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A NOTE ON RELATIVE T-NILPOTENCY

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This note gives some supplementary results of [6]. The first one shows an application of the idea in the proof of [6], Lemma 7 and gives a characterization of artinian rings. The second one gives a refinement of [6], Corollary 2 to Theorem A.2 and the final one is a special type of the exchange property.

Throughout we shall assume that R is a ring with identity and modules are unitary right R-modules. First, we shall recall definitions in [6].

Let $\{P_\alpha\}_I$ and $\{Q_\beta\}_J$ be two infinite sets of R-modules. We take a countable set $\{M_i\}_1^\infty$ such that $M_{2i-1}=P_{\alpha(2i-1)}\in\{P_\alpha\}_I$ and $M_{2j}=Q_{\beta(2j)}\in\{Q_\beta\}_J$. Further we take a set of non-isomorphisms $f_i\colon M_i\to M_{i+1}$. If for any element m in M_1 there exists n such that $f_nf_{n-1}\cdots f_1(m)=0$, we say $\{f_i\}_1^\infty$ is *locally T-nilpotent*. If for any countable sets $\{M_i\}_1^\infty$ above such that $\alpha(2i-1)\neq\alpha(2i'-1)$ ($\beta(2j)\neq\beta(2j')$) if $i\neq i'$ ($j\neq j'$) any sets $\{f_i\}$ of non-isomorphisms are always locally T-nilpotent, then we say $\{P_\alpha\}_I$ and $\{Q_\beta\}_J$ are *relatively and locally sami-T-nilpotent*. If we omit the assumptions $\alpha(2i-1)\neq\alpha(2i')$ ($\beta(2j)\neq\beta(2j')$) in the above, we say $\{P_\alpha\}_I$ and $\{Q_\beta\}_J$ are *relatively and locally T-nilpotent*. If $\{P_\alpha\}_I=\{Q_\beta\}_J$, we say $\{P_\alpha\}_I$ is *locally semi-T-nilpotent* or *T-nilpotent*, corresponding to the above cases. We shall assume that the definition of relatively semi-T-nilpotency contains a case of either I or J being finite. If $K=\sum_I\oplus P_\alpha=\sum_J\oplus Q_\beta$ and $\{P_\alpha\}_I$, $\{Q_\beta\}_J$ are locally and relatively T-nilpotent, then we say $\sum_I\oplus P_\alpha$ and $\sum_J\oplus Q_\beta$ are *relatively T-nilpotent decompositions* of K.

Finally, let $M=N\oplus P$ be R-modules and κ a cardinal number. If for any decomposition $M=\sum\oplus L_\alpha$ with κ-components there exist submodules L_α' of L_α such that $M=N\oplus\sum\oplus L_\alpha'$, then we say N has *the κ-exchange property* in M. In case κ is any cardinal, we say N has *the exchange property in M.*

1. T-nilpotent decompositions

First, we study a property of relative T-nilpotency. If the endomorphism ring of a module M is a local ring, then we call M *completely indecomposable.*

Lemma 1. *Let M be an R-module and f,g in* $\mathrm{End}_R(M)$. *If fg is isomorphic, $M=\mathrm{Im}\,g\oplus\mathrm{Ker}\,f$.*

Lemma 2. *Let P be an R-module. If P is itself locally T-nilpotent, P is a completely indecomposable module.*

Proof. Put $S=\mathrm{End}_R(P)$. If $e\in S$ and $e^2=e$, then $e=1$ or 0 by the assumption. Let x, y be elements in S with x non-unit. Then neither xy nor yx is unit in S from the above and Lemma 1. Furthermore, consider a sequence $\{x^n\}_1^\infty$ of non-units in S. For any element p in P, there exists $n=n(p)$ such that $x^{n(p)}(p)=0$ from the assumption. Therefore, $X=1+\sum_1^\infty x^i$ is an element in S and $((1-x)X)(p)=(1-x)(1+x+\cdots+x^{n(p)-1})(p)=(1-x^{n(p)})(p)=p$. Hence, $1-x$ is unit in S from Lemma 1. Let x, y be non-unit in S. We assume that $x+y$ is unit in S. Then we may assume $x+y=1$, which is a contradiction to the above. Therefore, S is a local ring.

Theorem 1. *Let M be an R-module and $M=\sum_I\oplus P_\alpha=\sum_J\oplus Q_\beta$ two relatively T-nilpotent decompositions of M. Then all P_α and Q_β are completely indecomposable modules and hence, those decompositions are unique up to isomorphism and every direct summand of M has the exchange property in M.*

Proof. We put $I_1=\{\alpha\in I\,|\,P_\alpha\approx Q_\beta$ for some $\beta\in J\}$ and $J_1=\{\beta\in J\,|\,Q_\beta\approx P_\alpha$ for some $\alpha\in I\}$. We first show $I_1\neq\phi$ and so $J_1\neq\phi$. We assume the contrary. Let p_α, q_β be projections of M to P_α and Q_β, respectively. Let $x_1\neq 0\in P_{\alpha_1}$. Then there exists $\beta_2\in J$ such that $q_{\beta_2}(x_1)=x_2\neq 0$. Again there exists $\alpha_3\in I$ such that $p_{\alpha_3}(x_2)=x_3\neq 0$. Repeating those arguments, we obtain a contradiction to the T-nilpotency, since $I_1=J_1=\phi$ (cf. [6], Lemma 7). Hence, $I_1\neq\phi$ and so $J_1\neq\phi$. Furthermore, $\{P_\alpha\}_{I_1}$, $\{Q_\beta\}_{J_1}$ are sets of completely indecomposable modules and locally T-nilpotent by Lemma 2 and the assumption. We put $M=\sum_{I_1}\oplus P_\alpha\oplus\sum_{I_1'}\oplus P_{\alpha'}=\sum_{J_1}\oplus Q_\beta\oplus\sum_{J_1'}\oplus Q_{\beta'}$, where $I_1'=I-I_1$ and $J_1'=J-J_1$. Let $\{P_{\alpha_i}\}_1^n$ be any finite subset of $\{P_\alpha\}_{I_1}$. Since $\sum_1^n\oplus P_{\alpha_i}$ has the exchange property by [2], Lemma 3.11 and [9], Proposition 1,

$$M=\sum_1^n\oplus P_{\alpha_i}\oplus\sum_{J_1}\oplus Q_\beta'\oplus\sum_{J_1'}\oplus Q_{\beta'}'\cdots(*),$$

where $Q_\beta=Q_\beta'\oplus Q_\beta''$. Then $\sum_{J_1'}\oplus Q_\beta''$ is isomorphic to a direct summand of $\sum_1^n\oplus P_{\alpha_i}$. If $Q_{\beta'}''\neq(0)$, $Q_{\beta'}''$ contains a direct summand isomorphic to some P_{α_j} by Krull-Remak-Schmidt's theorem, say $Q_{\beta'}''=X\oplus Y$; $X\overset{\varphi}{\approx}P_{\alpha_j}$. Since $\beta'\in J_1'$, $Y\neq(0)$. Let p be a projection of $Q_{\beta'}''$ to X and i the inclusion of X to $Q_{\beta'}''$, then p and i are not isomorphic. However, $\varphi pi\varphi^{-1}=1P_{\alpha_j}$ and neither $\varphi\,p$ nor $i\varphi^{-1}$ is isomorphic. Which contradicts the relative T-nilpotency. Accordingly, $Q_{\beta'}''=(0)$ and $(\sum_1^n\oplus P_{\alpha_i})\cap(\sum_{J_1'}\oplus Q_{\beta'})=(0)$. Therefore, $(\sum_{I_1}\oplus P_\alpha)\cap$

$(\sum_{J_1'}\oplus Q_\beta)=(0)$. Let $\bar{M}=M/(\sum_{J_1'}\oplus Q_{\beta'})$ and ψ the natural epimorphism of M to $\bar{M}$. Then $\bar{M}=\sum_{J_1}\oplus\psi(Q_\beta)\supseteq\sum_{I_1}\oplus\psi P_\alpha)$ (note $\psi(Q_\beta)\approx Q_\beta$ and $\psi(P_\alpha)\approx P_\alpha$). On the other hand, $\sum_{I_1}\oplus\psi(P_\alpha)$ is locally direct summand of $\bar{M}$ from (*) and $\{\psi(P_\alpha)\}_{I_1}$ is locally T-nilpotent. Hence, $\sum_{I_1}\oplus\psi(P_\alpha)$ is a direct summand of $\bar{M}$ by [3], Theorem 9 and [7], Corollary 2 to Lemma 2 and Lemma 3, since $\{\psi(Q_\beta)\}_{J_1}$ is a set of completely indecomposable modules. Furthermore, $\sum_{I_1}\oplus\psi(P_\alpha)$ has the exchange property in $\bar{M}$ by [4], Theorem 4 and so $\bar{M}=\sum_{I_1}\oplus\psi(P_\alpha)\oplus\sum_{J_1''}\oplus\psi(Q_\beta)$, where $J_1''\subseteq J_1$. Therefore, $M=\sum_{I_1}\oplus P_\alpha\oplus\sum_{J_1''}\oplus Q_{\beta'}\oplus\sum_{J_1'}\oplus Q_{\beta'}=\sum_{I_1}\oplus P_\alpha\oplus\sum_{I_1'}\oplus P_{\alpha'}$. Hence, $\sum_{J_1''}\oplus Q_{\beta'}\oplus\sum_{J_1'}\oplus Q_{\beta'}\approx\sum_{I_1'}\oplus P_{\alpha'}$. However, $\{P_{\alpha'}\}_{I_1'}$ and $\{Q_{\beta'}\}_{J_1'\cup J_1''}$ are locally and relatively T-nilpotent and so some $P_{\alpha'}$ is isomorphic to some $Q_{\beta'}$ by the first part, provided $P_{\alpha_1'}\neq(0)$. Therefore, $I_1'=J_1'=J_1''=\phi$. The remaining parts are clear from [1] and [4], Theorem 4.

REMARK. Theorem 1 does not remain valid if we replace the T-nilpotency by the semi-T-nilpotency in the assumption.

Corollary 1. *Let P be R-projective. Then P has two relatively T-nilpotent decompositions if and only if P is a perfect module.*

Proof. It is clear from Theorem 1 and [4], Theorem 6.

Corollary 2. *R is right artinian if and only if every projective modules and every injective modules have relatively T-nilpotent decompositions.*

Proof. It is clear from Corollary 1, [3], Corollary 1 to Proposition 1 and [8].

2. Exchange property

Let $\{M_\alpha\}_I$ be a set completely indecomposable modules and $M=\sum_I\oplus M_\alpha$. We consider a relation between the concept of the exchange property in M and that of the 2-exchange property in M for a direct summand N of M. If N is also a direct sum of indecomposable modules, those concepts are equivalent by [6], Theorem A.2. We do not know whether this fact is true without any assumptions.

Lemma 3. *Let $A=B\oplus C\oplus D$ be R-modules. If $B\oplus C$ has the 2-exchange property in A, then C has the same property in $C\oplus D$.*

Proof. We assume $C\oplus D=K\oplus L$. Then $A=B\oplus C\oplus D=B\oplus K\oplus L$. Hence, $A=B\oplus C\oplus K'\oplus L'$ for some $K'\subseteq K$ and $L'\subseteq L$ from [6], Lemma A.4. Since $C\oplus D\supseteq C\oplus K'\oplus L'$, $C\oplus D=C\oplus K'\oplus L'$.

Let M be as above and $M=S_1\oplus S_2$. Let $\{M_\alpha\}_J$ be the isomorphic representative classes of $\{M_\alpha\}_I$ and we shall denote it by $[M]$. We put $J'=\{\alpha\in J\,|\,M_\alpha$ is isomorphic to a direct summand of both S_1 and $S_2\}$ and $J''=\{\alpha\in J'\,|$ directsums of any finite copies of M_α are isomorphic to direct summands of both S_1 and $S_2\}$.

Theorem 2. *Let M, S_i, etc. be as above. Then S_1 has the exchange property in M if and only if S_1 has the 2-echange property in M, $\{M_\alpha\}_{J'}$ is locally semi-T-nilpotent and $\{M_\alpha\}_{J''}$ is locally T-nilpotent.*

Proof. Let N_i be a dense submodule of S_i and $N_i=\sum_{j\in J}\sum_{\beta\in I_{ij}}\oplus N_{ij\beta}$, where $i=1,2$ and $N_{ij\beta}$'s are isomorphic to some M_α in $\{M_\alpha\}_I$ and $N_{ij\beta}\approx N_{ij\beta'}\approx N_{i'j\beta}\approx N_{i'j\beta'}$ and $N_{ij\beta}\not\approx N_{ij'\beta}$ if $J\neq I'$ (see [4]). We put

$$\begin{aligned}
J_1 &= \{j\in J\,|\,I_{1j} \text{ and } I_{2j} \text{ are infinite}\}\\
J_2 &= \{j\in J\,|\,I_{1j} \neq \phi \text{ and } I_{2j} \text{ is finite}\}\\
J_3 &= \{j\in J\,|\,I_{1j} \text{ is finite and } I_{2j} \text{ is infinite}\}\\
J_3 &= \{j\in J\,|\,I_{1j} = \phi\} \text{ and}\\
J_5 &= \{j\in J\,|\,I_{2j} = \phi\}\,.
\end{aligned}$$

Then $J_1=J''$ and $J_1\cup J_2\cup J_3=J'$. Furthermore, we put $N_i(k)=\sum_{j\in J_k}\sum_{I_{ij}}\oplus N_{ij\beta}$. Then $N_1=\sum_{k\neq 4}\oplus N_1(k)$ and $N_2=\sum_1^4\oplus N_2(k)$. We shall show "if" part. $N_1(1)$, $N_1(3)$ and $N_2(2)$ are direct summands of M from the assumption and [4], Proposition 2. Since $N_1(1)\oplus N_1(3)\subseteq S_1$ and $N_2(2)\subseteq S_2$, $M=N_1(1)\oplus N_1(3)\oplus S_1'\oplus N_2(2)\oplus S_2'$, where $S_i'\subseteq S_i$. Furthermore, a dense submodule of S_1' (resp. S_2') is isomorphic to $N_i(2)\oplus N_1(5)$ (resp. $N_2(1)\oplus N_2(3)\oplus N_2(4)$). Since $S_1=N_1(1)\oplus N_1(3)\oplus S_1'$, $S_2=N_2(2)\oplus S_2'$ and S_1 has the 2-exchange property in M, S_1' has the same property in $S_1'\oplus N_2(2)\oplus S_2'$ $(=C)$ from Lemma 3. We assume $S_1'\oplus S_2'=A\oplus B$. Then $C=A\oplus B\oplus N_2(2)$. Hence, $C=S_1'\oplus A'\oplus B'\oplus N_2(2)'$ from [6], Lemma A.4, where $A'\subseteq A$, etc. Therefore, $S_1'\oplus A'\oplus B'=S_1'\oplus S_2'$, which means S_1' has the 2-exchange property in $S_1'\oplus S_2'$. Accordingly, S_1' has the exchange property in $S_1'\oplus S_2'$ by [6], Corollary 2 to Theorem A.2, since $N_1(1)\oplus N_1(3)\oplus N_2(2)$ has the exchange property in M by [5], Theorem 2 and so $S_1'\oplus S_2'$ is a directsum of completely indecomposable modules. Therefore, S_i' is also a directsum of completely indecomposable modules and hence, S_1 has the exchange property in M by [6], Theorem A.2. The converse is clear from [6], Theorem A.2.

Finally, we shall study some special properties concerning with the exchange property in M of a direct summand of M. We are interested in a relation between the exchange property in M and the relative semi-T-nilpotency. We assume $M=N_1\oplus N_2$ and $N_i=\sum_k\oplus N_i(k)$ as in the proof of Theorem 2.

We know from the proof above that N_1 has the exchange property in M if and only if $N_1(2)\oplus N_1(5)$ has the same property in $N_1(2)\oplus N_1(5)\oplus N_2(3)\oplus N_2(4)$ and $\{N_{1ij}\}$, $\{N_{2ij}\}$ are locally and relatively semi-T-nilpotent (cf. [6], the proof of Corollary 1 to Theorem). Hence, we may restrict ourselves to a case of $[N_1]\cap[N_2]=\phi$.

In the following we shall use the category $\underline{A}$ induced from a set of completely indecomposable modules and its factor category $\bar{\underline{A}}=\underline{A}/J'$ studied in [3]. We refer to [3] for the notations and results on $\underline{A}$.

Lemma 4. *Let M be in $\underline{A}$ and A, B two locally direct summands of M. If $[A]\cap[B]=\phi$, $A\cap B=(0)$ and $A\oplus B$ is a locally direct summand of M.*

Proof. Let i_A, i_B be the inclusions of A and B into M, respectively. Since $[A]\cap[B]=\phi$, $\operatorname{Im}\bar{i}_A\cap\operatorname{Im}\bar{i}_B=(0)$ from [3], Theorem 7 and [7], Lemma 3. Let $A\oplus B$ be the external direct sum and $i=(i_A, i_B)\colon A\oplus B\to M$. Then it is clear from the above that $\bar{i}$ is monomorphic in $\bar{A}$. Hence, $\operatorname{Im} i=A+B$ is a locally direct summand of M and $A\cap B=(0)$.

Lemma 5. *Let $M=S\oplus T$ and M in $\underline{A}$. For any element x in S there exists a finite set of indecomposable modules S_i such that $S=\sum_1^t\oplus S_i\oplus S'$, $x\in\sum_1^t\oplus S_i$ and S_i's are isomorphic to some in $[M]$.*

Proof. See [4], the proof of Proposition 3.

Let $M=N_1\oplus N_2=\sum\oplus S_\gamma$ and $N_i\in\underline{A}$ with $[N_1]\cap[N_2]=\phi$. We put $S_\gamma(i)=\sum S_\alpha$, where S_α runs through all indecomposable direct summands of S_γ which are isomorphic to some in $[N_i]$. By $[S_\gamma(i)]$ we denote the representative classes of such S_α's. Then $S_\gamma(i)$ is also the union of all locally direct summands A of S_γ with $[A]\subseteq[N_i]$. It is clear, from Lemma 5, $S_\gamma=S_\gamma(1)+S_\gamma(2)$. If N_1(or N_2) has the exchange property in M, $S_\gamma=S_{\gamma_1}\oplus S_{\gamma_2}$ where $S_{\gamma_i}\subseteq S_\gamma(i)$ and every indecomposable direct summand of S_{γ_i} is isomorphic to some in $[N_i]$. In the following, we shall study a case of $S_{\gamma_1}=S_\gamma(1)$.

The following lemma is a slight generalization of [6], Lemma 7.

Lemma 6. *Let $M=N_1\oplus N_2=\sum_K\oplus S_\gamma$ and $N_i=\sum_{\alpha\in I_i}\oplus N_{i\alpha}$; $N_{i\alpha}$'s are completely indecomposable modules. We assume $\{N_{1\alpha}\}_{I_1}$ and $\{N_{2\alpha}\}_{I_2}$ are locally and relatively semi-T-nilpotent and $[N_1]\cap[N_2]=\phi$. Then $M=\sum_K\oplus S_\gamma(1)+N_2=N_1+\sum_K\oplus S_\gamma(2)$.*

Proof. We shall give a sketch of the proof (cf. [6], Lemma 7). We assume I_1 and I_2 are infinite. Put $M^*=\sum_K\oplus S_\gamma(1)+N_2$. We assume $M\neq M^*$. Then there exists $N_{1\alpha}$ not contained in M^*. Let $x_1\in N_{1\alpha}-M^*$ and $x_{1\alpha}=\sum x_{\gamma_i}$; $x_{\gamma_i}\in S_{\gamma_i}$.

We may assume $x_{\gamma_1}\notin M^*$. Then from Lemma 5 we have $x_{\gamma_1}=\sum_{\beta} y_{i\beta}$; $y_{i\beta}\in A_\beta\langle\oplus S_{\gamma_1}$ and A_β's are indesompoosable. Since $x_{\gamma_1}\notin M^*$, there exists $y_{2\beta}\in A_\beta-M^*$ and so $[A_\beta]\in[N_2]$. Now, we can find $S_{\gamma_1'}$ which contains a direct summand $S_{\gamma_1'\alpha'}$ isomorphic to $N_{1\alpha}$. Since $S_{\gamma_1'\alpha'}$ has the exchange property by [9], Proposition 1,

$$M = S_{\gamma_1'\alpha'}\oplus N_1'\oplus N_2;\ N_1'\subseteq N_1\,,$$

since $[N_1]\cap[N_2]=\phi$. Let $y_{2\beta}=a+b+c$; $a\in S_{\gamma_1'\alpha'}$, $b\in N_1'$ and $c\in N_2$. Then $b\in N_1'-M^*$. Hence, we can find an indecomposable direct summand $N_{1\delta}$ of N_1' such that $b=x_{1\delta}+\cdots$, $x_{1\delta}\in N_{1\delta}-M^*$. On the other hand, since $[A_\beta]\in[N_2]$. there exists $N_{2\varepsilon}$ isomorphic to A_β and

$$M = N_{2\varepsilon}\oplus\sum_K\oplus S_\gamma';\ S_\gamma'\subseteq S_\gamma\,.$$

Let $x_{1\delta}=d+\sum f_\gamma$; $d\in N_{2\varepsilon}$, $f_\gamma\in S_\gamma'$. Again there exists $f_{\gamma_2}\in S_{\gamma_2}'-M^*$. Similarly to the above, we can find a direct summand A_η of S_{γ_2}' such that $[A_\eta]\in[N_2]$, $f_{\gamma_2}=\sum y_{i\eta'}$ and $y_{2\eta}\in A_\eta-M^*$. Furthermore, since $N_{1\delta}$ is a direct summand of N_1', there exists $S_{\gamma_2'}$ which contains a direct summand $S_{\gamma_2'\delta'}$ isomorphic to $N_{1\delta}$ such that

$$M = S_{\gamma_1''\alpha''}\oplus S_{\gamma_2'\delta'}\oplus N_1''\oplus N_2;\ N_1\subseteq N_1'$$

and $S_{\gamma_1''\alpha''}\approx S_{\gamma_1'\alpha'}$. Repeating those arguments we obtain a sequence $\{x_{1\alpha}, y_{2\beta}, x_{1\delta}, y_{2\eta}\cdots\}$, which contradicts the assumption of relative semi-T-nilpotency.

Theorem 3. *Let M be a direct sum of completely indecomposable modules and $M=N_1\oplus N_2=\sum_K\oplus S_\gamma$. We assume N_i is a direct sum of indecomposable modules $\{N_{i\alpha}\}_{I_i}$ such that $N_{1\alpha}\not\approx N_{2\beta}$ for any $\alpha\in I_1$, $\beta\in I_2$ and $\{N_{1\alpha}\}_{I_1}$, $\{N_{2\beta}\}_{I_2}$ are locally and relatively semi-T-nilpotent. Then the following conditions are equivalent.*

1) $M=\sum_K\oplus S_\gamma(1)\oplus N_2$

2) $\mathrm{Hom}_R([S_\gamma(1)], [S_\gamma(2)])=(0)$ *for every* γ

3) $S_\gamma(1)$ *is a direct summand of* S_γ *and* $[A]\in[N_1]$ *for every indecomposable direct summands* A *of* $S_\gamma(1)$, *where* $\mathrm{Hom}_R([S_\gamma(1)], [S_\gamma(2)])=(0)$ *means* $\mathrm{Hom}_R(B, C)=(0)$ *for all* $B\in[S_\gamma(1)]$ *and all* $C\in[S_\gamma(2)]$.

Proof. 1)$\to$2) and 3). 1) implies $S_\gamma=S_\gamma(1)\oplus S_\gamma'$ and $\sum_K\oplus_\gamma S(1)\approx N_1$, $\sum_K\oplus S_\gamma'\approx N_2$. Hence, we obtain 3). Furthermore, every C in $[S_\gamma(2)]$ is isomorphic to a direct summand C' of S_γ'. Let B be in $[S_\gamma(1)]$ and $f'\in\mathrm{Hom}_R(B, C)\overset{\varphi}{\approx}\mathrm{Hom}_R(B, C')$. Then $f=\varphi(f')$ is not isomorphic from the assumption. Put $B'=\{x+f(x)\,|\,x\in B\}$. Then $B'\approx B$ and B' is a direct summand of S_γ from [7], Lemma 3. Hence, $B'\subseteq S_\gamma(1)$. Therefore, $\mathrm{Im} f\subseteq (B+B')\cap S_\gamma'\subseteq S_\gamma(1)\cap S_\gamma' =(0)$.

2)$\rightarrow$1). Let x be in $S_\gamma(1)$ and $x=\sum x_i$; $x_i\in A_i\langle\oplus S_\gamma$ with $[A_i]\in[N_1]$. On the other hand, there exists, from Lemma 5, a direct summand $B=\sum_{j=1}^{t}\oplus B_j$ of S_γ such that $x\in B$ and $[B_j]\in[M]$. Let p_j be the projection of S_γ onto B_j. If $0\neq p_j(x)=\sum p_j(x_i),[B_j]\in[N_1]$ from 2). Hence, $x\in\sum_s\oplus B_{js}$ with $[B_{js}]\in[N_1]$. Now, let y be in $(\sum\oplus S_\gamma(1))\cap N_2$ and $y=\sum y_i$; $y_i\in S_{\gamma_i}(1)$. Then there exists a direct summand $\sum_{\gamma_i}\sum\oplus B_{js}$ containing y as above. Hence, $y\in(\sum\sum\oplus B_{js})\cap N_2=(0)$ by Lemma 4. Therefore, $M=\sum\oplus S_\gamma(1)\oplus N_2$ from Lemma 6.

3)$\rightarrow$1). Let $S_\gamma=S_\gamma(1)\oplus S_\gamma'$ and C an indecomposable direct summand of S_γ'. Then $C\in[N_2]$ otherwise $C\subseteq S_\gamma(1)$. Hence, for dense submodules B_{γ_1}, B_{γ_2} in $S_\gamma(1)$ and S_γ', respectively, we have $[B_{\gamma_i}]\subseteq[N_i]$. Since $\sum_K\oplus B_{\gamma_1}$ and $\sum_K\oplus B_{\gamma_2}$ are dense submodules of $\sum_K\oplus S_\gamma(1)$ and $\sum_K\oplus S_\gamma'$, respectively by [4], Theorem 1, $M=\sum\oplus S_\gamma(1)\oplus N_2$ by [6], Lemma 7.

Corollary 1. *Let M be as above and $M=N_1\oplus N_2$. If either* $\mathrm{Hom}_R(N_1, N_2) =0)$ *or* $\mathrm{Hom}_R(N_1, N_2)=(0)$, *then N_1 and N_2 have the exchange property in M,* (cf. [6], Corollary 5 to Theorem).

Proof. N_i is in $\underline{A}$ by [6], Corollary 5 to Theorem. The condition 2) in the theorem is satisfied for any decompositions $M=\sum_K\oplus S_\gamma$. Hence, N_1 and N_2 have the exchange property in M from Theorem 3 and [6], Theorem A.2.

Corollary 2. *Let M, N_i and S_γ be as in Theorem* 3. *Then the following conditions are equivalent.*

1) $M=(\sum_K\oplus S_\gamma(1))\oplus N_2=N_1\oplus(\sum_K\oplus S_\gamma(2))$
2) $\mathrm{Hom}_R([S_\gamma(1)], [S_\gamma(2)])=(0)=\mathrm{Hom}_R([S_\gamma(2)], [S_\gamma(1)])$
3) $S_\gamma=S_\gamma(1)\oplus S_\gamma(2)$.

Proof. 1)$\rightarrow$3). 1) implies $S_\gamma=S_\gamma(1)\oplus S_\gamma'=S_\gamma(2)\oplus S_\gamma''$. Hence, $S_\gamma(2)= S_\gamma'\oplus S_\gamma(1)\cap S_\gamma(2)$, since $S_\gamma'\subseteq S_\gamma(2)$. If $S_\gamma(1)\cap S_\gamma(2)=T\neq(0)$, T contains an indecomposable direct summand A of S_γ from Lemma 5. Then $[A]\in[N_1]\cap [N_2]$ from the first decompositions and 1). Therefore, $S_\gamma(1)\cap S_\gamma(2)=(0)$ and $S_\gamma=S_\gamma(1)\oplus S_\gamma(2)$. Other implications are clear from Theorem 3.

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