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A CHARACTERIZATION OF CONTIGUOUS PROBABILITY MEASURES IN THE INDEPENDENT IDENTICALLY DISTRIBUTED CASE

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1. Introduction. The purpose of the present paper is to give a characterization of contiguity of the sequences $\{P_{n,\theta}\}$ and $\{P_{n,\theta+\theta_n}\}$ under a circumstance where for each n $P_{n,\theta}$ is the distribution of independent identically distributed (iid) random variables (rv's).

Contiguity is a concept expressing nearness between the sequences of probability measures. Some characterizations and important consequences of this concept have been established by LeCam (1960), (1966), Roussas (1972) and Philippou and Roussas (1973). Roussas (1972) showed that for stationary Markov process the sequences $\{P_{n,\theta}\}$ and $\{P_{n,\theta+\theta_n}\}$ with $\theta_n=h_n/n^{1/2}$, $h_n\to h\in R^k$, are contiguous. Result similar to the above one has been established by Philippou and Roussas (1973) for the independent, but not necessarily identically distributed case.

In Section 2, we introduce necessary but not always sufficient conditions for contiguity already obtained by previous authors (see Roussas (1972) and Suzuki (1974)). In the succeeding sections we study the sufficiency of these conditions under several circumstances. In Section 3, we discuss this problem in the case that $P_{n,\theta}$ have a constant support under certain regularity conditions. Furthermore, in this case, a simplest condition that $|\theta_n|=0(n^{-1/2})$ is shown to be equivalent to contiguity. In Sections 4—7, we deal with the problem of location parameter. After a few preliminary results are established in Section 4, we consider two cases that

$$(1)\quad f(x)=0 \text{ if } x<a,\ f(x)>0 \text{ if } x>a \text{ and } f(a+0)=0\,,$$
$$(2)\quad f(x)=0 \text{ if } x<a,\ f(x)>0 \text{ if } x>a \text{ and } f(a+0)>0\,,$$

in Section 5 and Section 6, respectively, where $f(x)$ stands for a underlying probability density function (pdf). Furthermore, in Section 6, $|\theta_n|=o(n^{-1})$ is shown to be equivalent to contiguity. Finally, in Section 7, we mention some results which follow immediately from the previous results.

2. Necessary conditions for contiguity. For the purpose of completeness of discussions, we present the concept of contiguity introduced by LeCam (1960). Let $\{(\mathcal{X}, \mathcal{A}_n)\}$ be a sequence of measurable spaces, and let P_n and Q_n be probability measures on $\mathcal{A}_n$.

DEFINITION. $\{P_n\}$ and $\{Q_n\}$ are said to be *contiguous* if for any sequence $\{T_n\}$ of $\mathcal{A}_n$-measurable rv's on $\mathcal{X}$, $T_n \to 0$ in P_n-probability if and only if $T_n \to 0$ in Q_n-probability.

In order to avoid unnecessary repetitions, all limits are taken as $\{n\}$, or subsequences thereof, converges to infinity through the positive integers unless otherwise specified. Also, integrals without limits are understood to be taken over the entire space. If X is a random variable, its probability distribution for a probability measure P is denoted by $\mathcal{L}(X \mid P)$. Furthermore, we write $\mathcal{L}_n \Rightarrow \mathcal{L}$ if a sequence of probability measures $\{\mathcal{L}_n\}$ converges weakly to a probability measure $\mathcal{L}$.

For each n, let μ_n be a σ-finite measure dominating P_n and Q_n on $\mathcal{A}_n$ and write

$$f_n = dP_n/d\mu_n,\ g_n = dQ_n/d\mu_n\,. \tag{2.1}$$

Define the set B_n by

$$B_n = \{\omega \in \mathcal{X};\, f_n(\omega) g_n(\omega) > 0\} \tag{2.2}$$

and a rv Λ_n by

$$\begin{aligned} \Lambda_n &= \log\,(g_n/f_n), && \text{if}\ \ \omega \in B_n\,, \\ &= \text{arbitrary}, && \text{if}\ \ \omega \in B_n^{\,c}\,. \end{aligned} \tag{2.3}$$

The asymptotic distributions of Λ_n under both P_n and Q_n are independent on the value of Λ_n over $B_n^{\,c}$. Because we consider only the case that $\lim P_n (B_n^{\,c}) = \lim Q_n(B_n^{\,c}) = 0$ (see Theorem 2.2).

Moreover, use the notation

$$\mathcal{L}_n = \mathcal{L}(\Lambda_n \mid P_n)\,. \tag{2.4}$$

Theorem 2.1. (LeCam (1960)) *If $\{\mathcal{L}_n\}$ converges weakly to a normal distribution $N(-\frac{1}{2}\sigma^2, \sigma^2)$, then $\{P_n\}$ and $\{Q_n\}$ are contiguous.*

REMARK. For $\sigma^2 = 0$, $N(-\frac{1}{2}\sigma^2, \sigma^2)$ means the degenerate measure with mass 1 at the origin.

Let $\rho(P_n, Q_n)$ be the inner product:

$$\rho(P_n, Q_n) = \int \sqrt{f_n g_n}\, d\mu_n\,. \tag{2.5}$$

Theorem 2.2 *If $\{P_n\}$ and $\{Q_n\}$ are contiguous, then*

(1) $\lim P_n(B_n) = \lim Q_n(B_n) = 1$

and

(2) $\liminf \rho(P_n, Q_n) > 0$.

We shall omit the proof of theorem. Because the first assertion of the theorem is just the same as Lemma 5.1, Chapter 1, in Roussas (1972) and the second was proved for the case of independent observations in Suzuki (1974), whose proof extends immediately to the general case.

The conditions (1) and (2) of Theorem 2.2 are not necessarily sufficient for contiguity as seen in the following example.[1)]

EXAMPLE. Let $(\Omega, \mathcal{A}, (\mathcal{A}_t), P; (B_t))$ be a real Brownian motion with $B_0=0$. Define a stopping time τ by $\tau(\omega)=\inf\{t\geq 0; B_t=-1\}$. For any positive integer n, let P_n be the restriction of P to a sub σ-field $\mathcal{A}_{\tau\wedge n}$ defined by $\mathcal{A}_{\tau\wedge n}=\{A\in\mathcal{A};\ A\cap\{\tau(\omega)\leq n\}\in\mathcal{A}_n\}$ and let a probability measure Q_n be defined as follows $dQ_n=\exp\{B_{\tau\wedge n}-\frac{1}{2}\tau\wedge n\}dP_n$. Then, if $A_n=\{\omega; \tau(\omega)\geq n\}$, we have $P_n(A_n)\to 0$ while $Q_n(A_n)\not\to 0$. Thus $\{P_n\}$ and $\{Q_n\}$ are not contiguous by Theorem 6.1, Chapter 1, in Roussas (1972). However this example obviously satisfies the conditions (1) and (2) of Theorem 2.2.

3. A characterization of contiguous probability measures with constant support. Let Θ be an open neighborhood of the origin of the k-dimensional Euclidean space R^k and for each $\theta\in\Theta$, let p_θ be a probability measure on the Borel real line $(R, \mathcal{B})$. It is assumed that there is a σ-finite measure μ on $\mathcal{B}$ such that $p_\theta\ll\mu$, $\theta\in\Theta$, and we set $f(\cdot, \theta)=dp_\theta/d\mu$ for a special version of the Radon-Nikodym derivative. Set $(\mathcal{X},\mathcal{A})=\prod_{j=1}^{\infty}(R_j,\mathcal{B}_j)$, where every $(R_j, \mathcal{B}_j)=(R, \mathcal{B})$, and let P_θ be the product measure of p_θ induced on $\mathcal{A}$. Then, let $X_j, j\geq 1$, be the coordinate rv's of $\omega=(x_1, x_2, \cdots)$ defined on $(\mathcal{X}, \mathcal{A})$; i.e. $X_j(\omega)=x_j$ for ω. In other words, for each $\theta\in\Theta$ these rv's $X_1, X_2 \cdots$ are independent and the pdf of X_j is $f(\cdot, \theta)$. Furthermore, let $\mathcal{A}_n$ be the σ-field induced by the vector valued rv $(X_1, X_2, \cdots, X_n)$ and let $P_{n,\theta}$ be the restriction of the probability measure P_θ to $\mathcal{A}_n$.

For $\theta\in\Theta$, set

$$A_\theta = \{x\in R; f(x, \theta)>0\}. \tag{3.1}$$

We call A_θ the support of p_θ. Next, for $\theta\in\Theta$ we set

$$\varphi(x, \theta) = \left(\frac{f(x, \theta)}{f(x, 0)}\right)^{1/2}, \qquad \text{if } x\in A_0\cap A_\theta, \tag{3.2}$$

1) This example was orally informed to the author by Mr. Takashi Komatsu. He got it with a slight modification from the paper of Lipcer and Sirjaev (1972).

$$= 0 \qquad , \quad \text{if } x \in (A_0 \cap A_\theta)^c ,$$

and

(3.3) $$\Lambda_n(\omega, \theta) = 2 \sum_{j=1}^{n} \log \varphi(X_j, \theta) .$$

Hereafter, we consider the sequences $\{P_n\}$ and $\{Q_n\}$ on measurable spaces $\{(\mathcal{X}, \mathcal{A}_n)\}$ as follows

(3.4) $$P_n = P_{n,0} ,$$
$$Q_n = P_{n,\theta_n} ,$$

where θ_n belongs to Θ for all n. Then the pdf's of P_n and Q_n are given by

(3.5) $$f_n(\omega) = \prod_{j=1}^{n} f(X_j, 0) ,$$
$$g_n(\omega) = \prod_{j=1}^{n} f(X_j, \theta_n) ,$$

respectively. For simplicity, we write p instead of p_0 and P instead of P_0 and write

(3.6) $$\varphi_n = \varphi(x, \theta_n) ,$$

(3.7) $$\mathcal{E}\varphi_n = \int \varphi(x, \theta_n) dp = \int \varphi(X_j, \theta_n) dP$$

and

(3.8) $$\Lambda_n = \Lambda_n(\omega, \theta_n) .$$

Lemma 3.1. *For any constant a_n, $0 \leqq a_n \leqq 1$,*

$$\liminf (1-a_n)^n > 0 \text{ if and only if } a_n = 0(n^{-1}) .$$

In particular,

$$\lim (1-a_n)^n = 1 \text{ if and only if } a_n = o(n^{-1}) .$$

According to Lemma 3.1 the statements (1) and (2) of Theorem 2.2 are equivalent to the following assumptions (A.1) and (A.2), respectively.

Assumption A.

(A.1) $1 - \int_{B_{n1}} f(x, 0) d\mu = o(n^{-1})$ and $1 - \int_{B_{n1}} f(x, \theta_n) \, d\mu = o(n^{-1})$,
where $B_{n1} = A_0 \cap A_{\theta_n}$.

(A.2) $1 - \mathcal{E}\varphi_n = 0(n^{-1})$.

Furthermore, we need a set of assumptions for our investigation.

Assumption B.

(B.1) The set $\{\theta_n; n = 1, 2, \cdots\}$ is bounded and its closure is contained in Θ.

(B.2) For every $t \neq 0$,

$$\int |f(x,t)-f(x,0)|\,d\mu > 0 .$$

(B.3) For every $\theta \in \Theta$,

$$\lim_{t\to\theta}\int |f(x,\theta)-f(x,t)|\,d\mu = 0 .$$

(B.4) The set A_θ is independent of $\theta \in \Theta$.

(B.5) The function $\varphi(x,\theta)$ is differentiable in quadratic mean (qm) at $\theta=0$ when p is employed. That is, there is a k-dimensional function $\dot\varphi(x)$ such that

$$|\lambda^{-1}|\cdot|\varphi(\cdot,\lambda h)-1-\lambda h'\dot\varphi(\cdot)| \to 0 \text{ in qm } [p], \text{ as } \lambda \to 0 ,$$

uniformly on every bounded sets of $h \in R^k$, where h' denotes the transpose of h.

(B.6) $\Gamma = 4\mathcal{E}[\dot\varphi(X)\dot\varphi(X)']$ is positive definite.

We use the assumptions (B.1) to (B.3) only to show that $\theta_n \to 0$. If for every $\theta \in \Theta$, φ is differentiable in qm (see Philippou and Roussas (1973), assumption (A.2)), then (B.3) holds.

(B.4) to (B.6) are the assumptions under which Philippou and Roussas (1973) obtained an asymptotic expansion for the log-likelihood function in the independent, but not necessarily identically distributed case. We will make use of their result restricted to the iid case.

Lemma 3.2. *Under Assumptions* (A.2) *and* B, *we have* $|\theta_n| = 0(n^{-1/2})$, *where the symbol* $|\cdot|$ *stands for the Euclidean norm.*

Proof. By Lemma 1 in Kraft (1955), (A.2) implies

$$\int |f(x,0)-f(x,\theta_n)|\,d\mu \to 0 ,$$

so that $|\theta_n| \to 0$ by (B.1), (B.2) and (B.3). Since it is enough to consider the case that $\theta_n \neq 0$ for all n, we set $h_n = \dfrac{\theta_n}{|\theta_n|}$. Then we get $\theta_n = |\theta_n| h_n$ and $|h_n| = 1$. By taking λ to be $\{|\theta_n|\}$ and replacing h by $\{h_n\}$, (B.5) implies

$$\mathcal{E}\left|\frac{1}{|\theta_n|}(\varphi_n-1)-h_n'\dot\varphi\right|^2 \to 0 , \tag{3.9}$$

where $\dot\varphi$ denotes an abbreviation of $\dot\varphi(x)$. Since $\mathcal{E}(h_n'\dot\varphi)^2 \leqq M(<\infty)$ for all n, it follows from (3.9) and Schwarz inequality that

$$\mathcal{E}\left|\frac{1}{|\theta_n|^2}(\varphi_n-1)^2-(h_n'\dot\varphi)^2\right| \to 0 . \tag{3.10}$$

By the fact that $\mathcal{E}\varphi_n^2=1$ and $(B.6)$, (3.10) implies

(3.11) $$\frac{1}{|\theta_n|^2}(1-\mathcal{E}\varphi_n)-\frac{1}{8}h_n'\Gamma h_n \to 0\,.$$

But (B.6) says that $h_n'\Gamma h_n \geq \delta(>0)$ for all n. Thus (A.2) and (3.11) imply the desired result.

Theorem 3.1. *Let Assumption* B *be satisfied. Then the following statements are equivalent.*

(a) *The sequences* $\{P_n\}$ *and* $\{Q_n\}$ *are contiguous.*

(b) $\liminf \rho(P_n, Q_n)>0$.

(c) $|\theta_n|=0(n^{-1/2})$.

Proof. According to Theorem 2.2 and Lemma 3.2 it is enough to see that the statement (a) follows from (c). Philippou and Roussas (1973) showed that for $\theta_n=\gamma_n/n^{1/2}$, $\gamma_n\to\gamma\in R^k$,

$$\mathcal{L}(\Lambda_n|P)\Rightarrow N\left(-\frac{1}{2}\gamma'\Gamma\gamma,\ \gamma'\Gamma\gamma\right).$$

Thus by Theorem 2.1 $\{P_n\}$ and $\{Q_n\}$ are contiguous. This completes the proof of Theorem 3.1.

4. A characterization of contiguous probability measures with location parameter-preliminaries. Let Θ be an open neighborhood of the origin of the real line R, and let $f(x)$ be a pdf with respect to the Lebesgue measure μ on R. Furthermore, we set $f(x, \theta)=f(x-\theta)$ in order to use the same notations as the previous section.

The next two lemmas will be needed in the sequel.

Lemma 4.1. *Suppose that the following conditions* (1) *to* (4) *are satisfied.*

(1) $\lim n(1-\mathcal{E}\varphi_n)=\sigma^2(>0)$.

(2) $\lim n(1-\mathcal{E}\varphi_n^2)=0$.

(3) $\lim n\, p(|\varphi_n-1|\geq\varepsilon)=0$, *for every* $\varepsilon>0$.

(4) $\lim\limits_{M\to\infty}\sup\limits_{n\geq 1} n\int_{|\varphi_n-1|\geq M}(\varphi_n-1)^2dp=0$.

Then

$$\mathcal{L}(\Lambda_n|P)\Rightarrow N(-4\sigma^2,\ 8\sigma^2)$$

and consequently $\{P_n\}$ *and* $\{Q_n\}$ *are contiguous.*

Proof. For any $\tau>0$, we set

(4.1) $$a_n(\tau)=n\int_{|\varphi_n-1|<\tau}(\varphi_n-1)dp$$

and

(4.2) $$\sigma_n^2(\tau) = n\left\{\int_{|\varphi_n-1|<\tau}(\varphi_n-1)^2dp - \left[\int_{|\varphi_n-1|<\tau}(\varphi_n-1)\,dp\right]^2\right\}.$$

Then

$$a_n(\tau) = n\int(\varphi_n-1)dp - n\int_{|\varphi_n-1|\geqq\tau}(\varphi_n-1)dp\,.$$

But, by the fact that $\mathcal{E}\varphi_n^2 \leqq 1$

(4.3) $$\begin{aligned}\left\{n\int_{|\varphi_n-1|\geqq\tau}(\varphi_n-1)dp\right\}^2 &\leqq n^2 p(|\varphi_n-1|\geqq\tau)\int(\varphi_n-1)^2dp\\ &\leqq 2n^2p(|\varphi_n-1|\geqq\tau)\,(1-\mathcal{E}\varphi_n)\\ &\to 0\,, \qquad \text{by (1) and (3).}\end{aligned}$$

Therefore (1) gives

(4.4) $$\lim a_n(\tau) = -\sigma^2\,.$$

Next

$$n\int_{|\varphi_n-1|<\tau}(\varphi_n-1)^2dp = n\int(\varphi_n-1)^2dp - n\int_{|\varphi_n-1|\geqq\tau}(\varphi_n-1)^2dp\,.$$

But

$$\begin{aligned}n\int(\varphi_n-1)^2dp &= n\left\{(\mathcal{E}\varphi_n^2-1)+2(1-\mathcal{E}\varphi_n)\right\}\\ &\to 2\sigma^2\,, \qquad \text{by (1) and (2).}\end{aligned}$$

For every $M>\tau$, we have

$$\begin{aligned}&\left|n\int_{|\varphi_n-1|<M}(\varphi_n-1)^2dp - n\int_{|\varphi_n-1|<\tau}(\varphi_n-1)^2dp\right|\\ &\qquad = n\int_{\tau\leqq|\varphi_n-1|<M}(\varphi_n-1)^2dp\\ &\qquad \leqq M^2n\,p(|\varphi_n-1|\geqq\tau)\\ &\qquad \to 0\,, \qquad \text{by (3).}\end{aligned}$$

Therefore (4) gives

$$\lim n\int_{|\varphi_n-1|\geqq\tau}(\varphi_n-1)^2dp = 0\,,$$

so that

(4.5) $$\lim n\int_{|\varphi_n-1|<\tau}(\varphi_n-1)^2dp = 2\sigma^2\,.$$

Next we have

(4.6) $$\begin{aligned}&n\left\{\int_{|\varphi_n-1|<\tau}(\varphi_n-1)dp\right\}^2\\ &\qquad = n\left\{\int(\varphi_n-1)dp - \int_{|\varphi_n-1|\geqq\tau}(\varphi_n-1)dp\right\}^2\end{aligned}$$

$$\leqq 2n\left\{\left[\int(\varphi_n-1)dp\right]^2+\left[\int_{|\varphi_n-1|\geqq\tau}(\varphi_n-1)dp\right]^2\right\}$$
$$\to 0\,, \qquad \text{by (1) and (4.3).}$$

By (4.2) and (4.5), (4.6) implies

$$\lim \sigma_n^2(\tau) = 2\sigma^2\,. \tag{4.7}$$

From (3), (4.4), (4.7) and Normal Convergence Criterion (see Loève (1963), page 316),

$$\mathcal{L}(\sum_{j=1}^{n}(\varphi(X_j,\theta_n)-1)\,|\,P) \Rightarrow N(-\sigma^2, 2\sigma^2)\,.$$

By this fact and LeCam's second lemma (see Hájek and Šidák (1967), page 205), we have

$$\mathcal{L}(\Lambda_n\,|\,P) \Rightarrow N(-4\sigma^2, 8\sigma^2)\,.$$

The desired result then follows.

Lemma 4.2. *If* $\lim n(1-\mathcal{E}\varphi_n)=0$, *then we have*

$$\Lambda_n \to 0 \text{ in } P\text{-probability.}$$

Consequently $\{P_n\}$ *and* $\{Q_n\}$ *are contiguous.*

Proof. For any $\varepsilon>0$, we have

$$\begin{aligned}
P(|\sum_{j=1}^{n}\{(\varphi(X_j,\theta_n)-1)-\mathcal{E}(\varphi_n-1)\}|\geqq\varepsilon)
&\leqq \frac{1}{\varepsilon^2}n\mathcal{E}\{(\varphi_n-1)-\mathcal{E}(\varphi_n-1)\}^2\\
&\leqq \frac{1}{\varepsilon^2}n\mathcal{E}(\varphi_n-1)^2\\
&\leqq \frac{2}{\varepsilon^2}n(1-\mathcal{E}\varphi_n)\\
&\to 0\,.
\end{aligned}$$

By the assumption, we have

$$\sum_{j=1}^{n}(\varphi(X_j,\theta_n)-1) \to 0 \text{ in } P\text{-probability.}$$

It follows from LeCam's second lemma that

$$\Lambda_n \to 0 \text{ in } P\text{-probability,}$$

as was to be established.

5. Contiguous probability measures with location parameter-case 1. In this section, we shall assume the following conditions.

Assumption C.

(C.1) There exists a real number a such that $A_0=(a, \infty)$.

(C.2) The pdf f is continuous on (a, ∞).

(C.3) There exist positive numbers d and k such that

$$\lim_{x\downarrow a}\frac{f(x)}{(x-a)^k}=d\,.^{2)}$$

(C.4) $\lim\limits_{M\to\infty}\lim\limits_{h\to 0}\sup\sup\limits_{x\geqq M}\left|\dfrac{f(x+h)-f(x)}{f(x)}\right|=0\,.$

Theorem 5.1. *Under Assumptions* A *and* C, $\{P_n\}$ *and* $\{Q_n\}$ *are contiguous.*

Proof. From (A.2), there exist a subsequence $\{m\}\subset\{n\}$ and $\sigma^2\geqq 0$ such that $\lim m(1-\mathcal{E}\varphi_m)=\sigma^2$. If $\sigma^2=0$, then the theorem immediately follows from Lemma 4.2. Thus, assume that $\sigma^2>0$. Since (A.1) implies (2) in Lemma 4.1, it is enough to show that the conditions (3) and (4) in Lemma 4.1 are satisfied. From (A.1) and (C.3), we have

$$\theta_n=o(n^{-1/k+1})\,. \tag{5.1}$$

In order to show the validity of (3), we first show that for any given $\varepsilon>0$ there exists a positive number u such that

(5.2) $\{x;\ |\varphi_n-1|\geqq\varepsilon\}\subset(a, a+u|\theta_n|]\cup B_{n1}{}^c$, for all sufficiently large n.

Let $g(t, k)$ be defined by

$$g(t, k)=(1+t)^{1/k}-1\,.$$

Then we get

$$|g(t, k)|\leqq\frac{c}{k}|t|,\qquad\text{if}\quad |t|\leqq\frac{1}{2}\,, \tag{5.3}$$

where the constant c depends only on k. Let η be a positive number such that $\eta<\min(2, c/2k)$ and

$$\{x;\ x>0,\ |x-1|\geqq\varepsilon\}\subset\{x;\ |x^2-1|\geqq\eta\}\cap\{x;\ |x^{2/k}-1|\geqq\eta\}\,.$$

Also, let $\gamma=\dfrac{kd\eta}{4c-k\eta}$. From (C.3) there exists a positive number δ_1 such that

2) This form of assumption was inspired by a lecture by Professor Kei Takeuchi on estimation of location parameter.

$$\left|\frac{f(x)}{(x-a)^k}-d\right| \leqq \gamma, \qquad \text{for all } x,\ a<x<a+3\delta_1.$$

Furthermore, from (C.4) there exist a constant L and a positive number δ_2 such that

$$\text{(5.4)} \qquad \sup_{x\geqq L}\left|\frac{f(x+h)-f(x)}{f(x)}\right| < \eta, \qquad \text{for all } h,\ |h|<\delta_2.$$

Define the sets S_{n1}, S_{n2} and S_{n3} by

$$S_{n1} = (a, a+2\delta_1)\cap B_{n1},$$
$$S_{n2} = [a+2\delta_1, L]\cap B_{n1},$$
$$S_{n3} = (L, \infty)\cap B_{n1}.$$

Then we have

$$\text{(5.5)} \qquad \begin{aligned}\{x;\ |\varphi_n-1|\geqq\varepsilon\} &\subset \bigcup_{i=1}^{3}\{(x;\ |\varphi_n-1|\geqq\varepsilon)\cap S_{ni}\}\cup B_{n1}^c \\ &\subset \{(|\varphi_n^{2/k}-1|\geqq\eta)\cap S_{n1}\} \\ &\quad \cup\{\bigcup_{i=2}^{3}[(|\varphi_n^2-1|\geqq\eta)\cap S_{ni}]\}\cup B_{n1}^c.\end{aligned}$$

Suppose that $\theta_n>0$. If $x\in S_{n1}$, then

$$\begin{aligned}\varphi_n^{2/k}-1 &\leqq \left(\frac{d+\gamma}{d-\gamma}\right)^{1/k}-1 \\ &= g\left(\frac{2\gamma}{d-\gamma},\ k\right) \\ &\leqq \frac{c}{k}\,\frac{2\gamma}{d-\gamma} \qquad \text{(by (5.3))} \\ &\leqq \frac{2}{3}\eta\ (<\eta),\end{aligned}$$

and

$$\begin{aligned}\varphi_n^{2/k}-1 &\geqq \left(\frac{d-\gamma}{d+\gamma}\right)^{1/k}\left(1-\frac{\theta_n}{x-a}\right)-1 \\ &= \left[1+g\left(\frac{-2\gamma}{d+\gamma},\ k\right)\right]\left(1-\frac{\theta_n}{x-a}\right)-1 \\ &\geqq \left(1-\frac{\eta}{2}\right)\left(1-\frac{\theta_n}{x-a}\right)-1.\end{aligned}$$

Hence, if $x\in S_{n1}$ and $\varphi_n^{2/k}-1\leqq-\eta$, then $x\leqq a+\frac{2-\eta}{\eta}\theta_n$. In case $-\delta_1<\theta_n<0$, we similary obtain $x\leqq a+\frac{3+2\eta}{\eta}|\theta_n|$ if $x\in\{|\varphi_n^{2/k}-1|\geqq\eta\}\cap S_{n1}$. Hence, for $u=\frac{3+2\eta}{\eta}$ we have

(5.6) $\{|\varphi_n^{2/k}-1|\geqq\eta\}\cap S_{n1}\subset(a, a+u|\theta_n|]$, for all sufficiently large n.

Since $f(x)$ is uniformly continuous on $[a+2\delta_1, L]$ by (C.2) and $\inf\{f(x); x\in[a+2\delta_1, L]\}>0$, we have

$$\sup\left\{\left|\frac{f(x-\theta_n)-f(x)}{f(x)}\right|; x\in S_{n2}\right\}<\eta \text{ for all sufficiently large } n.$$

Therefore

(5.7) $\{|\varphi_n^2-1|\geqq\eta\}\cap S_{n2}=\phi$, for all sufficiently large n.

Furthermore (5.4) implies

(5.8) $\{|\varphi_n^2-1|\geqq\eta\}\cap S_{n3}=\phi$, for all sufficiently large n.

Therefore (5.2) follows from (5.5), (5.6), (5.7) and (5.8). By (5.1), (5.2) and (A.1), we get

$$\begin{aligned} n\,p(|\varphi_n-1|\geqq\varepsilon) &\leqq n\,p((a, a+u|\theta_n|])+n\,p(B_{n1}^c)\\ &\leqq n(d+\gamma)\int_{(a,a+u|\theta_n|]}(x-a)^k d\mu+n\,p(B_{n1}^c)\\ &=\frac{d+\gamma}{k+1}u^{k+1}n|\theta_n|^{k+1}+n\,p(B_{n1}^c)\\ &\to 0\,, \end{aligned}$$

from which (3) follows.

We next show the validity of (4). From (5.2), we have

$$\lim n\int_{|\varphi_n-1|\geqq 2}f(x-\theta_n)d\mu=0\,.$$

This implies

(5.9) $$\lim_{M\to\infty}\sup_{n\geqq 1} n\int_{|\varphi_n-1|\geqq M}f(x-\theta_n)d\mu=0\,.$$

Since under (3), (5.9) is equivalent to (4), the proof is completed.

From Lemma 4.1 and Theorem 5.1, we immediately have the following

Corollary 5.1. *Suppose that Assumptions* (A.1) *and* C *are satisfied. Suppose in addition that* $\lim n(1-\mathcal{E}\varphi_n)=\sigma^2(>0)$. *Then we have*

$$\mathcal{L}(\Lambda_n|P)\Rightarrow N(-4\sigma^2, 8\sigma^2)\,.$$

6. Contiguous probability measures with location parameter-case 2. In this section, we consider the other case. We assume the following conditions.

Assumption D.

(D.1) There exists a real number a such that $A_0 \subset [a, \infty)$ and $0 < f(a+0) = \lim_{x \downarrow a} f(x) < \infty$.

(D.2) The pdf f is differentiable in x on (a, ∞).

Let f' denote the first derivative of f.

(D.3) $\int_{(a,\infty)} |f'(x)| d\mu < \infty$.

Lemma 6.1. *Under Assumptions* (A.1) *and* D, *we have* $|\theta_n| = o(n^{-1})$.

Proof. Since

$$\int_{(a,a+|\theta_n|]} f(x) d\mu \geqq \inf \{f(x);\ a < x \leqq a + |\theta_n|\} \cdot |\theta_n| ,$$

it follows from (A.1) and (D.1) that $|\theta_n| = o(n^{-1})$.

Theorem 6.1. *Let Assumption* D *be satisfied. Then the following statements are equivalent.*

(a) $\{P_n\}$ *and* $\{Q_n\}$ *are contiguous.*

(b) $|\theta_n| = o(n^{-1})$.

(c) $\Lambda_n \to 0$ *in P-probability.*

Proof. According to Theorem 2.1, Theorem 2.2 and Lemma 6.1 it is enough to see that the statement (c) follows from (b). Let d_n be defined by

$$d_n = \int |f(x) - f(x - \theta_n)| d\mu .$$

Then

$$\begin{aligned} d_n &= \int |f(x) - f(x - |\theta_n|)| d\mu \\ &= \int_{(a,a+|\theta_n|]} f(x) d\mu + \int_{(a+|\theta_n|,\infty)} |f(x) - f(x - |\theta_n|)| d\mu . \end{aligned}$$

But

$$\begin{aligned} &\int_{(a+|\theta_n|,\infty)} |f(x) - f(x - |\theta_n|)| d\mu \\ &\leqq \int_{(a+|\theta_n|,\infty)} d\mu(x) \int_{[x-|\theta_n|,x]} |f'(z)| d\mu(z) \\ &\leqq \int_{(a,\infty)} |f'(z)| d\mu(z) \int_{[z,z+|\theta_n|]} d\mu(x) \\ &= |\theta_n| \int_{(a,\infty)} |f'(z)| d\mu . \end{aligned}$$

It follows from (b), (D.1) and (D.3) that $d_n = o(n^{-1})$. Hence by Lemma 1 in Kraft (1955), $1 - \mathcal{E}\varphi_n = o(n^{-1})$. The desired conclusion then follows from

Lemma 4.2.

REMARK. The argument given above shows that Theorem 6.1 remains valid even if the assumption (D.2) is replaced by the following assumption (D.2′).

(D.2′) The pdf f is continuous on (a, ∞) and the derivative f' exists except for finite points on (a, ∞).

7. Remarks. In this section, we mention results without proving which follow immediately from Section 5 and Section 6.

Assumption E.

(E.1) There exist real numbers a and b such that $a<b$ and $A_0=(a, b)$.

(E.2) The pdf f is continuous on (a, b).

(E.3) There exist positive numbers d_1, d_2, k_1 and k_2 such that

$$\lim_{x\downarrow a}\frac{f(x)}{(x-a)^{k_1}}=d_1 \text{ and } \lim_{x\nearrow b}\frac{f(x)}{(b-x)^{k_2}}=d_2 .$$

Theorem 7.1. *Under Assumptions* A *and* E, $\{P_n\}$ *and* $\{Q_n\}$ *are contiguous.*

Assumption F.

(F.1) There exist real numbers a and b with $a<b$ such that $A_0\subset[a, b]$, $0<f(a+0)<\infty$ and $f(b-0)=\lim_{x\nearrow b}f(x)<\infty$ (or else $f(a+0)<\infty$ and $0<f(b-0)<\infty$).

(F.2) The pdf f is continuous on (a, b) and f' exists except for finite points on (a, b).

(F.3) $\int_{(a,b)}|f'(x)|\,d\mu<\infty$.

Theorem 7.2. *Let Assumption* F *be satisfied. Then the following statements are equivalent.*

(a) $\{P_n\}$ *and* $\{Q_n\}$ *are contiguous.*

(b) $|\theta_n|=o(n^{-1})$.

(c) $\Lambda_n\to0$ *in P-probability.*

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