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SPHERICAL MEANS ON RIEMANNIAN MANIFOLDS

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1. Let X be a compact Riemannian manifold of dimension n, TX its tangent bundle and SX its unit sphere bundle. Denote by p: $SX \to X$ the canonical projection. Let G_t: $SX \to SX$ $(t \in \mathbf{R})$ be the geodesic flow.

The spherical mean (of radius t) L_t: $C^\infty(X) \to C^\infty(X)$ is defined by the following commutative diagram:

$$\begin{tikzcd} C^\infty(X) \arrow[r, "L_t"] \arrow[d, "p^*"'] & C^\infty(X) \\ C^\infty(SX) \arrow[r, "G_t^*"] & C^\infty(SX) \arrow[u, "p_!"'] \end{tikzcd}$$

Here p^* and G_t^* denote the maps induced, respectively, by p and G_t, and $p_!$ is the fibre integral defined by

$$p_! f(x) = \int_{p^{-1}x} f \omega_F\,, \qquad f \in C^\infty(SX)\,,$$

ω_F being the volume element on the fibre of p defined naturally by the Riemannian metric on X.

In this paper we prove the following

Theorem I. *For sufficiently small positive t, L_t is a Fourier integral operator of order $-\frac{1}{2}(n-1)$, which belongs to the class determined by the conormal bundle $\Lambda \subset T^*(X \times X) \setminus 0$ of $\Delta_t = \{(x, y);\ d(x, y) = t\} \subset X \times X$, d being the metric induced by the Riemannian metric.*

The author would like to express his gratitude to T. Sunada for suggesting the above result.

2. For convenience sake, we consider all the operators as acting on the spaces of half densities. Let $\Omega_{\frac{1}{2}}(X)$ denote the bundle of half densities on X and $C^\infty\Omega_{\frac{1}{2}}(X)$ the space of smooth cross-sections of $\Omega_{\frac{1}{2}}(X)$. The Riemannian metric of X induces canonical densities ω_X and ω_{SX}, respectively, on X and SX, which allow us to identify $C^\infty(X)$ with $C^\infty\Omega_{\frac{1}{2}}(X)$, $C^\infty(SX)$ with $C^\infty\Omega_{\frac{1}{2}}(SX)$, respectively, by the isomorphisms $f\mapsto f\sqrt{\omega_X}$ and $f\mapsto f\sqrt{\omega_{SX}}$, $\sqrt{\omega_X}$ and $\sqrt{\omega_{SX}}$ being the half densities that are the square roots of ω_X and ω_{SX}, respectively. Under these identifications, the operators of §1 are transformed into the operators on the spaces of half densities:

$$\begin{array}{ccc} C^\infty\Omega_{1/2}(X) & \xrightarrow{\tilde{L}_t} & C^\infty\Omega_{1/2}(X) \\ \tilde{p}^* \downarrow & & \uparrow \tilde{p}_! \\ C^\infty\Omega_{1/2}(SX) & \xrightarrow{\tilde{G}_t^*} & C^\infty\Omega_{1/2}(SX) \end{array} .$$

3. Let $K\in\mathcal{D}'(SX\times X, 1\boxtimes\Omega(X))$ be the distribution kernel of $p^*: C^\infty(X)\to C^\infty(SX)$. Here $\Omega(X)$ denotes the bundle of densities on X. We define $\tilde{K}\in\mathcal{D}'(SX\times X, \Omega_{\frac{1}{2}}(SX\times X))$ by

$$\tilde{K}(x, y) = \frac{K(x, y)\sqrt{\omega_{SX}(x)}}{\sqrt{\omega_X(y)}} .$$

Then, obviously, we have

Lemma 1. *The operators $\tilde{p}^*$ and $\tilde{p}_!$ have $\tilde{K}$ and $\tilde{K}'$ as the distribution kernels, respectively. Here $\tilde{K}'\in\mathcal{D}'(X\times SX, \Omega_{\frac{1}{2}}(X\times SX))$ is the distribution corresponding to $\tilde{K}$ under the transposition map $X\times SX\xrightarrow{\approx}SX\times X$.*

Moreover, we have

Lemma 2. *$\tilde{K}\in I^{-\frac{1}{4}(n-1)}(SX\times X, \Lambda)$, where $\Lambda\subset T^*(SX\times X)\backslash 0$ is the conormal bundle of the graph of p, that is, $\Lambda=\{(x, p^*\eta)\times(px, -\eta);\ x\in SX, \eta\in T^*_{px}X\backslash 0\}$.*

REMARK 1. As to the notation $I^m(X, \Lambda)=I^m_1(X, \Lambda)$, see [3].

REMARK 2. We denote a point e of a bundle $p: E\to B$ by (x, e), where $x=pe$.

This lemma follows from the following

Lemma 3. *Let M and N be manifolds of dimension m and n, respectively. Let $g: M\to N$ be a smooth mapping. Fixing non-vanishing half densities on M and N, we get $\tilde{g}^*: C^\infty\Omega_{\frac{1}{2}}(N)\to C^\infty\Omega_{\frac{1}{2}}(M)$ induced by g. Then*

$$\tilde{g}^*\in I^{\frac{1}{4}(n-m)}(M\times N, \Lambda_g) ,$$

where $\Lambda_g=\{(x, g^*\eta)\times(gx, -\eta);\ x\in M, \eta\in T^*_{gx}N\backslash 0\}$.

(See, for example, [1].)

This lemma also implies the following

Lemma 4. *$\tilde{G}_t^*\in I^0(SX\times SX, \Lambda_t)$, where $\Lambda_t=\{(x, G_t^*\xi)\times(G_tx, -\xi);\ x\in SX, \xi\in T^*_{G_tx}SX\backslash 0\}$.*

4. Now we quote a theorem concerning the composition of Fourier integral operators.

Let X and Y be manifolds. For any subset $\Lambda\subset T^*X\times T^*Y$, we define $\Lambda'=\{(x,\xi)\times(y,-\eta);\ (x,\xi)\times(y,\eta)\in\Lambda\}\subset T^*X\times T^*Y$. When $\Lambda\subset(T^*X\backslash 0)\times(T^*Y\backslash 0)$ is a conic Lagrangean submanifold, Λ' is nothing but a homogeneous canonical relation from T^*Y to T^*X in the sense of [3].

Theorem A ([3] Theorem 4.2.2). *Let X, Y and Z be smooth manifolds. Let $C_1\subset T^*X\times T^*Y$, $C_2\subset T^*Y\times T^*Z$ be homogeneous canonical relations which satisfy the following conditions:*

i) *$C_1\times C_2$ and $T^*X\times\Delta(T^*Y)\times T^*Z$ intersect transversally, $\Delta(T^*Y)$ being the diagonal of $T^*Y\times T^*Y$;*

ii) *the restriction π of the projection $T^*X\times T^*Y\times T^*Y\times T^*Z\to T^*X\times T^*Z$ to $C_1\times C_2\cap T^*X\times\Delta(T^*Y)\times T^*Z$ is injective and proper.*

Denote the image of π by $C_1\circ C_2$.

Then $C_1\circ C_2$ is a homogeneous canonical relation from T^*Z to T^*X. Moreover, for any $A_1\in I_\rho^{m_1}(X\times Y, C_1')$, $A_2\in I_\rho^{m_2}(Y\times Z, C_2')$, which are properly supported and $\rho>\frac{1}{2}$, we have

$$A_1\circ A_2\in I_\rho^{m_1+m_2}(X\times Z;\ (C_1\circ C_2)')\,.$$

Lemma 5. *$\tilde{G}_t^*\circ\tilde{p}^*\in I^{-\frac{1}{4}(n-1)}(SX\times X,\ C_2')$, where $C_2=\{(x,\ G_t^*p^*\eta)\times(pG_tx,\ \eta);\ x\in SX,\ \eta\in T^*_{pG_tx}X\backslash 0\}$.*

Proof. Λ_t' and Λ_p' are obviously homogeneous canonical relations, respectively, from T^*SX to T^*SX and from T^*X to T^*SX. The following sublemma shows that the conditions of Theorem A are satisfied in this case.

Sublemma. *Let X, Y and Z be manifolds, $g\colon X\to Y$ a diffeomorphism and $C\subset Y\times Z$ a submanifold. Denote by $C_g\subset X\times Y$ the graph of g. Then*

i) *$C_g\times C$ and $X\times\Delta(Y)\times Z$ intersect transversally;*

ii) *$p\colon C_g\times C\cap X\times\Delta(Y)\times Z\to C_g\circ C$ is a homeomorphism, p being the restriction of the projection $X\times Y\times Y\times Z\to X\times Z$.*

Proof. First, we show the assertion i). Fix $(x_0, y_0, y_0, z_0)\in C_g\times C\cap X\times\Delta(Y)\times Z$ and let $x=(x^i)$, $y=(y^j)$ and $z=(z^k)$ be local charts around x_0, y_0 and z_0, respectively. Since C_g has the parametrization $y\mapsto(g^{-1}y, y)$, $y\circ\pi$ gives a local chart of C_g around (x_0, y_0), where $\pi\colon X\times Y\to Y$ is the natural projection. Define $y_1=y\circ\pi_2$ and $y_2=y\circ\pi_3$, where π_j denotes the projection of $X\times Y\times Y\times Z$ on the j-th factor. In order to prove the assertion i), it suffices to show that the local equations $y_1^j-y_2^j=0$ of $X\times\Delta(Y)\times Z$ in $X\times Y\times Y\times Z$ restricted to $C_g\times C$ are independent near (x_0, y_0, y_0, z_0). But this follows trivially from the fact that the differentials dy^j are independent on C_g.

The assertion ii) follows from the fact that p has the inverse: $(x, z)\mapsto(x, gx, gx, z)$. Q.E.D.

This completes the proof of Lemma 5.

5. Let t_0 be a positive real number which satisfies the following conditions:

a) for each $x\in X$, the exponential mapping $T_xX\to X$ maps $\{\xi\in T_xX; \|\xi\|<3t_0\}$ diffeomorphically into X, whose image is denoted by $B(x; 3t_0)$;

b) for all $y, z\in B(x; 3t_0)$, there is a unique geodesic curve in $B(x; 3t_0)$ joining y and z.

Since X is compact, such t_0 exists. (cf [2].)

Theorem I′. *For $0<t\leq t_0$, we have $\tilde{L}_t\in I^{-\frac{1}{4}(n-1)}(X\times X, C')$. Here C' is the conormal bundle of Δ_t minus the zero section.*

Note that Δ_t is a submanifold of $X\times X$, since $t\leq t_0$.

Proof. We shall apply Theorem A in the situation where $X=X$, $Y=SX$, $Z=X$, $C_1=\{(px, \eta)\times(x, p^*\eta); x\in SX, \eta\in T^*_{px}X\backslash 0\}$, $C_2=\{(x, G^*_tp^*\eta)\times(pG_tx, \eta); x\in SX, \eta\in T^*_{pG_tx}X\backslash 0\}$, $A_1=\tilde{g}_!\in I^{-\frac{1}{4}(n-1)}(X\times SX, C_1')$, $A_2=\tilde{G}^*_t\circ\tilde{p}^*\in I^{-\frac{1}{4}(n-1)}(SX\times X, C_2')$.

1) First we determine the set $C=C_1\circ C_2$. Let $(x, \xi)\times(y, \eta)\in C$. By definition, there is a point $(z, \zeta)\in T^*SX$ such that $pz=x$, $pG_tz=y$, $p^*\xi=\zeta=G^*_tp^*\eta$. From $p^*\xi=G^*_tp^*\eta$, it follows that $\langle G^*_tp^*\eta, \delta x\rangle=\langle p^*\xi, \delta x\rangle=\langle\xi, p_*\delta x\rangle=0$ for any $\delta x\in T_z(p^{-1}x)$. Hence $\langle\eta, (p\, G_t)_*\delta x\rangle=0$ for all $\delta x\in T_z(p^{-1}x)$. Since $t\leq t_0$, $(pG_t)_*\,|\,T_z(p^{-1}x)$ is injective, whence $(pG_t)_*(T_z(p^{-1}x))$ is a hyperplane of T_yX. Denote by $\hat{\eta}$, the element of T_yX corresponding to η under the isomorphism $T_yX\cong T^*_yX$ defined by the Riemannian metric. Then $\hat{\eta}$ is orthogonal to the hyperplane $(p\, G_t)_*(T_z(p^{-1}x))$, in other words, $\hat{\eta}$ is a normal vector at y of the geodesic sphere $pG_t(p^{-1}x)$ of radius t with center x. $G_tz\in S_yX\subset T_yX$ being also a normal vector of this geodesic sphere, we have $\hat{\eta}=cG_tz$ for some $c\in\mathbf{R}-\{0\}$. Starting from $G^*_{-t}p^*\xi=p^*\eta$, we can argue in the same way to show that $\hat{\xi}=c'z$ for some $c'\in\mathbf{R}-\{0\}$. Now we shall show $c=c'$. Let V be the vector field on SX which generates the flow G_t. Recall that $p_*V(z)=z$ $(z\in SX)$. Then

$$\begin{aligned} c &= (G_tz, cG_tz)\\ &= \langle G_tz, \eta\rangle\\ &= \langle V(G_tz), p^*\eta\rangle\\ &= \langle G_{t*}V(z), p^*\eta\rangle\\ &= \langle V(z), G^*_tp^*\eta\rangle\\ &= \langle V(z), p^*\xi\rangle\\ &= \langle z, \xi\rangle \end{aligned}$$

$$= (z, c'z) = c' .$$

Hence if c is positive, then $\hat{\eta}=cG_t z=\tilde{G}_t(cz)=\tilde{G}_t(\hat{\xi})$. Here $\tilde{G}_t$: $TX\backslash 0 \to TX\backslash 0$ is the map defined by

$$\tilde{G}_t(\xi) = ||\xi|| G_t\left(\frac{\xi}{||\xi||}\right).$$

If c is negative, then $\hat{\eta}=(-c)(-G_t z)=(-c)G_{-t}(-z)=\tilde{G}_{-t}(cz)=\tilde{G}_{-t}(\hat{\xi})$. Thus $C=\Gamma_t \cup \Gamma_{-t}$, where Γ_t is the graph of the diffeomorphism of $T^*X\backslash 0$ which is induced from $\tilde{G}_t$ by the usual isomorphism: $T^*X \overset{\approx}{\rightarrow} TX$. Note that $\Gamma_t \cap \Gamma_{-t}=\phi$, since $t \le t_0$.

2) Next we show that the condition i) of Theorem A is satisfied in the present case. Fix $P_0=(x_0, \xi_0)\times(z_0, \zeta_0)\times(z_0, \zeta_0)\times(y_0, \eta_0)\in C_1\times C_2 \cap T^*X\times \Delta(T^*SX)\times T^*X$. Let $x=(x^i)$, $y=(y^i)$ be local charts of X around x_0 and y_0, respectively, and (x, ξ), (y, η) the local charts of T^*X induced by them. Furthermore let $z=(z^\kappa)$ be a local chart of SX around z_0 and (z, ζ) the local chart of T^*SX induced by z. We denote the functions $x\circ\pi_1$, $\xi\circ\pi_1$, $z\circ\pi_2$, $\zeta\circ\pi_2$, $z\circ\pi_3$, $\zeta\circ\pi_3$, $y\circ\pi_4$, $\eta\circ\pi_4$, respectively, by x, ξ, z_1, ζ_1, z_2, ζ_2, y, η, π_j being the natural projection of $W=T^*X\times T^*SX\times T^*SX\times T^*X$ onto the j-th factor. Since $C_1\times C_2$ has a local parametrization: $(z_1, \xi, z_2, \eta)\mapsto(pz_1, \xi)\times(z_1, p^*\xi)\times(z_2, G_t^*p^*\eta)\times(pG_t z_2, \eta)$, we can take (z_1, ξ, z_2, η) as a local chart of $C_1\times C_2$ around P_0.

Now the local equations of $T^*X\times\Delta(T^*SX)\times T^*X$ in W is given by

$$\begin{cases} z_1^\kappa - z_2^\kappa = 0 & 1\le\kappa\le 2n-1 \\ \zeta_1^\kappa - \zeta_2^\kappa = 0 & 1\le\kappa\le 2n-1 . \end{cases}$$

In order to verify the condition i), it suffices to show that these $2(2n-1)$ equations remain independent after restricted to $C_1\times C_2$. Obviously $dz_1^\kappa - dz_2^\kappa (1\le\kappa\le 2n-1)$ are linearly independent on $C_1\times C_2$ at P_0. Thus it suffices to see that $d(\zeta_1^\kappa-\zeta_2^\kappa)=d((p^*\xi)^\kappa-(G_t^*p^*\eta)^\kappa)$ $(1\le\kappa\le 2n-1)$ are linearly independent modulo dz_1^κ, $dz_2^\kappa(1\le\kappa\le 2n-1)$. We can write locally $(p^*\xi)^\kappa=\sum_j a_j^\kappa(z_1)\xi^j$, $(G_t^*p^*\eta)^\kappa=\sum_j b_j^\kappa(z_2)\eta^j$. Then

$$\begin{aligned} & d((p^*\xi)^\kappa-(G_t^*p^*\eta)^\kappa) \\ &= d(\sum_j (a_j^\kappa(z_1)\xi^j - b_j^\kappa(z_2)\eta^j)) \\ &\equiv \sum_j (a_j^\kappa(z_1)d\xi^j - b_j^\kappa(z_2)d\eta^j) \quad (\text{modulo } dz_1, dz_2). \end{aligned}$$

Hence $d((p^*\xi)^\kappa-(G_t^*p^*\eta)^\kappa)$ $(1\le\kappa\le 2n-1)$ are linearly independent modulo dz_1, dz_2 on $C_1\times C_2$ at P_0 if and only if the rank of the matrix $(a_j^\kappa(z_0), b_j^\kappa(z_0))$ is $2n-1$, which is equivalent to say that the dimension of the subspace $U=\{p^*\xi+G_t^*p^*\eta;\ \xi\in T_{x_0}^*X, \eta\in T_{y_0}^*X\}\subset T_{z_0}^*SX$ is $2n-1$. On the other hand, in 1), we have shown that the pair $(\xi, \eta)\in T_{x_0}^*X\times T_{y_0}^*X$ such that $p^*\xi=G_t^*p^*\eta$ in $T_{z_0}^*SX$ is determined by z_0 up to scalar multiplications. This, together with the injectivity of $G_t^*p^*: T_{y_0}^*X \to T_{z_0}^*SX$, implies that the dimension of U equals $2n-1$. Thus the condition i) is verified in the present case.

3) Now we check the condition ii) of Theorem A. For any $(x, \xi)\times(y, \eta) \in C$, we have either $\hat{\xi}=\tilde{G}_t(\hat{\eta})$ or $\hat{\xi}=\tilde{G}_{-t}(\hat{\eta})$. Let $(z, \zeta)\in T^*SX$ be such that $(x, \xi)\times(z, \zeta)\in C_1$ and $(z, \zeta)\times(y, \eta)\in C_2$. Then $\zeta=p^*\xi$, and from 1) it follows immediately that $z=\hat{\xi}/||\hat{\xi}||$ if $\hat{\xi}=\tilde{G}_t(\hat{\eta})$ and $z=-\hat{\xi}/||\hat{\xi}||$ if $\hat{\xi}=\tilde{G}_{-t}(\hat{\eta})$. Thus, in view of $\Gamma_t \cap \Gamma_{-t}=\phi$, $C_1\times C_2 \cap T^*X\times\Delta(T^*SX)\times T^*X \to C_1\circ C_2$ is a diffeomorphism.

4) Finally we show that $\Gamma_t \cup \Gamma_{-t}$ is the conormal bundle minus the zero section of $\Delta_t \subset X\times X$. It is obvious that the projection $T^*(X\times X)\to X\times X$ maps $\Gamma_t \cup \Gamma_{-t}$ onto Δ_t. The fibre of $\Gamma_t \cup \Gamma_{-t} \to \Delta_t$ is easily seen to be $\mathbf{R}-\{0\}$. Since $\Gamma_t \cup \Gamma_{-t}$ is a Lagrangean submanifold of $T^*(X\times X)$, it follows that $\Gamma_t \cup \Gamma_{-t}$ is contained as an open set in the conormal bundle of Δ_t, whence $\Gamma_t \cup \Gamma_{-t}$ must be the conormal bundle of Δ_t minus the zero section.

This completes the proof of Theorem I$'$.

6. Using Theorem I$'$, we get some information about the regularity of $\tilde{L}_t$. We quote the following

Theorem B ([3] Theorem 4.3.2). *Let X and Y be smooth manifolds and $C\subset T^*X\times T^*Y$ a homogeneous canonical relation satisfying the following conditions:*

i) *the projections $C\to X$, $C\to Y$ have surjective differentials;*

ii) *the differentials of the projections $C\to T^*X$ and $C\to T^*Y$ have rank at least $k+\dim X$ and $k+\dim Y$, respectively.*

Let $m\leq \frac{1}{4}(2k-\dim X-\dim Y)$. Then every $A\in I^m(X\times Y, C')$ is a continuous operator: $L_c^2(Y, \Omega_{\frac{1}{2}})\to L_{\mathrm{loc}}^2(X, \Omega_{\frac{1}{2}})$.

From this, it follows immediately the following

Corollary. *Under the assumptions of Theorem B, A is continuous: $H_c^s(Y, \Omega_{\frac{1}{2}}) \to H_{\mathrm{loc}}^{s+r}(X, \Omega_{\frac{1}{2}})$, where $r=-m+\frac{1}{4}(2k-\dim X-\dim Y)$.*

In the case of the operator $\tilde{L}_t$, the condition i) is trivially satisfied and the condition ii) is valid with $k=n$, since Γ_t and Γ_{-t} are the graphs of diffeomorphisms: $T^*X\backslash 0\to T^*X\backslash 0$. Furthermore, $r=\frac{1}{2}(n-1)-\frac{1}{4}(2n-n-n)=\frac{1}{2}(n-1)$. Hence we have

Theorem II. *The spherical mean $\tilde{L}_t(0<t\leq t_0)$ is a continuous operator: $H^s(X, \Omega_{\frac{1}{2}})\to H^{s+\frac{1}{2}(n-1)}(X, \Omega_{\frac{1}{2}})$ for each $s\in\mathbf{R}$.*

Corollary. *If* $n \geq 2$, *then the operator* $\tilde{L}_t$: $L^2(X, \Omega_{\frac{1}{2}}) \rightarrow L^2(X, \Omega_{\frac{1}{2}})$ *is compact for* $0 < t \leq t_0$.

Corollary. *If* $n \geq 2$, *then the eigenfunctions of non-zero eigenvalues of the operator* $\tilde{L}_t$: $L^2(X, \Omega_{\frac{1}{2}}) \rightarrow L^2(X, \Omega_{\frac{1}{2}})$ *are smooth functions, for* $0 < t \leq t_0$.

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