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ON MULTIPLY TRANSITIVE GROUPS XIII

Dedicated to Professor Mutuo Takahasi on his 60th birthday

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1. Introduction

In this paper we shall prove the following

Theorem. *Let G be a 4-fold transitive group on $\Omega=\{1, 2, \cdots ' n\}$. If the order of the stabilizer of four points in G is not divisible by three, then G is one of the following groups: S_4, S_5, S_6, A_6, M_{11} or M_{12}.*

In the proof of this theorem we shall use the following two lemmas, which will be proved in the section 3 and 4.

Lemma 1. *Let G be a permutation group on $\Omega=\{1, 2, \cdots, n\}$ satisfying the following two conditions.*

(i) *The order of the stabilizer of any four points in G is even and not divisible by three.*
(ii) *Any involution fixing at least four points fixes exactly four or six points.*

Then $G=S_6$ or M_{12}.

Lemma 2. *Let G be a permutation group on $\Omega=\{1, 2, \cdots, n\}$ satisfying the following three conditions.*

(i) *The order of the stabilizer of any four points in G is even and not divisible by three.*
(ii) *Any involution fixing at least four points fixes exactly four or twelve points.*
(iii) *For any 2-subgroup X fixing exactly twelve points, $N(X)^{I(X)}\leq M_{12}$.*

Then $G=S_6$ or M_{12}.

We shall use the same notation as in [4].

2. Proof of the theorem

Let G be a group satisfying the assumption of the theorem. If the order of the stabilizer of four points in G is odd and not divisible by three, then G is S_4, S_5, A_6 or M_{11} by a theorem of M. Hall ([1], Theorem 5.8.1). Hence we may consider only the case in which the stabilizer of four points in G is of even order.

Let P be a Sylow 2-subgroup of G_{1234}. Then $P\neq 1$. If P is semiregular on $\Omega-I(P)$, then G is S_6 or M_{12} by Theorem of [3] and the assumption. Hence from now on we assume that P is not semiregular on $\Omega-I(P)$ and prove the theorem by way of contradiction.

By Corollary of [5] and Theorem of [7], $|I(P)|=4$ or 5. We treat these cases separately.

Case I. $|I(P)|=4$.

(1) *There is a point t in $\Omega-I(P)$ such that $|I(P_t)|=6$ or 12 and $N(P_t)^{I(P_t)}=S_6$ or M_{12} respectively. In particular if t is a point of a minimal P-orbit, then $N(P_t)^{I(P_t)}$ is one of the groups listed above.*

Proof. Since G has no element of order three fixing at least four points, this follows from Corollary of [6].

(2) *Any element of order three fixes no point or exactly three points.*

Proof. By (1), there is a point t in $\Omega-I(P)$ such that $N(P_t)^{I(P_t)}=S_6$ or M_{12}. Then $N(P_t)$ has a 3-element whose restriction on $I(P_t)$ has exactly three fixed points. Since any element of order three fixes at most three points, $|\Omega|\equiv 0 \pmod 3$ and any element of order three fixes no point or exactly three points.

(3) *If G has a 2-subgroup Q such that $|I(Q)|=6$ and $N(Q)^{I(Q)}=S_6$, then there is no 2-subgroup R such that $|I(R)|=12$ and $N(R)^{I(R)}=M_{12}$.*

Proof. Suppose by way of contradiction that there are 2-subgroups Q and R such that $|I(Q)|=6$, $N(Q)^{I(Q)}=S_6$, $|I(R)|=12$ and $N(R)^{I(R)}=M_{12}$. Let $\bar{Q}$ be a Sylow 2-subgroup of $G_{I(Q)}$. Then $|I(\bar{Q})|=6$ and $N(\bar{Q})^{I(\bar{Q})}=S_6$. Similarly let $\bar{R}$ be a Sylow 2-subgroup of $G_{I(R)}$. Then $|I(\bar{R})|=12$ and $N(\bar{R})^{I(\bar{R})}\geq M_{12}$. If $N(\bar{R})^{I(\bar{R})}\neq M_{12}$, then $N(\bar{R})^{I(\bar{R})}\geq A_{12}$. Hence $N(\bar{R})^{I(\bar{R})}$ has an element which is of order three and fixes nine points, contrary to (2). Thus $N(\bar{R})^{I(\bar{R})}=M_{12}$. Hence we may assume that Q and R are Sylow 2-subgroups of $G_{I(Q)}$ and $G_{I(R)}$ respectively.

Since G is 4-fold transitive on Ω, we may assume that P contains Q and R. Then set $I(Q)=\{1, 2, 3, 4, i_1, i_2\}$ and $I(R)=\{1, 2, 3, 4, j_1, j_2, \cdots, j_8\}$. Since $N(Q)^{I(Q)}=S_6$, for any point i of $\{i_1, i_2\}$ $P_i=Q$ and Q is a Sylow 2-subgroup of G_{1234i}. Similarly since $N(R)^{I(R)}=M_{12}$, for any point j of $\{j_1, j_2, \cdots, j_8\}$ $P_j=R$ and R is a Sylow 2-subgroup of G_{1234j}. Hence the G_{1234}-orbit Δ containing i is different from the G_{1234}-orbit Γ containing j. Since $N(Q)^{I(Q)}=S_6$ and

$N(R)^{I(R)}=M_{12}$, $\{i_1, i_2\}\subseteq\Delta$ and $\{j_1, j_2, \cdots, j_8\}\subseteq\Gamma$.

Since $N(Q)^{I(Q)}=S_6$, there is an element

$$x=(1\ 2\ 3)\ (4)\ (i_1)\ (i_2)\cdots.$$

Then $x\in N(G_{1234})$. Hence x induces a permutation on the set of G_{1234}-orbits. Since $\{i_1, i_2\}\subseteq\Delta$ and $\{i_1, i_2\}^x=\{i_1, i_2,\}$ $\Delta^x=\Delta^x$. Since the order of G_{1234} is not divisible by three, the lengts of G_{1234}-orbits in $\Omega-\{1, 2, 3, 4\}$ are not divisible by three. By (2), $I(x)=\{4, i_1, i_2\}$ and so x has no fixed point in $\Omega-(\{1, 2, 3, 4\}\cup\Delta)$. Thus $\Gamma^x\neq\Gamma$. On the other hand since $N(R)^{I(R)}=M_{12}$, there is an element

$$y=(1\ 2\ 3)\ (4)\ (j_1)\ (j_2)\ (j_3\ j_4\ j_5)\ (j_6\ j_7\ j_8)\cdots.$$

Then $y\in N(G_{1234})$. Since $\{j_1, j_2, \cdots, j_8\}\subseteq\Gamma$ and $\{j_1, j_2, \cdots, j_8\}^y=\{j_1, j_2, \cdots, j_8\}$, $\Gamma^y=\Gamma$. Hence $\Gamma^{yx^{-1}}=\Gamma^{x^{-1}}\neq\Gamma$. This is a contrdiction since $yx^{-1}\in G_{1234}$ and Γ is a G_{1234}-orbit. Thus we complete the proof.

(4) *Suppose that P has a subgroup Q such that $|I(Q)|=6$ and $N(Q)^{I(Q)}=S_6$ ($|I(Q)|=12$ and $N(Q)^{I(Q)}=M_{12}$). Let $\bar{Q}$ be a subgroup of P such that the order of $\bar{Q}$ is maximal among all subgroups of P fixing more than six (twelve) points. Set $N=N(\bar{Q})^{I(\bar{Q})}$. Then M satisfies the following conditions.*

(i) *The order of the stabillzer of any four points in N is even and not divisi- by three.*

(ii) *Any involution of N fixing at least four points fixes exactly four or six (twelve) points.*

(iii) *N has an involution fixing exactly six (twelve) points.*

(iv) *When P has a subgroup Q such that $|I(Q)|=12$ and $N(Q)^{I(Q)}=M_{12}$, for any 2-subgroup X of N fixing exactly twelve points, $N_N(X)^{I(X)}\leq M_{12}$.*

Proof. (i), (ii) and (iv) are obvious. (iii) follows immediatly from Theorem 1 in [6].

(5) By Lemma 1 and 2, which will be proved in the section 4, there is no such group N as in (4). Thus we complete the proof of Case I.

Case II. $|I(P)|=5$.

(1) *Let t be a point of a minimal P-orbit in $\Omega-I(P)$. Then $|I(P_t)|=7,9$ or 13. In particular if $|I(P_t)|=9$ or 13, then $N(P_t)^{I(P_t)}\leq A_8$ or $N(P_t)^{I(P_t)}=S_1\times M_{12}$ respectively.*

Proof. This is Theorem of [6].

(2) $|I(P_t)|\neq 7$.

Proof. If $|I(P_t)|=7$, then $N(P_t)^{I(P_t)}$ is one of the groups listed in (2) of Case II in the section 3 of [6]. But these groups have an element of order three fixing four points. Thus $|I(P_t)|\neq 7$.

(3) $|I(P_t)|\neq 9$.

Proof. Suppose by way of contradiction that $|I(P_t)|=9$. Then we may assume that $I(P_t)=\{1, 2, \cdots, 9\}$. Set $N=N(P_t)^{I(P_t)}$. Then for any four points i, j, k and l of $I(P_t)$, N_{ijkl} has an involution fixing exactly five points.

First assume that N is primitive. Then since N is a subgroup of A_9 and has an involution fixing five points, $N=A_9$ (see [9]). But this is a contradiction since N has no element which is of order three and fixes six points.

Next assume that N is transitive but imprimitive. Then N has three blocks $\{i_1, i_2, i_3\}$, $\{j_1, j_2, j_3\}$ and $\{k_1, k_2, k_3\}$ of length three. Let x be an inovlution fixing i_1, i_2, j_1 and j_2. Then x fixes i_3, j_3 and one more point of $\{k_1, k_2, k_3\}$. Thus x is a transposition. This is a contradiction since $N\leq A_9$.

Finally assume that N is intransitive. Then one of the N-orbits is of length less than five.

Suppose that N has an orbit of length one, say $\{1\}$. Then for any four point i, j, k and l of $\{2, 3, \cdots, 9\}$, there is an involution in N fixing exactly five points $1, i, j, k$ and l. Then by a lemma of D. Livingstone and A. Wagner [2], N_1 is 4-fold transitive on $\{2, 3, \cdots, 9\}$. Thus $N=S_1\times A_8$. This is a contradiction since N has no element which is of order three and fixes six points.

Suppose that N has an orbit of length two, say $\{1, 2\}$. Then for any three points i, j and k of $\{3, 4, \cdots, 9\}$, there is an involution in N fixing exactly five points $1, 2, i, j$ and k. Thus by a lemma of D. Livingstone and A. Wagner [2], N_{12} is 3-fold transitive on $\{3, 4, \cdots, 9\}$. Hence by [9], $N_{12}=A_7$. This is a contradiction since N has no element which is of order three and fixes six points.

Suppose that N has an orbit of length three, say $\{1, 2, 3\}$. Set $\Delta=\{4, 5, \cdots, 9\}$. Then for any four points of Δ, there is an involution in N^{Δ} fixing exactly these four points. Hence by a lemma of D. Livingstone and A. Wagner [2], N^{Δ} is 4-fold transitive on Δ and so $N^{\Delta}=S_6$. Thus N has an element

$$x=(4)(5\ 6)(7\ 8\ 9)\cdots.$$

Since $N\leq A_9$, x is an even permutation. Hence x has one more 2-cycle on $\{1, 2, 3\}$. Thus x^2 is of order three and fixes six points, which is a contradiction.

Suppose that N has an orbit of length four, say $\{1, 2, 3, 4\}$. Set $\Delta=\{5, 6, \cdots, 9\}$. Then for any three points i, j and k of Δ, N has an involution fixing i, j, k and two more points of $\{1, 2, 3, 4\}$. Thus by a lemma of D. Livingstone and A. Wagner [2], N^{Δ} is 3-fold transitive on Δ and so $N^{\Delta}=S_5$. Thus N has an element

$$x=(5\ 6)\ (7\ 8\ 9)\ \cdots .$$

Since $N \leq A_9$, x is an even permutation. Hence x has one 2-cycle and two fixed points, or one 4-cycle on $\{1, 2, 3, 4\}$. Thus x^4 is of order three and fixes six points, which is a contradiction.

Thus $|I(P_t)| \neq 9$.

(4) If $|I(P_t)|=13$, then $N(P_t)^{I(P_t)}=S_1 \times M_{12}$. Hence $N(P_t)^{I(P_t)}$ has an element of order three fixing four points, which is a contradiction.

Thus we complete the proof of Case II and so complete the proof of Theorem.

3. Proof of Lemma 1

Let G be a permutation group satisfying the assumptions of Lemma 1. If G has no involution fixing six points, then $G=S_6$ or M_{12} by Theorem 1 in [6] and the assumptions. Hence from now on we assume that G has an involution fixing exactly six points and prove Lemma 1 by way of contradiction. Then we may assume that G has an involution a fixing exactly six points $1, 2, \cdots, 6$ and

$$\mathrm{a}=(1)\ (2) \cdots (6)\ (7\ 8) \cdots .$$

Set $\mathrm{T}=\mathrm{C}(\mathrm{a})_{7\,8}$.

(1) *For any two points i and j of $I(a)$, there is an involution in $T_{i\,j}$. Any involution of T is not the identity on $I(a)$.*

Proof. Since a normalizes $G_{7\,8\,i\,j}$ and $G_{7\,8\,i\,j}$ is of even order, $G_{7\,8\,i\,j}$ has an involution x commuting with a. Then $x \in T_{i\,j}$. Since $|I(a)|=6$ and $I(x) \supseteq \{7, 8\}$, any involution of T is not the identity on $I(a)$ by (ii).

(2) *Any element of order three of T has no fixed points in $I(a)$.*

Proof. If an element u of order three of T has fixed points in $I(a)$, then since $|I(a)|=6$, u fixes at least three points of $I(a)$. This contradicts (i) since $I(u) \supseteq \{7, 8\}$. Thus any element of order three of T has no fixed point in $I(a)$.

(3) *We may assume that $(T^{I(a)})_{1\,2\,3\,4}=1$.*

Proof. By (2), $T^{I(a)} \neq S_6$. Hence there is four points in $I(a)$ such that the stabilizer of these four points in $T^{I(a)}$ is the identity. Hence we may assume that $(T^{I(a)})_{1\,2\,3\,4}=1$.

(4) $T^{I(a)}$ *is one of the following groups.*
(*a*) $T^{I(a)}$ *is intransitive and one of the* $T^{I(a)}$*-orbits is of length one, two or three.*
(*b*) $T^{I(a)}$ *is a transitive but imprimitive group with three blocks of length two or two blocks of length three.*
(*c*) $T^{I(a)}$ *is primitive.*

Proof. This is clear.

(5) $T^{I(a)}$ *has no orbit of length one.*

Proof. Suppose by way of contradiction that $T^{I(a)}$ has an orbit of length one.

First assume that a $T^{I(a)}$-orbit of length one is contained in $\{1, 2, 3, 4\}$. Then we may assume that $\{1\}$ is a $T^{I(a)}$-orbit of length one. By (1), T_{23} has an involution x_1. By (3), we may assume that

$$x_1 = (1)(2)(3)(5)(4\ 6)\cdots .$$

Similarly T_{24} has an involution x_2 of the form

$$x_2 = (1)(2)(4)(5)(3\ 6)\cdots \text{ or } (1)(2)(4)(6)(3\ 5)\cdots .$$

If x_2 is of the first from, then $x_1 x_2 = (1)(2)(5)(3\ 6\ 4)\cdots$, contrary to (2). Thus x_2 is of the second form. Similarly T_{34} has an invloution x_3 of the form

$$x_3 = (1)(3)(4)(5)(2\ 6)\cdots \text{ or } (1)(3)(4)(6)(2\ 5)\cdots .$$

If x_3 is of the first form, then $x_1 x_3 = (1)(3)(5)(2\ 6\ 4)\cdots$, contrary to (2). If x_3 is of the second form, then $x_2 x_3 = (1)(4)(6)(2\ 5\ 3)\cdots$, contrary to (2).

Let $\{i\}$ be a $T^{I(a)}$-orbit of length one. Then as is shown above, for any three points j, k and l of $I(a)-\{i\}$ $(T^{I(a)})_{i\,j\,k\,l} \neq 1$. Hence by a lemma of *D.* Livingstone and *A.* Wagner [2], $(T^{I(a)})_i$ is 3-fold transitive on $I(a)-\{i\}$. Hence $(T^{I(a)})_i = S_5$. Then T has an element which is of order three and has fixed points in $I(a)$, contrary to (2). Thus $T^{I(a)}$ has no orbit of length one.

(6) $T^{I(a)}$ *has neither orbit of length two nor block of length two.*

Proof. Suppose by way of contradiction that $T^{I(a)}$ has an orbit of length two or three blocks of length two.

First assume that $\{1, 2, 3, 4\}$ contains an orbit of length two or a block of length two. Then we may assume that $\{1, 2\}$ is an orbit or a block. By (1), T_{13} has an involution x_1. By (3), we may assume that

$$x_1 = (1)(2)(3)(5)(4\ 6)\cdots .$$

Let x_2 be an involution of T_{14}. Then similarly

$$x_2=(1)\ (2)\ (4)\ (5)\ (3\ 6)\cdots \text{ or } (1)\ (2)\ (4)\ (6)\ (3\ 5)\cdots.$$

If x_2 is of the first form, then $x_1 x_2=(1)(2)(5)(3\ 6\ 4)\cdots$, contrary to (2). Thus x_2 is of the second form. Hence when $T^{I(a)}$ is imprimitive, $\{1, 2\}$, $\{3, 5\}$ and $\{4, 6\}$ form a complete block system. Let x_3 be an involution of T_{34}. When $T^{I(a)}$ is imprimitive

$$x_3=(1\ 2)\ (3)\ (4)\ (5)\ (6)\cdots.$$

When $T^{I(a)}$ has an orbit $\{1, 2\}$, x_3 is of this form or $x_3=(1\ 2)\ (3)\ (4)\ (5\ 6)\cdots$. But if $x_3=(1\ 2)\ (3)\ (4)\ (5\ 6)\cdots$, then $(x_1 x_3)^2=(1)\ (2)\ (3)\ (4\ 6\ 5)\cdots$, contrary to (2). Thus in any case x_3 is of the same form on $I(a)$.

Set $\Delta=\{1, 2, \cdots, 8\}$. Let Q be a Sylow 2-subgroup of $\langle a, x_1, x_2, x_3\rangle$. Then $a\in Z(Q)$, $Q^{\Delta}=\langle a, x_1, x_2, x_3\rangle^{\Delta}$ and $Q_{\Delta}=1$. Hence $Q=\langle a, \bar{x}_1, \bar{x}_2, \bar{x}_3\rangle$, where $\bar{x}_i^{\Delta}=x_i^{\Delta}$ and $\bar{x}_i$ is conjugate to x_i, $i=1, 2, 3$. Thus we may assume that $\langle a, x_1, x_2, x_3\rangle$ is a 2-group. Then $\langle a, x_1, x_2, x_3\rangle$ is elementary abelian. Since $|I(ax_1)|\le 6$, $\langle a, x_1\rangle^{\Omega-\Delta}$ has at most one orbit of length two and the remaining orbits are of length four.

Suppose that $\langle a, x_1\rangle$ has an orbit of length four. Then we may assume that $\{9, 10, 11, 12\}$ is an orbit of length four and

$$a=(1)\ (2)\cdots(6)\ (7\ 8)\ (9\ 10)\ (11\ 12)\cdots,$$
$$x_1=(1)\ (2)\ (3)\ (5)\ (4\ 6)\ (7)\ (8)\ (9\ 11)\ (10\ 12)\cdots.$$

Suppose that x_2 fixes $\{9, 10, 11, 12\}$. Then since $|I(ax_2)|\le 6$ and $|I(x_1x_2)|\le 6$, $x_2=(9\ 12)(10\ 11)$ on $\{9, 10, 11, 12\}$. Hence $\langle a, x_1, x_2\rangle_{9\,10\,11\,12}=\langle ax_1x_2\rangle$ and $I(ax_1x_2)=\{1, 2, 9, 10, 11, 12\}$. Thus $\langle a, x_1, x_2\rangle$ has exactly one orbit $\{9, 10, 11, 12\}$ of length four. Then since x_3 normalizes $\langle a, x_1, x_2\rangle$, x_3 fixes $\{9, 10, 11, 12\}$. Then by the same argument as is used for x_2, x_3 is of the same form as x_2 on $\{9, 10, 11, 12\}$. Hence $I(x_2x_3)\ge\{4, 6, 7, 8, 9, 10, 11, 12\}$, contrary to (ii). Thus x_2 does not fix any $\langle a, x_1\rangle$-orbit of length four. Hence $\langle a, x_1, x_2\rangle^{\Omega-\Delta}$ has at most one orbit of length two and the remaining orbits are of length eight. Hence $\langle a, x_1, x_2, x_3\rangle$-orbits whose lengths are not two are of length eight or sixteen. If $\langle a, x_1, x_2, x_3\rangle$ has an orbit of length eight, then $\langle a, x_1, x_2, x_3\rangle$ has an involution fixing at least eight points of this orbit, contrary to (ii). Thus $\langle a, x_1, x_2, x_3\rangle^{\Omega-\Delta}$ has at most one orbit of lenght two and is semiregular on the set consisting of the remaining points. Since $\langle a, x_1\rangle$ nomralizes $G_{9\,10\,11\,12}$ and $G_{9\,10\,11\,12}$ is of even order, there is an involution y in $G_{9\,10\,11\,12}$ commuting with a and x_1. Then y fixes $\{1, 2, 3, 5\}$, $\{4, 6\}$ and $\{7, 8\}$. Suppose that $y^{\Delta}\in\langle a, x_1, x_2, x_3\rangle^{\Delta}$. Then since $\langle a, x_1, x_2, x_3, y\rangle_{\Delta}$ is of odd order, $\langle a, x_1, x_2, x_3\rangle$ is a Sylow 2-subgroup of $\langle a, x_1, x_2, x_3, y\rangle$. Hence $\langle a, x_1, x_2, x_3\rangle$ has an element which is conjugate to y in $\langle a, x_1, x_2, x_3, y\rangle$. This is a contradiction since any involution of $\langle a, x_1, x_2, x_3\rangle$ fixes at most two points of $\Omega-\Delta$. Thus $y^{\Delta}\notin\langle a, x_1, x_2, x_3\rangle^{\Delta}$. Hence $\{1, 2\}^{y}=\{3, 5\}$. On the other hand since y fixes $\{7, 8\}$, y or ya is contained in T. Thus $\{1, 2\}$ is not a T-orbit. Then $T^{I(a)}$ is imprimitive and we may assume that $y=(1\ 3)\ (2\ 5)$ on $\{1, 2, 3, 5\}$. Then $x_2 y$ is of order $4m$, where m is odd. Set $z=(x_2 y)^{2m}$. Then

$$z=(1\ 2)\ (3\ 5)\ (4)\ (6)\ (7)\ (8)\cdots$$

and z centralizes $\langle a, x_1, x_2, y\rangle$. Since $|I(y)|\leq 6$, y fixes exactly four points 9, 10, 11 and 12 in $\Omega-\Delta$. Hence z fixes $\{9, 10, 11, 12\}$. Thus the $\langle a, x_1, x_2, z\rangle$-orbit containing $\{9, 10, 11, 12\}$ is of length eight. Since $\langle a, x_1, x_2, z\rangle$ is abelian and of order sixteen, there is an involution fixing this $\langle a, x_1, x_2, z\rangle$-orbit of length eight pointwise, contrary to (ii). Thus $\langle a, x_1\rangle$ has no orbit of length four. Since $|I(ax_1)|\leq 6$, $|\Omega|=8$ or 10.

Suppose that $|\Omega|=8$. Then by (i), there is an involution x in G fixing 1, 3, 4 and 7. If x fixes 8, then $x\in T$. Hence x fixes 2. Then $x^{I(a)}\in(T^{I(a)})_{1\,2\,3\,4}$ and $x^{I(a)}\neq 1$, contrary to (3). Hence $x=(1)\ (3)\ (4)\ (7)\ (8\ i)\cdots$, $i\in\{2, 5, 6\}$. Then $(ax)^2=(7\ 8\ i)$, contrary to (i).

Suppose that $|\Omega|=10$. Then

$$\begin{aligned} a &= (1)\ (2)\cdots(6)\ (7\ 8)\ (9\ 10)\,,\\ x_1 &= (1)\ (2)\ (3)\ (5)\ (4\ 6)\ (7)\ (8)\ (9\ 10)\,,\\ x_2 &= (1)\ (2)\ (3\ 5)\ (4)\ (6)\ (7)\ (8)\ (9\ 10)\,. \end{aligned}$$

By (i), there is an involution x in G fixing 1, 3, 4 and 7. Assume that x fixes 8. If x commutes with a, then $x\in T$. Hence x fixes 2. Then $x^{I(a)}\in(T^{I(a)})_{1\,2\,3\,4}$ and $x^{I(a)}\neq 1$, contrary to (3). Thus x does not commute with a and so $\{9, 10\}^{x}\neq\{9, 10\}$. If x fixes 9, then $x=(9)\ (10\ i)\cdots$, $i\in\{2, 5, 6\}$. Hence $(ax)^2=(9\ 10\ i)$, contrary to (i). Similarly x does not fix 10. Thus $x=(9\ i)\ (10\ j)$, $\{i, j\}\subset\{2, 5, 6\}$. Then $(x_1 x_2 x)^2$ is of order three and fixes at lesat four points, contrary to (i). Thus x does not fix 8. Hence $x=(1)\,(3)\,(4)\,(7)\,(8\ i)\cdots$, $i\in\{2, 5, 6, 9, 10\}$. If $i\in\{2, 5, 6\}$, then $ax=(1)\ (3)\ (4)\ (8\ 7\ i)\cdots$. Since $|\Omega|=10$, a suitable power of ax is of order three and fixes at least four points, contrary to (i). If $i\in\{9, 10\}$, then $ax_1x=(1)\ (3)\ (8\ 7\ i)\cdots$. Then similarly we have a contradiction. Hence $\{1, 2\}$ is neither orbit nor block.

Let $\{i, j\}$ be an orbit or a block of $T^{I(a)}$. Then by what we have proved above, for any two points k and l of $\{1, 2, \cdots, 6\}-\{i, j\}$ there is an involution in $(T^{I(a)})_{ijkl}$. Hence by a lemma of D. Livingstone and A. Wagner [2], $(T^{I(a)})_{i\,j}$ is doubly transitive on $I(a)-\{i, j\}$. Hence $(T^{I(a)})_{i\,j}=S_4$. Then $(T^{I(a)})_{i\,j}$ has an element of order three, contrary to (2). Thus $T^{I(a)}$ has neither orbit of length two nor block of length two.

(7) *$T^{I(a)}$ has neither orbit of length three nor block of length three.*

Proof. Suppose by way of contradiction that $T^{I(a)}$ has an orbit of length three or two blocks of length three. When $T^{I(a)}$ is intransitive, $T^{I(a)}$ has two orbits of length three by (5) and (6). Let $\{i_1, i_2, i_3\}$ and $\{j_1, j_2, j_3\}$ be the two orbits or the two blocks. Then $T_{i_1 i_2}$ has an involution

$$x = (i_1)(i_2)(i_3)(j_1)(j_2 j_3)\cdots .$$

Since $\{j_1, j_2, j_3\}$ is an orbit or a block and $x \in T_{i_1 i_2 i_3}$, $(T^{I(a)})_{i_1 i_2 i_3} = S_3$. Thus $(T^{I(a)})_{i_1 i_2 i_3}$ has an element of order three, contrary to (2). Hence $T^{I(a)}$ has neither orbit of length three nor block of length three.

(8) *We show that $T^{I(a)}$ is not primitive and complete the proof.*

Proof. Suppose by way of contradiction that $T^{I(a)}$ is primitive. Then since any element of order three in $T^{I(a)}$ has no fixed point, $T^{I(a)} = PSL(2, 5)$ or $PGL(2, 5)$ (see [9]). Let u be an element of order three of T. Since u commutes with a, if u has a fixed point in $\Omega - (I(a) \cup \{7, 8\})$, then u fixes at least two points of $\Omega - (I(a) \cup \{7, 8\})$, contrary to (i). Thus $I(u) = \{7, 8\}$ and so $|\Omega| \equiv 2 \pmod 3$. Furthermore this shows that any element of order three fixes exactly two points of Ω. Hence $N(G_{I(a)})^{I(a)}$ has no element consisting of exactly one 3-cycle. Thus $N(G_{I(a)})^{I(a)} \not\geq A_6$. Then since $T^{I(a)} = PSL(2, 5)$ or $PGL(2, 5)$, $N(G_{I(a)})^{I(a)} = PSL(2, 5)$ or $PGL(2, 5)$. Furthermore this shows that for any involution v fixing exactly six points, $N(G_{I(v)})^{I(v)} = PSL(2, 5)$ or $PGL(2, 5)$.

Suppose that G has an involution x fixing exactly four points. Then x is of the form

$$x = (i_1)(i_2)(i_3)(i_4)(j_1 j_2)\cdots .$$

For any two points i_r and i_s of $\{i_1, i_2, i_3, i_4\}$ x normalizes $G_{j_1 j_2 i_r i_s}$. Hence by (i), $G_{j_1 j_2 i_r i_s}$ has an involution y commuting with x. If y fixes $I(x)$ pointwise, then $I(y) = I(x) \cup \{j_1, j_2\}$. Thus $|I(y)| = 6$ and $x^{I(y)} = (j_1 j_2)$. This is a contradiction since $N(G_{I(y)})^{I(y)} = PSL(2, 5)$ or $PGL(2, 5)$. Hence y fixes exactly two points i_r and i_s in $I(x)$. Hence by a lemma of D. Levingstone and A. Wagner [2], $(C(x)_{j_1 j_2})^{I(x)} = S_4$. Thus $C(x)_{j_1 j_2}$ has a 3-element of the form $(i_1 i_2 i_3)(i_4)(j_1)(j_2)\cdots$. This is a contradiction since every element of order treee fixes exactly two points. Thus G has no involution fixing exactly four points.

Let x be an involution of T_{12}. Then we may assume that

$$a = (1)(2)\cdots(6)(7\,8)(9\,10)\cdots ,$$
$$x = (1)(2)(3\,4)(5\,6)(7)(8)(9)(10)\cdots .$$

Let $(i\ j)$ be any 2-cycle of a. Then $(C(a)_{i\ j})^{I(a)}=PSL\,(2, 5)$ or $PGL(2, 5)$. Since $N(G_{I(a)})^{I(a)}$ is also $PSL\,(2, 5)$ or $PGL\,(2, 5)$, $T^{I(a)}=(C(a)_{i\ j})^{I(a)}$ or one of these two groups is a subgroup of the other. Hence there are 3-elements u and u' in T and $C(a)_{i\ j}$ respectively such that $u^{I(a)}=u'^{I(a)}$. Then u and u' normalize $G_{I(a)}$, $I(u)=\{7, 8\}$ and $I(u')=\{i, j\}$. Let Γ be the $G_{I(a)}$-orbit containing $\{7, 8\}$. Then since $\{7, 8\}^u=\{7, 8\}$, $\Gamma^u=\Gamma$. Suppose that $\{i, j\}$ is contained in a $G_{I(a)}$-orbit different form Γ. Since the order of $G_{I(a)}$ is not divisible by three, $|\Gamma|$ is not divisible by three. Hence $\Gamma^{u'}\neq\Gamma$. Thus $\Gamma^{uu'^{-1}}=\Gamma^{u'^{-1}}\neq\Gamma$. This is a contradiction since $uu'^{-1}\in G_{I(a)}$. Thus $\{i, j\}\subset\Gamma$. Since $(i\ j)$ is any 2-cycle of a, $G_{I(a)}$ is transitive on $\Omega-I(a)$. From the same reason, $G_{I(x)}$ is transitive on $\Omega-I(x)$. Then since $I(\langle G_{I(a)}, G_{I(x)}\rangle)=\{1, 2\}$, $G_{1\,2}$ is transitive on $\Omega-\{1, 2\}$. Since $N(G_{I(a)})$ is doubly transitive on $I(a)$, G is 3-fold transitive on Ω.

Let Q be a Sylow 2-subgroup of $G_{I(a)}$. Since $N(Q)^{I(a)}=N(G_{I(a)})^{I(a)}$, $(N(Q)^{I(a)})_{1\,2\,3}=1$. Hence Q is a Sylow 2-subgroup of $G_{1\,2\,3}$. Since $|I(Q)|=6$, G is not 4-fold transitive by Theorem of [4]. On the other hand $G_{I(a)}$ is transitive on $\Omega-I(a)$. Hence there is a point i_1 in $\{4, 5, 6\}$ such that i_1 does not belong to the $G_{1\,2\,3}$-orbit containing $\Omega-I(a)$. Since Q is a Sylow 2-subgroup of $G_{1\,2\,3}$, the length of the $G_{1\,2\,3}$-orbit containing i_1 is not two. Moreover the length of the $G_{1\,2\,3}$-orbit containing i_1 is not three since $G_{1\,2\,3}$ has no element of order three. Thus $G_{1\,2\,3}$ fixes i_1. Since Q is a Sylow 2-subgroup of $G_{1\,2\,3}$, $\{4, 5, 6,\}-\{i_1\}$ is not a $G_{1\,2\,3}$-orbit. Similarly since $|\{4, 5, \cdots, n\}-\{i_1\}|$ is even, $\{4, 5, \cdots, n\}-\{i_1\}$ is not a $G_{1\,2\,3}$-orbit. Hence $G_{1\,2\,3}$-orbits on $\Omega-\{1, 2, 3\}$ are $\{4\}$, $\{5\}$, $\{6\}$ and $\{7, 8, \cdots, n\}$ or $\{i_1\}$, $\{i_2\}$ and $\{i_3, 7, 8, \cdots, n\}$, where $\{i_1, i_2, i_3\}=\{4, 5, 6\}$. First assume that $\{4\}$, $\{5\}$, $\{6\}$ and $\{7, 8, \cdots, n\}$ are $G_{1\,2\,3}$-orbits. By (i), $G_{1\,2\,3\,7}$ has an involution y. Then $y\in G_{1\,2\,3}$. Hence $I(y)\supset\{1, 2, \cdots, 7\}$, contraty to (ii). Next assume that $\{i_1\}$, $\{i_2\}$ and $\{i_3, 7, 8, \cdots, n\}$ are $G_{1\,2\,3}$-orbits. Since G is 3-fold transitive on Ω, $G_{1\,2\,3}=G_{1\,2\,i_1}=G_{1\,2\,i_2}$ and $G_{1\,2\,3}\neq G_{1\,2\,i_3}$. Thus $G_{1\,2\,i_3}$ fixes exactly two points of $\Omega-\{1, 2, \cdots, 6\}$. This is a contradiction since $a\in G_{1\,2\,i_3}$ and a has no fixed point in $\Omega-\{1, 2, \cdots, 6\}$.

Thus we complete the proof of Lemma 1.

4. Proof of Lemma 2

The proof of Lemma 2 is similar to the proof of Lemma 1. Let G be a permutation group satisfying the assumptions of Lemma 2. If G has no involution fixing twelve points, then $G=S_6$ or M_{12} by Theorem 1 and the assumptions. Hence from now on we assume that G has a involution fixing exactly twelve points and prove Lemma 2 by way of contradiction. Then we may assume that G has an involution a fixing exactly twelve points $1, 2, \cdots, 12$ and

$$a=(1)\,(2)\cdots(12)\,(13\ 14)\cdots.$$

Set $T=C(a)_{13\,14}$.

(1) *For any two points i and j of* $I(a)$, *there is an involution in* $T_{i\,j}$. *Any involution of* T *is not the identity on* $I(a)$.

(2) *Any element of order three in* T *has no fixed point on* $I(a)$.

The proofs of (1) and (2) are similar to the proofs of (3.1) and (3.2) respectively.

(3) $T^{I(a)}$ *is one of the following groups.*
(a) $T^{I(a)}$ *is intransitive and one of the* $T^{I(a)}$*-orbits is of length one, two, three, four, five or six.*
(b) $T^{I(a)}$ *is a transitive but imprimitive group with six blocks of length two, four blocks of length three, three blocks of length four or two blocks of length six.*
(c) $T^{I(a)}$ *is primitive.*

Proof. This is clear.

(4) $T^{I(a)}$ *is not primitive.*

Proof. If $T^{I(a)}$ is primitive, then by (iii) $T^{I(a)}$ is $PSL\,(2, 11)$, M_{11} or M_{12}, which are of degree twelve (see [9]). But since $T^{I(a)}$ has an involution fixing at least two points by (1), $T^{I(a)}\neq PSL\,(2, 11)$. Furthermore since any element of order three of $T^{I(a)}$ has no fixed point by (2), $T^{I(a)}\neq M_{11}$, M_{12}. Thus $T^{I(a)}$ is not primitive.

(5) $T^{I(a)}$ *has no orbit of length one.*

Proof. If $T^{I(a)}$ has an orbit $\{i\}$ of length one, then $T^{I(a)-\{i\}}$ is one of the groups of (4) of Lemma 4 in [5]. But all these groups have an element of order three which has fixed points, contrary to (2). Thus $T^{I(a)}$ has no orbit of length one.

(6) $T^{I(a)}$ *has neither orbit of length three nor block of length three.*

Proof. Suppose by way of contradiction that $T^{I(a)}$ has an orbit of length three or a block of length three, say $\{1, 2, 3\}$. Let x_1 be an involution of $T_{1\,2}$. Then we may assume that

$$x_1=(1)(2)(3)(4)(5\,6)(7\,8)(9\,10)(11\,12)\cdots.$$

When $T^{I(a)}$ is transitive but imprimitive, we may assume that the block containing 4 is $\{4, 5, 6\}$. Assume that $T^{I(a)}$ is intransitive. If the length of the orbit containing 4 is not divisible by three, then $(T^{I(a)})_4$ has an element of order three, contrary to (2). If the length of the orbit containing 4 is nine, then $(T^{I(a)})_1$ has an element of order three, contrary to (2). Thus the length of the orbit containing 4 is three or six. On the other hand x_1 fixes exactly one point 4 in the orbit containing 4. Hence the length of the orbit containing 4 is three. Thus we may assume that $\{4, 5, 6\}$ is an orbit.

Let x_2 be an involution of $T_{1\,5}$. Then x_2 fixes $\{1, 2, 3\}$ and $\{4, 5, 6\}$. If $x_2=(1)(5)(4\ 6)\cdots$, then $x_1x_2=(1)(4\ 6\ 5)\cdots$, contrary to (2). Hence x_2 fixes $\{4, 5, 6\}$ pointwise. Since $|I(x_2^{I(a)})|=4$,

$$x_2 = (1)\,(2\ 3)\,(4)\,(5)\,(6)\cdots .$$

Let x_3 be an involution of $T_{2\,5}$. Then by the same argument as is used for x_2,

$$x_3 = (2)\,(1\ 3)\,(4)\,(5)\,(6)\cdots .$$

Then $x_2\,x_3=(1\ 3\ 2)\,(4)\,(5)\,(6)\cdots$, contrary to (2). Thus $T^{I(a)}$ has neither orbit of length three nor block of length three.

(7) *$T^{I(a)}$ has no subgroup which is isomorphic to the following group $\langle x_1, x_2, x_3\rangle$ as a permutation group.*

$$\begin{aligned} x_1 &= (1)\ (2)\ (3)\ (4)\ (5\ 6)\ (7\ 8)\ (9\ 10)\ (11\ 12)\,, \\ x_2 &= (1)\ (2)\ (3\ 4)\ (5)\ (6)\ (7\ 8)\ (9\ 11)\ (10\ 12)\,, \\ x_3 &= (1\ 2)\ (3\ 4)\ (5)\ (6)\ (7)\ (8)\ (9\ 10)\ (11\ 12)\,. \end{aligned}$$

Proof. This follows from the same argument as in the proof of (3.3) in [8].

(8) *$T^{I(a)}$ has neither orbit of length four nor block of length four.*

Proof. Suppose by way of contradiction that $T^{I(a)}$ has an orbit of length four or a block of length four, say $\{1, 2, 3, 4\}$.

First assume that T has an involution x_1 fixing $\{1, 2, 3, 4\}$ pointwise. Then we may assume that

$$x_1 = (1)(2)(3)(4)(5\,6)(7\,8)(9\,10)(11\,12)\cdots .$$

Let x_2 be an involution of $T_{1\,5}$. Then x_2 fixes $\{1, 2, 3, 4\}$ and so $x_2^{I(a)}$ commutes with $x_1^{I(a)}$. Hence we may assume that

$$x_2 = (1)(2)(3\,4)(5)(6)(7\,8)(9\,11)(10\,12)\cdots .$$

Let x_3 be an involution of $T_{3\,5}$. Then similarly $x_3^{I(a)}$ commutes with $x_1^{I(a)}$. Hence $x_3^{I(a)}$ fixes 3, 5 and 6. Since $|I(x_3^{I(a)})|=4$, $x_3^{I(a)}$ fixes one more point of $\{1, 2, 4\}$. If x_3 fixes 1 or 2, then $x_2\,x_3=(1)\,(2\ 4\ 3)\,(5)\,(6)\cdots$ or $(2)\,(1\ 4\ 3)\,(5)\,(6)\cdots$ respectively, contrary to (2). Thus x_3 fixes 4. Then $x_3^{I(a)}$ commutes with $x_2^{I(a)}$ and so

$$x_3 = (1\,2)(3)(4)(5)(6)(7\,8)(9\,12)(10\,11)\cdots .$$

This is a contradiction since $T^{I(a)}$ has no such subgroup as $\langle x_1, x_2, x_3\rangle^{I(a)}$ by (7).

Next assume that T has no involution fixing $\{1, 2, 3, 4\}$ pointwise. Let x_1 be an involution of $T_{1\,2}$. Then

$$x_1 = (1)\,(2)\,(3\ 4)\,\cdots .$$

Let x_2 be an involution of $T_{1\,3}$. Then

$$x_2 = (1)\,(3)\,(2\ 4)\,\cdots .$$

Then $x_1 x_2 = (1)\ (2\ 4\ 3)\cdots$, contrary to (2). Thus $T^{I(a)}$ has neither orbit of length four nor block of length four.

(9) *$T^{I(a)}$ has no orbit of length five.*

Proof. If $T^{I(a)}$ has an orbit Δ of length five, then $T^{I(a)}$ has an involution fixing exactly three points of Δ. Thus $T^{\Delta}=S_5$ (see [9]). Then T^{Δ} has an element of order three fixing two points, contrary to (2). Thus $T^{I(a)}$ has no orbit of length five.

(10) *$T^{I(a)}$ has no orbit of length two. If $T^{I(a)}$ is a transitive but imprimitive group with six blocks of length two, then $T^{I(a)}$ is also a transitive but imprimitive group with two blocks of length six.*

Proof. Suppose that $T^{I(a)}$ has an orbit of length two or a block of length two, say $\{1, 2\}$, Since $(T^{I(a)})_{1\,2}$ is a subgroup of M_{10} and has no element of order three, the order of $(T^{I(a)})_{1\,2}$ is $2^r 5^s$, where $4 \geq r \geq 1$ and $s=0$ or 1.

Assume that $s=0$. Then the subgroup H of T fixing $\{1, 2\}$ as a set is a 2-group on $I(a)$. Since $(T^{I(a)})_{1\,2}$ is a normal subgroup of $H^{I(a)}$, $T_{1\,2}$ has an involution x_1 whose restriction on $I(a)$ is a central involution of $H^{I(a)}$. Then we may assume that

$$x_1 = (1)\,(2)\,(3)\,(4)\,(5\ 6)\,(7\ 8)\,(9\ 10)\,(11\ 12)\,\cdots .$$

When $T^{I(a)}$ is imprimitive, $\{3, 4\}$ is a block of $T^{I(a)}$ since $I(x_1{}^{I(a)})=\{1, 2, 3, 4\}$.

Let x_2 be an involution of $T_{1\,5}$. Then $x_2 \in T_{1\,2}$. Hence $x_2{}^{I(a)}$ commutes with $x_1{}^{I(a)}$. Hence we may assume that

$$x_2 = (1)\,(2)\,(3\ 4)\,(5)\,(6)\,(7\ 8)\,(9\ 11)\,(10\ 12)\,\cdots.$$

Let x_3 be an involution of $T_{3\,5}$. When $\{1, 2\}$ is a T-orbit, $x_3{}^{I(a)}$ commutes with $x_1{}^{I(a)}$. Hence $x_3=(1\ 2)\ (3)\ (4)\ (5)\ (6)\cdots$. Hence $x_3{}^{I(a)}$ commutes with $x_2{}^{I(a)}$. When $T^{I(a)}$ is imprimitive, $\{5, 6\}$ is a block of $T^{I(a)}$ since $I(x_2{}^{I(a)})=\{1, 2, 5, 6\}$. Hence x_3 fixes $\{3, 4, 5, 6\}$ pointwise. Hence $x_3{}^{I(a)}$ commutes with $x_1{}^{I(a)}$ and $x_2{}^{I(a)}$. Thus in any case

$$x_3=(1\,2)(3)(4)(5)(6)(7\,8)(9\,12)(10\,11)\cdots.$$

Then since T has no such subgroup as $\langle x_1, x_2, x_3\rangle$ by (7), we have a contradiction.

Thus $s=1$. Since the order of $(T^{I(a)})_{12}$ is $2^r\,5$, $(T^{I(a)})_{12}$ is solvable. Let N be a minimal normal subgroup of $(T^{I(a)})_{12}$. Then N is elementary abelian. Let u be an element of T_{12} such that the order of $u^{I(a)}$ is five. Suppose that N is a 2-group. Since N is an elementary abelian subgroup of M_{10}, the order of N is two or four. Hence $u^{I(a)}$ centralizes N. This is a contradiction since $u^{I(a)}$ consists of two 5-cycles on $I(a)-\{1, 2\}$ and any involution of N has exactly two fixed points in $I(a)-\{1, 2\}$. Thus N is a 5-group. Hence $\langle u\rangle^{I(a)}$ is normal in $(T^{I(a)})_{12}$ and so the unique Sylow 5-subgroup of $(T^{I(a)})_{12}$.

Suppose that $\{1, 2\}$ is a T-orbit. Then $(T^{I(a)})_{12}$ is normal in $T^{I(a)}$. Since $\langle u\rangle^{I(a)}$ is the unique Sylow 5-subgroup of $(T^{I(a)})_{12}$, $\langle u\rangle^{I(a)}$ is normal in $T^{I(a)}$. Let Δ be a $\langle u\rangle^{I(a)}$-orbit of length five. Then for any two points i and j of Δ, T_{ij} has an involution x, which fixes Δ. Since $|I(x^{I(a)})|=4$ and $|\Delta|=5$, $|I(x)\cap\Delta|=3$. Thus the subgroup of T fixing Δ as a set is S_5 on Δ. Hence T has an element of order three fixing two points of Δ, contrary to (2). Thus $T^{I(a)}$ has no orbit of length two.

Suppose that $T^{I(a)}$ is imprimitive. Let x_1 be an involution of T_{13}. Then we may assume that

$$x_1=(1)(2)(3)(4)(5\,6)(7\,8)(9\,10)(11\,12)\cdots.$$

Since $(\langle u\rangle^{x_1})^{I(a)}=\langle u\rangle^{I(a)}$ and x_1 is of order two, $(u^{x_1})^{I(a)}=u^{I(a)}$ or $(u^{-1})^{I(a)}$. Since x_1 fixes exactly two points of $I(a)-\{1, 2\}$ and u has no fixed point in $I(a)-\{1, 2\}$, $(u^{x_1})^{I(a)}\neq u^{I(a)}$. Thus $(u^{x_1})^{I(a)}=(u^{-1})^{I(a)}$. Hence we may assume that

$$u=(1)(2)(3\,5\,7\,8\,6)(4\,9\,11\,12\,10)\cdots.$$

Since $T^{I(a)}$ is an imprimitive group with blocks of length two and x_1 fixes a block containing 3, $\{3, 4\}$ is a block. Then $\{3, 4\}^{u^i}$, $0\leq i\leq 4$, is also a block. Thus $\{1, 2\}$, $\{3, 4\}$ $\{5, 9\}$, $\{7, 11\}$, $\{8, 12\}$ and $\{6, 10\}$ are a complete block system of $T^{I(a)}$.

Since $u\in T_{12}$, $(T^{I(a)})_{12}$ is transitive or has two orbits of length five on $I(a)-\{1, 2\}$. Suppose that $(T^{I(a)})_{12}$ is transitive on $I(a)-\{1, 2\}$. Then since $\langle u\rangle^{I(a)}$ is a normal subgroup of $(T^{I(a)})_{12}$, T_{12} has a 2-element x such that $\{3, 5, 7, 8, 6\}^x=\{4, 9, 11, 12, 10\}$. Then $|I(x)\cap I(a)|=2$ and so $x^{I(a)}$ is of order eight. Then $(x^4)^{I(a)}$ is of order two and fixes exactly two points of $I(a)-\{1, 2\}$. Hence $(u^{x^4})^{I(a)}=(u^{-1})^{I(a)}$. Hence $x^{I(a)}$ induces an automorphism of order eight of $\langle u\rangle^{I(a)}$ by conjugation. This is a contradiction since the order of $\langle u\rangle^{I(a)}$ is five. Hence $(T^{I(a)})_{12}$ has two orbits of length five on $I(a)-\{1, 2\}$. Then since $(T^{I(a)})_1=(T^{I(a)})_{12}$, $(T^{I(a)})_1$ has three orbits $\{2\}$, $\{3, 5, 6, 7, 8\}$ and

$\{4, 9, 10, 11, 12\}$ on $I(a)-\{1\}$.

Let x_2 be an involution of T_{56}. Since $\{5, 9\}$ and $\{6, 10\}$ are blocks of $T^{I(a)}$, x_2 fixes 5, 9, 6 and 10. Hence $x_2^{I(a)}$ commutes with $x_1^{I(a)}$. Then x_2 fixes $\{1, 2, 3, 4\}$. If $x_2=(1\ 2)\ (3\ 4)\ (5)\ (6)\ (9)\ (10)\cdots$, then x_2 normalizes T_{12} and $(\langle u\rangle^{x_2})^{I(a)}\neq\langle u\rangle^{I(a)}$. This is a contradiction since $\langle u\rangle^{I(a)}$ is the unique Sylow 5-subgroup of $(T^{I(a)})_{12}$. Hence we may assume that

$$x_2=(1\,3)(2\,4)(5)(6)(9)(10)(7\,8)(11\,12)\cdots.$$

Then $\langle T_1, x_2\rangle^{I(a)}$ has two orbits $\{1, 3, 5, 6, 7, 8\}$ and $\{2, 4, 9, 10, 11, 12\}$. Thus $T^{I(a)}$ is also an imprimitive group with blocks of length six.

(11) *We show that $T^{I(a)}$ has neither orbit of length six nor block of length six and complete the proof.*

Proof. Suppose by way of contradiction that $T^{I(a)}$ has an orbit of length six or a block of length six, say $\{1, 2, \cdots, 6\}$. Set $\Delta=\{1, 2, \cdots, 6\}$.

Assume that T has an involution fixing exactly four points of Δ. Then we may assume that T has an involution

$$x_1=(1)(2)(3)(4)(5\,6)(7\,8)(9\,10)(11\,12)\cdots.$$

Let x_2 be an involution of T_{15}. Then x_2 fixes Δ. If $x_2=(1)\ (5)\ (6\ i)\cdots, i\in\{2, 3, 4\}$, then $x_1x_2=(1)\ (5\ i\ 6)\cdots$, contrary to (2). Hence x_2 fixes 6. Then x_2 fixes $\{1, 2, 3, 4\}$ and so $x_2^{I(a)}$ commutes with $x_1^{I(a)}$. Hence we may assume that

$$x_2=(1)(2)(3\,4)(5)(6)(7\,8)(9\,11)(10\,12)\cdots.$$

Let x_3 be an involution of T_{35}. Then by the same argument as is used for x_2, $x_3^{I(a)}$ commutes with $x_1^{I(a)}$ and $x_3=(1\ 2)\ (3)\ (4)\ (5)\ (6)\cdots$. Hence $x_3^{I(a)}$ commutes with $x_2^{I(a)}$. Hence

$$x_3=(1\,2)(3)(4)(5)(6)(7\,8)(9\,12)(10\,11)\cdots.$$

Then since T has no such subgroup as $\langle x_1, x_2, x_3\rangle$ by (7), we have a contradiction.

Thus T has no involution fixing four points of Δ. Then we may assume that T has an involution

$$x_1=(1)(2)(3\,4)(5\,6)(7)(8)(9\,10)(11\,12)\cdots.$$

Since $I(x_1)\supset\{1, 2, 7, 8, 13, 14\}$, $|I(x_1)|=12$ by (i). Hence we may assume that

$$\begin{aligned}a&=(1)\ (2)\cdots(12)\ (13\ 14)\ (15\ 16)\ (17\ 18)\ (19\ 20)\cdots,\\ x_1&=(1)\ (2)\ (3\ 4)\ (5\ 6)\ (7)\ (8)\ (9\ 10)\ (11\ 12)\ (13)\ (14)\cdots(20)\cdots.\end{aligned}$$

Let x_2 be an involution of T_{13}. Then x_2 fixes Δ and $I(x_2)\cap\Delta=\{1, 3\}$. If $x_2=(1)\ (3)\ (2\ 4)\cdots$, then $x_1 x_2=(1)\ (2\ 4\ 3)\cdots$, contrary to (2). Hence we may assume that $x_2=(1)\ (3)\ (2\ 5)\ (4\ 6)\cdots$. Then $x_1 x_2=(1)\ (2\ 5\ 4\ 3\ 6)\cdots$. Thus $(x_1 x_2)^{I(a)}$ is of order five and so $(x_1 x_2)^{I(a)}$ has one more fixed points in $I(a)-\Delta$. Hence we may assume that

$$x_2 = (1)\ (3)\ (2\ 5)\ (4\ 6)\ (7)\ (8\ 11)\ (10\ 12)\cdots.$$

Hence

$$x_1 x_2 = (1)\ (2\ 5\ 4\ 3\ 6)\ (7)\ (8\ 11\ 10\ 9\ 12)\cdots.$$

Thus the subgroup of T fixing Δ as a set is doubly transitive on Δ and on $I(a)-\Delta$.

Since the order $T^{I(a)}$ is divisible by three, T has an element u of order three. Then by (2), u has no fixed point in $I(a)$. Thus u fixes exactly two points 13 and 14 in $I(a)\cup\{13, 14\}$. Since u commutes with a, if u has fixed points in $\Omega-(I(a)\cup\{13, 14\})$, then u fixes at least two points of $\Omega-(I(a)\cup\{13, 14\})$, contrary to (ii). Thus u has no fixed point in $\Omega-(I(a)\cup\{13, 14\})$ and so $I(u)=\{13, 14\}$. This shows that $|\Omega|\equiv 2 \pmod 3$. Hence any element of order three has exactly two fixed points.

Now we consider $N(G_{I(a)})$. Let H be the subgroup of $N(G_{I(a)})$ fixing Δ as a set and $\bar{H}$ the subgroup of T fixing Δ as a set. Since $\bar{H}$ is doubly transitive on Δ, H is doubly transitive on Δ. Hence $H^{\Delta}=S_6$, A_6, $PGL(2, 5)$ or $PSL(2, 5)$ (see [9]). Since any element of order three fixes exactly two points and $|I(a)|=12$, any element of order three of $N(G_{I(a)})$ has no fixed point in $I(a)$. Hence $H^{\Delta}=PGL(2, 5)$ or $PSL(2, 5)$. Thus $\bar{H}^{I(a)}=H^{I(a)}$ or the index of $\bar{H}^{I(a)}$ in $H^{I(a)}$ is two. If $N(G_{I(a)})$ is transitive on $I(a)$, then by the same argument as is used in the proof of (4) $N(G_{I(a)})^{I(a)}$ is imprimitive. Then $(N(G_{I(a)})^{I(a)})_1$ is not transitive on $I(a)-\{1\}$. Moreover since any element of order three of $N(G_{I(a)})$ has no fixed point in $I(a)$, $(N(G_{I(a)})^{I(a)})_1$ has no orbit of length six. Hence $(N(G_{I(a)})^{I(a)})_1$-orbits are $\{7\}$, $\Delta-\{1\}$ and $I(a)-(\Delta\cup\{7\})$ on $I(a)-\{1\}$, which are $(T^{I(a)})_1$-orbits. Thus when $N(G_{I(a)})^{I(a)}$ is imprimitive, $N(G_{I(a)})^{I(a)}$ has two blocks of length six, which are orbits or blocks of $T^{I(a)}$. This implies that for any involution x fixing exactly twelve points $N(G_{I(x)})^{I(x)}$ satisfies the same condition as $N(G_{I(a)})^{I(a)}$.

Let $(i\,j)$ be any 2-cycle of a. Then $T^{I(a)}$ and $(C(a)_{i\,j})^{I(a)}$ are subgroups of $N(G_{I(a)})^{I(a)}$. Hence there are 3-elements v and v' in T and $C(a)_{i\,j}$ respectively such that $v^{I(a)}=v'^{I(a)}$. Then v and v' normalizes $G_{I(a)}$, $I(v)=\{13, 14\}$ and $I(v')=\{i, j\}$. Let Γ be the $G_{I(a)}$-orbit containing $\{13, 14\}$. Then since $\{13, 14\}^v=\{13, 14\}$, $\Gamma^v=\Gamma$. Suppose that $\{i, j\}$ is contained in a $G_{I(a)}$-orbit different from Γ. Since the order of $G_{I(a)}$ is not divisible by three, $|\Gamma|$ is not divisible by three. Hence $\Gamma^{v'}\neq\Gamma$. Thus $\Gamma^{vv'^{-1}}=\Gamma^{v'^{-1}}\neq\Gamma$. This is a contradiction since $vv'^{-1}\in G_{I(a)}$. Thus $\{i,j\}\subset\Gamma$. Since (ij) is any 2-cycle of a, $G_{I(a)}$ is transitive on $\Omega-I(a)$. From the same reason, $G_{I(x_1)}$ is transitive on $\Omega-I(x_1)$. Then since $I(\langle G_{I(a)}, G_{I(x_1)}\rangle)=\{1,2,7,8\}$, G_{1278} is transitive on $\Omega-\{1,2,7,8\}$.

Let Q be a Sylow 2-subgroup of $G_{I(a)}$. Since $N(Q)^{I(a)}=N(G_{I(a)})^{I(a)}$, $(N(Q)^{I(a)})_{12783}=1$. Hence Q is a Sylow 2-subgroup of G_{12783}. Then since G_{1278} is transitive on $\Omega-\{1,2,7,8\}$, $(N(Q)^{I(a)})_{1278}$ is transitive on $I(a)-\{1,2,7,8\}$ by a lemma of *E.* Witt [10]. This is a contradiction since $N(Q)^{I(a)}=N(G_{I(Q)})^{I(a)}$ and $(N(G_{I(Q)})^{I(a)})_{1278}$ is intransitive on $I(a)-\{1,2,7,8\}$.

Thus we complete the proof of Lemma 2.

Appendix

In Theorem of [8] we assumed that Q was a Sylow 2-subgroup of $G_{I(Q)}$. But this assumption is not necessary since if there is a 2-subgroup R satisfying $|I(R)|=t$ and $N(R)^{I(R)}=A_t$ or S_t, then a Sylow 2-subgroup of $G_{I(R)}$ satsifies the assumption of Theorem of [8]. Hence we have the following

Theorem. *Let G be a 4-fold transitive group on $\Omega=\{1,2,\cdots,n\}$ and t be the maximal number of fixed points of involutions of G. Assume that G has a 2-subgroup Q such that $|I(Q)|=t$ and $N(Q)^{I(Q)}=S_t$ or A_t, then G is one of the following groups:* S_n $(n\geq 4)$, A_n $(n\geq 6)$ *or* M_n $(n=11, 12, 23, 24)$.

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