end{aligned} \tag{3.1}$$

Since $u^i_{n,k}$ is continuous, piecewise absolutely continuous on $[0, T]$ and there exist positive constants a_1 and a_2 such that

$$\varphi^t u \geqq -a_1 |u| - a_2 , \qquad \text{for any} \quad u \in H , t \in [0, T]$$

(see [2]), we get by [1] p. 156 Lemma A.5

$$|u^i_{n,k}(t)| \leqq C_1 , \tag{3.2}$$

where C_1 is a positive constant independent of n, k, i and t.
By (3.1) and (3.2), we have

$$\int_{t_1}^{t_2} (c(s)\varphi^s_n + I_{\bar{D}}) u^i_{n,k} ds \leqq C_2 \tag{3.3}$$

where C_2 is a positive constant independent of n, k, i, t_1, t_2. Since $u^i_{n,k}$, $u^i_{m,k}(m \geqq n)$ are piecewise strong solutions, we have

$$(c(s)\varphi^s_n + I_{\bar{D}}) u^i_{m,k} - (c(s)\varphi^s_n + I_{\bar{D}}) u^i_{n,k} \geqq (f^i_k - \frac{d}{ds} u^i_{n,k}, u^i_{m,k} - u^i_{n,k})$$

$$(c(s)\varphi^s_m + I_{\bar{D}}) u^i_{n,k} - (c(s)\varphi^s_m + I_{\bar{D}}) u^i_{m,k} \geqq (f^i_k - \frac{d}{ds} u^i_{m,k}, u^i_{n,k} - u^i_{m,k})$$

$$a.a \quad [0, T].$$

By adding these two inequalities, we get

$$\begin{aligned} &(c(s)\varphi^s_n + I_{\bar{D}}) u^i_{m,k}(s) - (c(s)\varphi^s_m + I_{\bar{D}}) u^i_{m,k}(s) \\ &\quad + (c(s)\varphi^s_m + I_{\bar{D}}) u^i_{n,k}(s) - (c(s)\varphi^s_n + I_{\bar{D}}) u^i_{n,k}(s) \\ &\geqq \frac{1}{2} \frac{d}{ds} |u^i_{m,k}(s) - u^i_{n,k}(s)|^2 . \end{aligned}$$

Since $u^i_{m,k}$, $u^i_{n,k}\in C([0, T]: H)$ are piecewise absolutely continuous, we have

$$\int_0^T \{(c(s)\varphi_n^s+I_{\bar D})u^i_{m,k}(s)-(c(s)\varphi_m^s+I_{\bar D})u^i_{m,k}(s)\}\, ds$$
$$+\int_0^T \{(c(s)\varphi_m^s+I_{\bar D})u^i_{n,k}(s)-(c(s)\varphi_m^s+I_{\bar D})u^i_{n,k}(s)\}\, ds$$
$$\geqq \frac{1}{2}|u^i_{m,k}(t)-u^i_{n,k}(t)|^2, \qquad \text{for any} \quad t\in[0, T]. \tag{3.4}$$

In view of (3.2), (3.3), (3.4) and (0.2), for any given $\varepsilon>0$ there exists an integer $n_0>0$ such that

$$|u^i_{m,k}(t)-u^i_{n,k}(t)|\leqq\sqrt{\varepsilon}\, C_2 \qquad \text{for any} \quad m\geqq n\geqq n_0,$$

where C_2 is a positive constant independent of m, n, k, i and t.
Then we have

$$|u_m(t)-u_n(t)|\leqq\sqrt{\varepsilon}\, C_3 \qquad \text{for any} \quad m\geqq n\geqq n_0 \quad \text{and} \quad t\in[0, T]. \tag{3.5}$$

Hence there exists $u\in C([0, T]: H)$ such that

$$u_n\to u \quad \text{as} \quad n\to\infty \quad \text{in} \quad C([0, T]: H). \tag{3.6}$$

By (3.2) and (3.3), we also have

$$|u_n(t)|\leqq C_1 \qquad \text{for any} \quad n \quad \text{and} \quad t\in[0, T], \tag{3.7}$$

$$\int_{t_1}^{t_2}(c(s)\varphi_n^s+I_D)u_n(s)\, ds\leqq C_2 \qquad \text{for any} \quad n \quad \text{and} \quad 0\leqq t_1\leqq t_2\leqq T. \tag{3.8}$$

From the construction of u_n, we have

$$u_n(t)\in\bar D \quad \text{on} \quad [0, T], \ (c(t)\varphi_n^t+I_{\bar D})u_n(t)\in L^1(0, T: H)$$

for any $[t_1, t_2]\subset[0, T]$

$$\int_{t_1}^{t_2}(c(s)\phi_n^s+I_{\bar D})v(s)\, ds-\int_{t_1}^{t_2}(c(s)\varphi_n^s+I_D)u_n(s)\, ds$$
$$\geqq\int_{t_1}^{t_2}\left(f(s)-\frac{d}{ds}v(s),\, v(s)-u_n(s)\right) ds$$
$$+\frac{1}{2}|v(t_2)-u_n(t_2)|^2-\frac{1}{2}|v(t_1)-u_n(t_1)|^2$$

for any $v\in W^{1,1}(t_1, t_2: H)$ such that $v(t)\in\bar D$ on $[t_1, t_2]$ and $(c(t)\varphi^t+I_D)v(t)\in L^1(t_1, t_2)$.

Using (3.6), (3.7), (3.8) and (0,2), we prove the existence of a solution.

Now, we get the uniqueness of solutions by the following Lemma.

Lemma 3.2. *Let $f, g \in L^1(0, T: H)$ and u, v be two weak solutions of* (1.1), (1.2) *respectively.*
Then we get the inequalities same as (1.3), (1.4).

Proof. Fix $z \in \bar{D}$, we define the operator K_z from $L^1(0, T: H)$ into $C([0, T]: H)$ such that $K_z f$ is the weak solution of (0. 1) with the initial value z which is actually constructed in Lemma 3.1. We know that K_z is a singlevalued operator and $D(K_z)=L^1(0, T: H)$. By using Lemma 2.3, we see that K_z is a continuous operator from $L^1(0, T: H)$ to $C([0, T]: H)$. For any $f \in L^1(0, T: H)$, we denote by $M_z f$ the set of weak solutions of (0. 1) with the initial deta z. Thus $D(M_z)=L^1(0, T: H)$, $K_z \subset M_z$. Let $[f, u] \in K_z$, $[f_1, u_1] \in M_z$ and $\{u_n\}$, $\{u_{n,k}\}$, $\{u^i_{n,k}\}$ be the sequences defined in the proof of Lemma 3.1 with u_0 replaced by z. We know $u^i_{n,k} \to u_{n,k}$ as $i \to \infty$, $u_{n,k} \to u_n$ as $k \to \infty$ and $u_n \to u$ as $n \to \infty$ in $C([0, T]: H)$. Noting that $u^i_{n,k}$ is a strong solution of

$$\frac{d}{dt} u^i_{n,k}(t) + \partial(c(t)\varphi^i_n + I_{\bar{D}})u^i_{n,k} \ni f^i_k(t),\ u^i_{n,k}(0) = z$$

$$\text{on } t^{n,k,i}_{j-1} \leqq t \leqq t^{n,k,i}_j,\ j = 1, 2, \cdots, N_{n,k,i}\,, \tag{3.9}$$

and following the proof of Lemma 2.3 we get

$$\begin{aligned}
&\int_0^T (c(s)\varphi^s_n + I_{\bar{D}})u_1\,ds - \int_0^T (c(s)\varphi^s + I_{\bar{D}})u_1\,ds \\
&\quad + \int_0^T (c(s)\varphi^s + I_{\bar{D}})u^i_{n,k}\,ds - \int_0^T (c(s)\varphi^s_n + I_{\bar{D}})u^i_{n,k}\,ds \\
&\geqq \int_0^T (f_1 - f^i_k,\ u^i_{n,k} - u_1)\,ds + \frac{1}{2}|u^i_{n,k}(T) - u_1(T)|^2 \\
&\quad - \frac{1}{2}|u^i_{n,k}(0) - u_1(0)|^2\,.
\end{aligned} \tag{3.10}$$

In view of (3.2), (3.3), (3.10) and (0.2), for any $\varepsilon > 0$ we have

$$\begin{aligned}
&\varepsilon \cdot C_4 \cdot \left\{\int_0^T (c(s)\varphi^s + I_{\bar{D}})u_1(s)\,ds + C_5\right\} \\
&\quad + \int_0^T (f^i_k - f_1,\ u^i_{n,k} - u_1)\,ds \geqq 0
\end{aligned}$$

if n is sufficiently large where C_4 and C_5 are positive constants independent of n, k and i. So we have

$$\int_0^T (f - f_1,\ u - u_1)\,ds \geqq 0\,,$$

for any $[f, u] \in K_z$ and $[f_1, u_1] \in M_z$. We can complete the proof by the method used at the last part of Lemma 2.3.

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