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ON BOUNDED KRULL PRIME RINGS

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In [7], the author defined the concept of non commutative Krull rings on prime Goldie rings by using perfect additive topologies, and gave some properties of such rings.

The main purpose of this paper is to investigate the ideal theory in bounded Krull prime rings (cf. Section 1 for the definition).

In Section 1, it is shown that bounded Krull prime rings are maximal orders in the sense of Asano [1]. Combining this with a result of [3] we shall show that the group of v-ideals of a bounded Krull prime ring becomes a direct product of infinite cyclic subgroups generated by minimal prime ideals in the ring. Further it is established that a bounded Krull prime ring is a Dedekind prime ring if and only if nonzero prime ideals of the ring are maximal.

In Section 2, we shall determine maximal orders equivalent to a fixed bounded Krull prime ring and shall show that such maximal orders are also bounded Krull prime rings (cf. Theorem 2.6).

In Section 3, it is shown that a bounded Krull prime ring with only a finite number of minimal prime ideals is a right and left principal ideal ring. By using this result we shall generalize two theorems on ideals in maximal orders over Krull domains to the case of bounded Krull prime rings.

Section 4 contains some results on Krull orders over commutative integral domains.

This paper is a continuation of [7]. Concerning the notations and terminolgy not defined in this paper we refer to [7].

1. Groups of v-ideals

Throughout this paper R will denote a prime Goldie ring with identity element and Q will denote the two-sided quotient ring of R, where Q is a simple and artinian ring (cf. [5]). Let F be a right additive topology on R. We will denote by R_F the ring of quotients of R with respect to F (cf. [11]). An overring R' of R is said to be *right essential* if it satisfies the following two conditions:

(1) There is a perfect right additive topology F on R such that $R'=R_F$ (cf. p. 74 of [11]).

(2) If $I \in F$, then $R'I = R'$.

If R_F is a right essential overring of R, then F consists of all right ideals I of R such that $IR_F = R_F$. So any element of F is an essential right ideal of R. Hence $R_F = \varinjlim \mathrm{Hom}(I, R)$ $(I \in F)$.

An overring R' of R is said to be *essential* if it is right and left essential. For the convenience of the reader we repeat the definition of Krull ring.

A prime Goldie ring R is said to be a *Krull ring* if there are families $\{R_i\}_{i \in I}$ and $\{S_j\}_{j \in J}$ of essential overrings of R such that

(K1) $R = \cap_i R_i \cap \cap_j S_j$,

(K2) each R_i is a noetherian, local, Asano order, each S_j is a noetherian, simple ring and the cardinal number of $\boldsymbol{J}$ is finite, and

(K3) for every regular element c in R we have $cR_i \neq R_i$ for only finitely many i in $\boldsymbol{I}$ and $R_k c \neq R_k$ for only finitely many k in $\boldsymbol{I}$.

If $\boldsymbol{J} = \phi$, then R is said to be *bounded*.

Lemma 1.1. *Let R be a prime Goldie ring and let $T \supseteq S$ be overrings of R. If T is a right essential overring of R, then it is a right essential overring of S.*

Proof. By assumption, there is a perfect right additive topology F_0 such that $T = R_{F_0}$. We put $F = \{I \mid IT = T,\ I \text{ is a right ideal of } S\}$. First we shall prove that a right ideal I of S is an element in F if and only if $I \cap R \in F_0$. If $I \cap R \in F_0$, then it is evident that $IT = T$. Conversely assume that $IT = T$. Write $1 = \sum x_i y_i$, where $x_i \in I$ and $y_i \in T$. There is an element $I_0 \in F_0$ such that $y_i I_0 \subseteq R$. Hence $I_0 \subseteq I \cap R$ and $I \cap R \in F_0$. Next we shall prove that F is a perfect right additive topology on S. (i) If $I \in F$ and $s \in S$, then we must prove that $s^{-1}I = \{x \in S \mid sx \in I\} \in F$. Since $IT = T$ and $R \to T$ is a flat epimorphism (cf. Theorem 13.10 of [11]), we obtain easily that $S/I \otimes_R T = 0$. So $(s^{-1}I)T = T$, because $S/s^{-1}I \cong (sS + I)/I$. Therefore $s^{-1}I \in F$. (ii) If $I \in F$ and J is a right ideal of S such that $a^{-1}J \in F$ for all $a \in I$, then we have

$$T \supseteq JT \supseteq \sum_{a \in I} a(a^{-1}J)T = \sum aT = IT = T .$$

Hence $J \in F$ so that F is a right additive topology on S.

If $I_0 \in F_0$, then $I_0 S \in F$ and so $S_F \supseteq T$. Conversely let x be any element of S_F. Then there exists $I \in F$ such that $xI \subseteq S$. So $x \in xT = xIT \subseteq T$. Therefore $T = S_F$. Thus F is perfect by Theorem 13.1 of [11]. For any $I \in F$, we have $T = T(R \cap I)$, because $R \cap I \in F_0$. Hence T is a right essential overring of S.

From Lemma 1.1, we have

Proposition 1.2. *If $R = \cap R_i \cap \cap S_j$ $(i \in \boldsymbol{I}, j \in \boldsymbol{J})$ is a Krull prime ring and if $\boldsymbol{I}_0$, $\boldsymbol{J}_0$ are subsets of $\boldsymbol{I}$, $\boldsymbol{J}$ respectively, then $S = \cap R_i \cap \cap S_j$ $(i \in \boldsymbol{I}_0, j \in \boldsymbol{J}_0)$ is Krull.*

In particular, if $J_0=\phi$, then S is bounded.

Let $S\subseteqq T$ be rings. Then S is an *order* in T if T is the two-sided quotient ring of it. If R_1 and R_2 are orders in Q, then they are *equivalent* if there exist regular elements a_1, b_1, a_2, b_2 of Q such that $a_1R_1b_1\subseteqq R_2$, $a_2R_2b_2\subseteqq R_1$. An order R in Q is said to be *maximal* when it is a maximal element in the set of orders which are equivalent to R. If I is a right (left) R-ideal of Q, then $O_l(I)=\{x\in Q\mid xI\subseteqq I\}$ is an order in Q and is equivalent to R. Similarly $O_r(I)=\{x\in Q\mid Ix\subseteqq I\}$ is an order in Q and is equivalent to R. They are called a *left order* and a *right order* of I respectively. We define the *inverse of* I to be $I^{-1}=\{q\in Q\mid IqI\subseteqq I\}$. Evidently $I^{-1}=\{q\in Q\mid Iq\subseteqq O_l(I)\}=\{q\in Q\mid qI\subseteqq O_r(I)\}$. A prime Goldie ring is said to be *Dedekind* if it is a maximal order, and is a right and left hereditary.

Proposition 1.3. *Let T be a prime Goldie ring with quotient ring Q. If $T=\cap T_i$, where T_i are essential overrings of T and are Dedekind prime rings, then T is a maximal order in Q.*

Proof. By Satz 1.2 of [1], it is enough to prove that $O_l(A)=T=O_r(A)$ for every T-ideal A of Q. It is clear that $O_l(A)\supseteqq T$. Conversely let x be any element in $O_l(A)$. Then $xA\subseteqq A$ so that $xA(T_iA)^{-1}\subseteqq(T_iA)\,(T_iA)^{-1}=T_i$, because T_i is a Dedekind prime ring. Write $1=\sum x_jy_j$, where $x_j\in T_iA$ and $y_j\in(T_iA)^{-1}$. Since $T_i=T_{F_{i_l}}$, where F_{i_l} is a perfect left additive topology on T, we have $Jx_j\subseteqq A$ for some $J\in F_{i_l}$. Hence $xJ\subseteqq(xA)(T_iA)^{-1}\subseteqq A(T_iA)^{-1}\subseteqq T_i$ and thus $x\in xT_i=xJT_i\subseteqq T_i$, because T_i is an essential overring of T. Therefore $x\in\cap T_i=T$ so that $T=O_l(A)$. Similarly $T=O_r(A)$.

Corollary 1.4. *If R is a bounded Krull prime ring, then it is a maximal order in Q.*

In [7], we defined the concepts of w-operations and v-operations on one-sided R-ideals (cf. §4 of [7]). Let R be a maximal order in Q and let I be a (right) R-ideal. Then $I^{-1}=(R:I)_l=\{q\in Q\mid qI\subseteqq R\}$ so that $I_v=(I^{-1})^{-1}$.

If $I=I_v$, then it is said to be a *(ritht) v-ideal.* For any right R-ideal I, we note that I^{-1} is a left v-ideal and that $(I^{-1}I)_v=R$ (cf. Proposition 4.1 and Corollary 4.2 of [7]). In particular, the set $D(R)$ of all v-ideals becomes an abelian group under the multiplication "$\circ$" defined by $A\circ B=(AB)_v$ for any v-ideals A and B (cf. Theorem 4.2 of [3]).

Let A be an integral R-ideal. We will denote by $C(A)$ those elements of R which are regular in R/A. If R satisfies the Ore condition with respect to $C(A)$, then we will denote by R_A the ring of quotients of R with respect to $C(A)$.

Let $R=\cap R_i(i\in I)$ be a bounded Krull prime ring. Throughout this paper P_i' will denote a unique maximal ideal of R_i and $P_i=P_i'\cap R$. By Proposition 1.1 of [7], P_i is a prime ideal of R and $R_i=R_{P_i}$. Note that $BR_{P_i}=R_{P_i}$ for any

integral R-ideal $B \subsetneqq P_i$ by Proposition 1.1 of [7] and Goldie's theorem. This fact is frequently used in this paper. We will denote by $\boldsymbol{P}$ the set $\{P_i \mid i \in I\}$.

Lemma 1.5. *Let R be a bounded Krull prime ring and let $P \in \boldsymbol{P}$. Then*

(1) $R \subsetneqq P^{-1}$.

(2) $P^{-1}PR_P = R_P = R_P P^{-1}P$ *and* $PP^{-1}R_P = R_P = R_P PP^{-1}$.

(3) *P is a v-ideal.*

Proof. (1) By Proposition 1.1 of [7], $P = PR_P \cap R$. So it follows that P is a w-ideal i.e., $P = \cap PR_{P_i} (P_i \in \boldsymbol{P})$. Since R_P is a principal right and left ideal ring, we have $PR_P = pR_P$ for some $p \in P$. On the other hand, since the integral right w-ideals satisfy the maximum condition, there are a finite number of elements $x_1, \cdots, x_n$ in P such that $P = (x_1R + \cdots + x_nR)_w$. Write $x_i = pb_i$, where $b_i \in R_P$, and $b_i = c^{-1}r_i$, where $r_i \in R$, $c \in C(P)$. Then we have $P = [pc^{-1}(r_1R + \cdots + r_nR)]_w = pc^{-1}(r_1R + \cdots + r_nR)_w \subseteqq pc^{-1}R$. Hence $cp^{-1}P \subseteqq R$ and $cp^{-1} \in P^{-1}$. If $cp^{-1} \in R = O_l(P)$, then $cp^{-1}P \subseteqq P$ and $cp^{-1}PR_P \subseteqq PR_P$. Since PR_P is invertible, $cp^{-1} \in R_P$ and so $p^{-1} \in R_P$. Hence $1 = pp^{-1} \in P'$, a contradiction. Thus we have $P^{-1} \supsetneqq R$.

(2) If $P^{-1}P = P$, then $P^{-1} = R$, a contradiction, since $D(R)$ is a group. Hence $P^{-1}P \supsetneqq P$ and so $P^{-1}PR_P = R_P$. The other cases are similar.

(3) By Propositon 1.1 of [7], $PR_P = R_P P$ and so $R_P P^{-1} = P^{-1}R_P$ by (2). So again by $P_v R_P = R_P P_v$, because $P_v \supseteqq P$, so that $P_v R_P$ is an ideal of R_P. Hence $PR_P = P_v R_P$ or $P_v R_P = R_P$. But if $R_P = P_v R_P$, then $R_P = PP^{-1}R_P = PP^{-1}P_v R_P = PR_P$, a contradiction, because $R_P = P^{-1}PR_P = P^{-1}P_v R_P = R_P$. Hence $PR_P = P_v R_P$ and so $P = PR_P \cap R \supseteqq P_v$. Therefore $P = P_v$, as desired.

Lemma 1.6. *Let R be a bounded Krull prime ring. If I is an integral right R-ideal, then I contains a intersection of powers of a finite number of elements in $\boldsymbol{P}$.*

Proof. Let c be a regular element in I. Then $cR = \cap cR_{P_i} (P_i \in \boldsymbol{P})$, and $cR_{P_i} \subsetneqq R_{P_i}$ for finitely many P_i in $\boldsymbol{P}$ only $(1 \leqq i \leqq k)$, Since R_{P_i} is regular in the sense of [1] and the ideals of R_{P_i} are only the powers of P_i'. Hence $cR_{P_i} \supseteqq P_i'^{n_i}$ for some positive integer n_i and so we have $I \supseteqq cR = \cap cR_P (P \in \boldsymbol{P}) \supseteqq P_1^{n_1} \cap \cdots \cap P_k^{n_k}$.

Proposition 1.7 *Let R be a bounded Krull prime ring and let P be a non-zero prime ideal of R. Then P is minimal prime if and only if $P \in \boldsymbol{P}$.*

Proof. If P is minimal prime, then it is evident that $P \in \boldsymbol{P}$ by Lemma 1.6. Conversely assume that $P \in \boldsymbol{P}$. Then it is a prime element in the commutative lattice ordered group $D(R)$ and is a maximal element in $D(R)$ by a rseult in [3, *p.* 11]. Combining this with Lemma 1.6, we have that P is a minimal prime ideal of R.

Theorem 1.8. *Let R be a bounded Krull prime ring. Then*

(1) *$D(R)$ is an abelian group and is a direct product of infinite cyclic subgroups generated by minimal prime ideals of R.*

(2) *$D(R)\cong \prod F(R_P)$, where P ranges over all minimal prime ideals of R and $F(R_P)$ denotes the group of all fractional R_P-ideals.*

Proof. (1) If $P\in \boldsymbol{P}$, then it is a prime element in $D(R)$. Conversely if P_0 is a prime element in $D(R)$, then it is a prime ideal of R and hence $P_0\supseteqq P$ for some P in $\boldsymbol{P}$. But P is a maximal element in $D(R)$ and so $P=P_0$. Hence the assertion follows from Theorem 1.2 of [3] and Proposition 1.7.
(2) is evident.

In §2 of [7] we considered the following condition on bounded Krull prime rings;

(K4): $P_i \not\supseteqq P_j$ and $P_i \not\subseteq P_j$ for any $P_i, P_j\in \boldsymbol{P}$ and $P_i \neq P_j$.

We know from Proposition 1.7 that bounded Krull prime rings satisfy the condition (K4).

Lemma 1.9. *Let R be a bounded Krull prime ring and let I be any right R-ideal. Then*

(1) $I^{-1} = \cap\, (IR_P)^{-1}(P\in \boldsymbol{P})$.

(2) $R_P I^{-1} = (IR_P)^{-1}$ *for any P in $\boldsymbol{P}$.*

Proof. (1) is evident. (2) follows from Lemma 2.1 of [7] and (1).

Proposition 1.10. *Let R be a bounded Krull prime ring. If I is a right R-ideal, then $I_v=I_w$.*

Proof. $I_v=(I^{-1})^{-1}=\cap (R_P I^{-1})^{-1}=\cap [(IR_P)^{-1}]^{-1}=\cap IR_P=I_w$ by Lemma 1.9.

Corollary 1.11. *Let R be a bounded Krull prime ring. If A is an R-ideal, then AR_P is an R_P-ideal for every P in $\boldsymbol{P}$.*

Proof. $A_w=(P^n P_1^{n_1}\cdots P_k^{n_k})_w$ by Theorem 1.8 and Proposition 1.10. Hence we have $AR_P=A_w R_P=(P^n P_1^{n_1}\cdots P_k^{n_k})_w R_P=P^n P_1^{n_1}\cdots P_k^{n_k} R_P=P^n R_P$. So, by Proposition 1.1 of [7], AR_P is an R_P-ideal.

Lemma 1.12. *Let R be a bounded Krull prime ring and let $P\in \boldsymbol{P}$. Then $R_P=\varinjlim B^{-1}$, where B ranges over integral R-ideals such that $B\not\subseteq P$.*

Proof. If B is an integral R-ideal not contained in P, then it is clear that $B\cap C(P)\neq \phi$ (cf. Proposition 1.1 of [7]). Hence $R_P\supseteqq \varinjlim B^{-1}$. Conversely let

c be any element in $C(P)$. Then $cR \supseteq P^n P_1^{n_1} \cdots P_k^{n_k}$, where $P_i \in \boldsymbol{P}$ and n, n_i are integers. So $R_P = cR_P = P^n R_P$. Thus $n=0$ so that $cR \supseteq P_1^{n_1} \cdots P_k^{n_k}$. Hence $R_P \subseteq \varinjlim B^{-1}$ and so $R_P = \varinjlim B^{-1}$.

Theorem 1.13. *Let R be a bounded Krull prime ring. Then R is a Dedekind prime ring if and only if any nonzero prime ideal is maximal.*

Proof. If R is a Dedekind prime ring, then the result is known [1]. Conversely, if any nonzero prime ideal is maximal, then the elements in $\boldsymbol{P}$ are only maximal ideals of R by Lemma 1.6. First we shall prove that maximal ideals of R are invertible. Let P be any maximal ideal of R. Then $R = P^{-1}P$ or $P^{-1}P = P$. If $P^{-1}P = P$, then $PR_P = R_P$ by Lemma 1.5, a contradiction. Hence $P^{-1}P = R$. Similarly $PP^{-1} = R$. Next we shall prove that R is a right and left noetherian ring. To prove this let I be an integral right R-ideal such that $I \supseteq P$ and $\bar{I}\bar{R}_P = \bar{R}_P$, where $\bar{I} = I/P$ and $\bar{R}_P = R_P/P'$. This implies that $IR_P + P' = R_P$. Since P' is the Jacobson radical of R_P, we have $IR_P = R_P$. Write $1 = \sum x_i y_i$, where $x_i \in I$ and $y_i \in R_P$. By Lemma 1.12, there exists an integral R-ideal B such that $B \nsubseteq P$ and $y_i B \subseteq R$. So $B \subseteq I$ and $R = B + P \subseteq I$. Hence $R = I$. This implies that R/P is an artinian ring by Proposition 1.1 of [7] and Goldie's theorem, and so R/P^n is also an artinian ring, because P is invertible. For any finite members of elements $P_1, \cdots, P_k$ in $\boldsymbol{P}$ and any positive integers $n_1, \cdots, n_k$ we have $R/(P_1^{n_1} \cap \cdots \cap P_k^{n_k}) \cong R/P_1^{n_1} \oplus \cdots \oplus R/P_k^{n_k}$. So, by Lemma 1.6, the integral right R-ideals satisfy the maximum condition. Therefore R is right noetherian, because R has a finite dimension in the sense of Goldie. Similarly R is left noetherian. Now, since any maximal ideal is invertible and R is noetherian, we obtain that R is an Asano order (see the proof of Theorem 2.6 of [6]). Further, since R is bounded, it is a Dedekind prime ring by Theorem 3.5 of [8].

2. Maximal orders equivalent to a bounded Krull prime ring

In this section we shall prove that any maximal order equivalent to a bounded Krull prime ring is also a bounded Krull prime ring. For this we need some Lemmas.

Lemma 2.1. *Let R be a bounded Krull prime ring, let I be a right R-ideal and let $P \in \boldsymbol{P}$. Then*

(1) $O_l(IR_P) = IR_P I^{-1}$.

(2) $I^{-1}IR_P = R_P = R_P I^{-1} I$.

(3) $IR_P = O_l(IR_P)I$.

Proof. (1) From Lemma 1.9 we have $IR_P I^{-1} = IR_P R_P I^{-1} = IR_P (IR_P)^{-1} = O_l(IR_P)$.

(2) Since $(I^{-1}I)_v=R$, we have $R_P=(I^{-1}I)_vR_P=(I^{-1}I)_wR_P=I^{-1}IR_P$ by Proposition 1.10. Similarly $R_P=R_PI^{-1}I$.

(3) $O_l(IR_P)I = IR_PI^{-1}I = IR_P$.

If R is a local, noetherian, Asano order with unique maximal ideal P and if I is a right R-ideal, then it follows that $O_l(I)$ is also a local, noetherian, Asano order with a unique maximal ideal IPI^{-1} (cf. Lemmas 2.3 and 3.1 of [9]).

Lemma 2.2. *Let R be a bounded Krull prime ring and let I be a right v-ideal. Then*

(1) $O_l(I)=\cap O_l(IR_P)$, *where P ranges over all elements in $\boldsymbol{P}$ and $O_l(IR_P)$ is a local, noetherian, Asano order with unique maximal ideal $IP'I^{-1}$.*

(2) $O_l(I)$ *satisfies the condition (K3).*

Proof. (1) By Proposition 1.10, $I=I_w$ and so it is evident that $O_l(I)=\cap O_l(IR_P)(P\in\boldsymbol{P})$. Since $IP'I^{-1}=(IR_P)P'((IR_P)^{-1}$, $O_l(IR_P)$ is a local, noetherian, Asano order with unique maximal ideal $IP'I^{-1}$.

(2) Let c be any regular element in $O_l(I)$. Then it follows from (K3) that $cIR_P=R_P=IR_P$ for almost all P in $\boldsymbol{P}$ and $cIR_PI^{-1}=IR_PI^{-1}$. Hence, by Lemma 2.1, $cO_l(IR_P)=O_l(IR_P)$ for almost all P in $\boldsymbol{P}$. By Corollary 4.2 of [7], I^{-1} is a left v-ideal. So from Lemma 1.9 and (1) we get: $O_r(I^{-1})=\cap O_r(R_PI^{-1})=\cap O_r((IR_P)^{-1})=\cap O_l(IR_P)=O_l(I)$. Hence, applying to I^{-1} the above discussion we have $O_l(IR_P)c=O_l(IR_P)$ for almost all P in $\boldsymbol{P}$.

Let I be any right v-ideal of R and let A be any right $O_l(I)$-ideal. Then we will denote by A_w the right $O_l(I)$-ideal $\cap AO_l(IR_P)$ $(P\in\boldsymbol{P})$.

Lemma 2.3. *Let R be a bounded Krull prime ring, let I be a right v-ideal and let A be an integral right $O_l(I)$-ideal. Then*

(1) *If P is an elemet in $\boldsymbol{P}$, then $AO_l(IR_P)=O_l(IR_P)$ if and only if $A_w\supseteq IBI^{-1}$ for some integral R-ideal B with $B\not\subseteq P$.*

(2) *If a is any element in $O_l(I)$, then $(a^{-1}A)_w=a^{-1}(A_w)$.*

Proof. (1) If $A_w\supseteq IBI^{-1}$, where B is an integral R-ideal with $B\not\subseteq P$, then we get: $AO_l(IR_P)=A_wO_l(IR_P)\supseteq IBI^{-1}O_l(IR_P)=IBI^{-1}IR_PI^{-1}=IR_PI^{-1}=O_l(IR_P)$. Hence $AO_l(IR_P)=O_l(IR_P)$. To prove the converse we may assume that $AIR_{P_i}I^{-1}\subsetneq IR_{P_i}I^{-1}$ for finitely many P_i in $\boldsymbol{P}$ only $(1\leqq i\leqq k)$ by Lemma 2.2. By assumption, $P_i\neq P$. There are positive integers n_i $(1\leqq i\leqq k)$ such that $IP_i'^{n_i}I^{-1}\subseteq AIR_{P_i}I^{-1}$. Hence we have $IP_1^{n_1}\cdots P_k^{n_k}I^{-1}\subseteq A_w$ and $P_1^{n_1}\cdots P_k^{n_k}\not\subseteq P$.

(2) Let x be an element in $O_l(I)$. Then we have the following implications: $x\in(a^{-1}A)_w\Leftrightarrow x\in(a^{-1}A)IR_PI^{-1}$ for all P in $\boldsymbol{P}\Leftrightarrow xIB_PI^{-1}\subseteq a^{-1}A$ for some integral R-ideal $B_P\not\subseteq P$ by Lemmas 1.12 and 2.1$\Leftrightarrow axIB_PI^{-1}\subseteq A\Leftrightarrow ax\in AIR_PI^{-1}$ for all $P\Leftrightarrow ax\in A_w\Leftrightarrow x\in a^{-1}A_w$. Hence $(a^{-1}A)_w=a^{-1}A_w$.

Lemma 2.4. *Let R be a bounded Krull prime ring and let I be a right v-ideal. Then $O_l(IR_P)$ is an essential overring of $O_l(I)$ for any P in $\boldsymbol{P}$.*

Proof. The lemma will be proved in the following four steps.

(a) Let $F_P=\{A \mid AO_l(IR_P)=O_l(IR_P)$ and A is a right ideal of $O_l(I)\}$. First we shall prove that F_P is a right additive toplogy on $O_l(I)$. (i) if $A\in F_P$ and if $x\in O_l(I)$, then there is an integral R-ideal B such that $IBI^{-1}\subseteqq A_w$ and $B\not\subseteq P$. Since IBI^{-1} is an $O_l(I)$-ideal, we have $x^{-1}A_w\supseteqq IBI^{-1}$. Hence $x^{-1}A\in F_P$ by Lemma 2.3. (ii) If $A\in \boldsymbol{F}_P$ and B is a right ideal of $O_l(I)$ such that $x^{-1}B\in F_P$ for all x in A, then we have $B\in F_P$ in a similar way as in the proof of Lemma 1.1. Hence F_P is a right additive topology.

(b) By Lemmas 1.12 and 2.1, $O_l(IR_P)=\varinjlim IBI^{-1}$, where B ranges over all integral R-ideals such that $B\not\subseteq P$. From this and (a) we easily obtain that $O_l(IR_P)=O_l(I)_{F_P}$. Hence F_P is perfect.

(c) Let A be any element of F_P. Then there is an integral R-ideal $B(\not\subseteq P)$ such that $A_w\supseteqq IBI^{-1}$. Hence $IR_PI^{-1}=IR_PI^{-1}A_w$. Write $1=\sum x_iy_i$, where $x_i\in IR_PI^{-1}$ and $y_i\in A_w\subseteqq AIR_PI^{-1}$. There is an integral R-ideal $C(\not\subseteq P)$ such that $y_iICI^{-1}\subseteqq A$ and hence $ICI^{-1}\subseteqq IR_PI^{-1}A$. So we have $IR_PI^{-1}=IR_PI^{-1}A$. Therefore $O_l(IR_P)$ is a right essential overring of $O_l(I)$.

(d) Let $F_{P_l}=\{A \mid O_l(IR_P)A=O_l(IR_P)$ and A is a left ideal of $O_l(I)\}$. By similar way as in (a), (b) and (c) we easily obtain that $O_l(IR_P)=O_l(I)_{F_{P_l}}$ and that it is a left essential overring of $O_l(I)$. This completes the proof.

Lemma 2.5. *Let R be a bounded Krull prime ring and let R' be any order equivalent to R. Then R' is a maximal order if and only if $R'=O_l(I)$ for some right v-ideal I of R.*

Proof. If $R'=O_l(I)$ for some right v-ideal I, then it is a maximal order by Satz 1.3 of [1]. This also follows from Proposition 1.3, Lemmas 2.2 and 2.4. Conversely assume that R' is a maximal order, then there are regular elements c, d in R such that $cR'd\subseteqq R$ and so $R'dR$ is a right R-ideal. Put $I=(R'dR)_v$, then $I=\cap(R'dR)R_P$ $(P\in\boldsymbol{P})$ by Proposition 1.10. This implies that I is a left R'-module, so that $O_l(I)\supseteqq R'$. Therefore $O_l(I)=R'$.

Theorem 2.6. *Let R be a bounded Krull prime ring. If R' is a maximal order equivalent to R, then it is a bounded Krull prime ring and $D(R)\cong D(R')$.*

Proof. This is evident from Theorem 1.8, Lemmas 2.2, 2.4 and 2.5.

In case maximal orders over commutative Krull domains, the second assertion of the theorem was proved by R.M. Fossum (cf. Theorem 2.1 of [4]).

3. The generalization of some results on maximal orders over commutative Krull domains

Lemma 3.1. *Let R be a bounded Krull prime ring, let $P_1, \cdots, P_k \in \boldsymbol{P}$ and let $A=P_1 \cap \cdots \cap P_k$. Then*

(1) $C(A)=C(P_1) \cap \cdots \cap C(P_k)$. *Hence each element of $C(A)$ is regular.*

(2) *R satisfies the Ore condition with respect to $C(A)$.*

(3) $R_A=R_{P_1} \cap \cdots \cap R_{P_k}=\varinjlim B^{-1}$, *where B ranges over all integral R-ideals such that $B \not\subseteq P_i$ $(1 \leqq i \leqq k)$.*

Proof. (1) Let c be any element in $C(A)$. If $cx \in P_i$, then $cxP_1 \cdots P_{i-1}P_{i+1} \cdots P_k \subseteq A$ and so $xP_1 \cdots P_{i-1}P_{i+1} \cdots P_k \subseteq A \subseteq P_i$. Hence $x \in P_i$ and $c \in C(P_1) \cap \cdots \cap C(P_k)$. The converse inclusion is evident. Since each element of $C(P_i)$ is regular in R (cf. Proposition 1.1 of [7]), so is each element of $C(A)$.

(2) Let I be an integral right R-ideal. First we shall prove that $I \cap C(A) \neq \phi$ if and only if there is an integral R-ideal $B \not\subseteq P_i$ $(1 \leqq i \leqq k)$ such that $B \subseteq I$. If c is an element in $I \cap C(A)$. Then $cR_{P_i}=R_{P_i}$ $(1 \leqq i \leqq k)$ by (1) and hence we have $I \supseteq cR \supseteq P_{k+1}^{n_{k+1}} \cdots P_{k+l}^{n_{k+l}}$ (cf. the proof of Lemma 1.6), where $P_j \in \boldsymbol{P}$ $(k+1 \leqq j \leqq k+l)$, $P_j \neq P_i$ $(1 \leqq i \leqq k)$ and n_j are positive integres. Conversely assume that $I \supseteq B$ and $P_i \not\supseteq B$ $(1 \leqq i \leqq k)$. Then $BP_1 \cdots P_{i-1}P_{i+1} \cdots P_k \not\subseteq P_i$ and so there is an element c_i such that $c_i \in BP_1 \cdots P_{i-1}P_{i+1} \cdots P_k \cap C(P_i)$. Put $c=c_1+\cdots+c_k$. Then we have $c \in C(A) \cap I$, as desired. Now let $S=R_{P_1} \cap \cdots \cap R_{P_k}$ and let x be any element in S. Then $xB_i \subseteq R$ for some integral R-ideal $B_i \not\subseteq P_i$ by Lemma 1.12. Put $B=\sum_{i=1}^{k} B_i$. Then $B \not\subseteq P_i$ for all i $(1 \leqq i \leqq k)$ and $xB \subseteq R$. So there is an element c' in $B \cap C(A)$ and r' in R such that $xc'=r'$. Now let c be any element in $C(A)$ and let r be any element in R. By (1), $c^{-1} \in S$ and so $c^{-1}rd=s$ for some $d \in C(A)$ and $s \in R$. Hence $rd=cs$. This implies that R satiefies the Ore condition with respect to $C(A)$ and $R_A=S$.

(3) is evident from the discussion of (2).

Lemma 3.2. *Under the same notation as in Lemma* 3.1, *if $A^*=P_1R_A \cap \cdots \cap P_kR_A$, then*:

(1) *R_A/A^* is the quotient ring of R/A.*

(2) *R_A/P_iR_A is a simple artinian ring.*

Proof. It is evident that $P_iR_A=P_i' \cap R_A$. Hence it follows that $A=A^* \cap R$ and that both P_iR_A and A^* are integral R_A-ideals. So $R/A \subseteq R_A/A^*$ as rings. Further regular elements in R/A are only $\{[c+A] \mid c \in C(A)\}$ and R_A is the quotient ring of R with respect to $C(A)$. So R_A/A^* is the quotient ring of R/A. Since $(P_1+(P_2 \cap \cdots \cap P_k)) \cap C(A) \neq \phi$, we get $(P_1+(P_2 \cap \cdots \cap P_k))R_A=R_A$ and $P_1R_A+(P_2R_A \cap \cdots \cap P_kR_A)=R_A$. This implies that $P_1R_A, \cdots, P_kR_A$ are comaximal. Hence we obtain the following diagram:

$$
\begin{array}{ccccccc}
R/A & \subseteq & R/P_1 & \oplus \cdots \oplus & R/P_k \\
\| & & \| & & \| \\
R_A/A^* & \simeq & R_A/P_1R_A & \oplus \cdots \oplus & R_A/P_kR_A \\
 & & \| & & \| \\
 & & R_{P_1}/P_1' & \oplus \cdots \oplus & R_{P_k}/P_k' \ .
\end{array}
$$

Since R_{P_i}/P_i' is the quotient ring of R/P_i by Proposition 1.1 of [7], we have $R_A/P_iR_A=R_{P_i}/P_i'$ and hence R_A/P_iR_A is a simple artinian ring.

Lemma 3.3. *Under the same notation as in Lemma* 3.1, R_A *is a right and left principal ideal ring.*

Proof. By Proposition 1.2, Theorem 1.13, Lemmas 3.1 and 3.2, R_A is a bounded Dedekind prime ring with the maximal ideals $P_1R_A, \cdots, P_kR_A$. Then it is well known that R_A is a right and left principal ideal ring (cf. Satz 2.8 of [2] or Remark 3.3 of [8, p. 437]).

Now by the validity of Lemma 3.3, the following two theorems are obtained by the same way as the corresponding ones for maximal orders over commutative Krull domains (cf. Theorems 3.5 and 4.5 of [7]).

Theorem 3.4 (Approximation theorem for bounded Krull prime rings). *Let R be a bounded Krull prime ring, let $P_1, \cdots, P_k \in \boldsymbol{P}$ and let $n_1, \cdots, n_k$ be any integers. Then there is a unit x in Q such that $xR_{P_i}=P_i'^{n_i}(1\leqq i\leqq k)$ and $x\in R_{P_j}$ for all $P_j\in\boldsymbol{P}$ with $P_j\neq P_i$.*

Theorem 3.5. *Let R be a bounded Krull prime ring and let I be any right R-ideal. Then there are units c, d in Q such that $I_v=(cR+dR)_v$.*

4. Krull orders over commutative Krull domains

Let $\mathfrak{o}$ be a commutative integral domain and let K be its field of quotients. Suppose that $\sum$ is a central simple K-algebra with finite dimension over K and that Λ is an $\mathfrak{o}$-order (cf. §1 of [4], for the definition $\mathfrak{o}$-orders). Then we have

Proposition 4.1. *Let Λ be a Krull prime ring and let $\mathfrak{o}^*$ be the center of it. Then*

(i) Λ *is a bounded and maximal $\mathfrak{o}^*$-order.*

(ii) $\mathfrak{o}^*$ *is a Krull domain.*

Proof. Let $\Lambda=\cap_i \Lambda_i$, where Λ_i are essential overrings of Λ. We shall prove that each Λ_i is not simple. Let x be any regular element but not unit in Λ_i. Then, by the same way as in the proof of Proposition 3.1 of [7], there is an element $a\neq 0$ in $\mathfrak{o}$ contained in $x\Lambda_i$ and so $a\Lambda_i$ is a proper ideal of Λ_i. Hence Λ_i is not simple so that it is a local, noetherian, Asano order with unique maximal ideal P_i'. Thus Λ is bounded. Let $\mathfrak{o}_i=\Lambda_i\cap K$. Then we have $\mathfrak{o}^*=\Lambda\cap K=(\cap_i\Lambda_i)\cap K=\cap\mathfrak{o}_i$. Let x be any nonzero element in K. Then $x\Lambda_i=P_i'^{n_i}$ for some integer n_i. The mapping $v_i: K\to \boldsymbol{Z}$ defined by $v_i(x)=n_i$ is a dicrete valuation, where $\boldsymbol{Z}$ is the ring of integers. It is evident that $\mathfrak{o}_i=\{x\in K \mid v_i(x)\geqq 0\}$. Let a be any nonzero element in $\mathfrak{o}^*$. Then $a\Lambda_i=\Lambda_i$ for almost all i. Hence

$v_i(a)=0$ for almost all i. Hence $\mathfrak{o}^*$ is a Krull domain by Theorem 3.5 of [10]. Since $\mathfrak{o}^* \supseteqq \mathfrak{o}$, Λ is an $\mathfrak{o}^*$-order and it is a maximal order as ring by Proposition 1.3. If Γ is an $\mathfrak{o}^*$-order such that $\Gamma \supseteqq \Lambda$, then we have $c\Gamma \subseteqq \Lambda$ for some $0 \neq c \in \mathfrak{o}^*$, because there exists a finitely generated free $\mathfrak{o}^*$-submoduole in $\sum$ which contains Γ by Proposition 1.1 of [4]. So Γ is equivalent to Λ. Hence $\Gamma=\Lambda$ and therefore Λ is a maximal order as $\mathfrak{o}^*$-algebras.

Proposition 4.2. *Let $\mathfrak{o}$ be a Krull domain and let Λ be an $\mathfrak{o}$-order. Then following conditions are equivalent:*

(i) *Λ is a Krull prime ring.*

(ii) *Λ is a maximal order as rings.*

(iii) *Λ is a maximal order as $\mathfrak{o}$-algebras.*

Proof. (i)⇒(ii): This follows from Corollary 1.4 and Proposition 4.1.

(ii)⇒(iii): This is clear from the proof of Proposition 4.1.

(iii)⇒(i) : This follows from Proposition 3.1 of [7].

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References

[1] K. Asano: *Zur Arithemetik in Schiefringen* I, Osaka Math. J. **1** (1949), 98–134.

[2] K. Asano: *Zur Arithemetik in Schiefringen* II, J. Inst. Polytec. Osaka City Univ. **1** (1950), 1–27.

[3] K. Asano and K. Murata: *Arithemetical ideal theory in semigroups*, J. Inst. Polytec. Osaka City Univ. **4** (1953), 9–33.

[4] R.M. Fossum: *Maximal orders over Krull domains*, J. Algebra **10** (1968), 321–332.

[5] A.W. Goldie: *Semi-prime rings with maximum condition*, Proc. London Math. Soc. **10** (1960), 201–220.

[6] C.R. Hajarnavis and T.H. Lenagan: *Localization in Asano orders*, J. Algebra **21** (1972), 441–449.

[7] H. Marubayashi: *Non commutative Krull rings*, Osaka J. Math. **12** (1975), 703–714

[8] G.O. Michler: *Asano orders*, Proc. London Math. Soc. **19** (1969), 421–443.

[9] J.C. Robson: *Non-commutative Dedekind rings*, J. Algebra **9** (1968), 249–265.

[10] P. Samuel: Lectures on unique factorization domains, Tata Inst. for Fundamental Research, Bombay, 1964.

[11] B. Stenström: Rings and Modules of Quotients, Springer, 1971.