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PROJECTIVE DIMENSION OF BROWN-PETERSON HOMOLOGY WITH MODULO $(p, v_1, \cdots, v_{n-1})$ COEFFICIENTS

Dedicated to the memory of Professor Taira Honda

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Let BP denote the Brown-Peterson spectrum for a fixed prime p. This spectrum gives us a multiplicative homology theory $BP_*(\ )$ defined on the category of CW-spectra, with coefficient ring $BP_*=Z_{(p)}[v_1, \cdots, v_n, \cdots]$. Recall that the only invariant prime ideals of BP_* are those of the form $I_n=(v_0, v_1, \cdots, v_{n-1})$, $0\leqq n\leqq\infty$, where we put $v_0=p$. Using Baas-Sullivan technique we can kill the generators $v_0, v_1, \cdots, v_{n-1}, v_{m+1}, v_{m+2}, \cdots$ of BP_* to construct a homology theory $BP[n, m+1)_*(\ )$ represented by a certain BP-module spectrum $BP[n, m+1)$, where $0\leqq n\leqq m+1\leqq\infty$. In particular $BP[n, \infty)$ is the Brown-Peterson spectrum with modulo I_n coefficients, denoted by $P(n)$ [4]. According to Morava's geometric computation, $P(n)_*(X)$ becomes a module over the coherent ring $P(n)_*=BP_*/I_n$. Morava and also Johnson-Wilson[4] have obtained rich results concerned with the BP-module spectrum $P(n)$ and its operations.

Define hom $\dim_{P(n)_*}M$ to be the projective dimension of M as $P(n)_*$-module. Johnson-Wilson[3] gave completely satisfactory conditions under which hom $\dim_{BP_*}BP_*(X)\leqq m$, where they assumed that X is a based finite CW-complex. The proof was based on Wilson's splitting theorem. Later Landweber[7] introduced the category $\mathcal{BP}_0$ of coherent comodules over $BP_*(BP)$ which has $BP_*(X)$ as object for every based finite CW-complex X. He then proved algebraically the result of Johnson-Wilson, applying two powerful tools called Filtration theorem and Exact functor theorem. The purpose of this note is to give the $P(n)_*(\ )$-version of the result of Johnson-Wilson without the finiteness assumption on X, *i.e.*, the characterization of numerical invariant hom $\dim_{P(n)_*}P(n)_*(X)$ for a connective CW-spectrum X (Theorem 4.8). In the proof we will use Landweber's methods.

Let X be a connective CW-spectrum. Using structure theorem of $P(n)^*(P(n))$ we can see easily that $P(n)_*$-modules $P(n)_*(X)$ become comodules over $BP_*(BP)$ when $n<2(p-1)$ (see [4, Remark 2.14]). But the author doesn't know whether $P(n)_*(X)$ are always so for all n. So we must construct a nice abelian category in which all $P(n)_*(X)$ are contained as objects. Fortunately $P(n)_*(X)$ possesses a certain structure like a comodule over $BP_*(BP)$. Denote by $\mathcal{P}(n)$ (or $\mathcal{P}(n)_0$) the category of connective (or coherent) $P(n)_*$-modules equipped with the certain additional structure of operations. Of course our category $\mathcal{P}(0)_0$ coincides with the category $\mathcal{BP}_0$ of coherent comodules over $BP_*(BP)$. For this category $\mathcal{P}(n)$ (or $\mathcal{P}(n)_0$) we will give $\mathcal{P}(n)$-versions of Landweber filtration theorem and exact functor theorem (Theorems 3.4 and 4.2). These are useful in giving a number of characterizations of homological dimension in $\mathcal{P}(n)$ (Theorem 4.5).

In §1 we study homological properties of connective modules over a graded polynomial ring $k[x_1, \cdots, x_i, \cdots]$, particularly in the case when k is a field or a subring of the rationals Q, by arguments of the sort given by Landweber[7]. In §2 we investigate conditions that $P(n)_*(X)$ is $P(n)_*$-free and then indicate that [4, Lemma 4.10] implies the existence of a geometric $P(n)_*$-resolution. We next construct a natural spectral sequence

$$E^2_{s,*} = \mathrm{Tor}^{P(n)_*}_{s,*}(BP[k, l+1)_*, P(n)_*(X)) \Rightarrow BP[k, l+1)_*(X)$$

where X is connective and $0 \leqq n \leqq k \leqq l+1 \leqq \infty$. This spectral sequence is needed to give necessary and sufficient conditions that $\mu^{k,l}_n: P(n)_*(X) \to BP[k, l+1)_*(X)$ is epic (Theorem 2.7).

The pairing $m_n: BP \wedge P(n) \to P(n)$ makes $P(n)^*(P(n))$ into a $BP^*(BP)$-comodule. In §3 we first observe behavior of coaction on $P(n)^*(P(n))$ and also product formula of $P(n)$-operations. These allow us to introduce the abelian category $\mathcal{P}(n)$ which has enough projectives. Then, using Landweber technique of [5, Lemma 3.3] we can reduce the proof of Filtration theorem in $\mathcal{P}(n)_0$ to that of Landweber's. In §4 we prove Exact functor theorem on $\mathcal{P}(n)$ applying our Filtration theorem as Landweber did. This useful tool is applied to the homology theory $P(n)_*(\ \ )$ so that we study homological properties of connective modules $P(n)_*(X)$. Finally we discuss $P(n)_*(\ \ )$-versions of Johnson's result[2] dealt with low projective dimension.

The author would like to acknowledge to Professors Johnson and Wilson for sending him their preprint[4], and particularly to the former for several suggestions which have been very helpful in the development of this paper. He would also like to thank Mr. Yagita who indicated to him a program for proving our main result. Since they have jointly proved $P(n)_*$-versions of Filtration theorem and Exact functor theorem along the program, some of our results will be restated with minor alternations by Yagita [10].

1. Connective modules over a polynomial ring $k[x_1, x_2, \cdots]$

1.1. Let k be a commutative ring with unit and Λ be a graded polynomial ring over k in infinitely many variables x_i with degree $x_i \geqq i$. We shall deal throughout with graded Λ-modules on which the gradings are bounded below, called *connective* Λ-modules. We first introduce ideals $J[n, m+1)$ in Λ defined by $J[n, m+1)=(x_1, \cdots, x_{n-1}, x_{m+1}, x_{m+2}, \cdots)$ for $1 \leqq n \leqq m+1 \leqq \infty$. Putting $\Lambda[n, m+1)=\Lambda/J[n, m+1)$ we have an isomorphism $\Lambda[n, m+1) \cong k[x_n, \cdots, x_m]$. In particular $\Lambda[1, \infty)=\Lambda$ and $\Lambda[n, n) \cong k$. Obviously there exist short exact sequences

$$(1.1) \quad \begin{array}{l} 0 \to \Lambda[n, m+1) \xrightarrow{\cdot x_n} \Lambda[n, m+1) \to \Lambda[n+1, m+1) \to 0 \\ 0 \to \Lambda[n, m+1) \xrightarrow{\cdot x_m} \Lambda[n, m+1) \to \Lambda[n, m) \to 0 \end{array}$$

where $\cdot x_i$, $i=n$ or m, acts as multiplication by x_i.

Notice that each generator x_i has positive degree. As is easily checked, we get

(1.2) *If a connective* Λ*-module* M *satisfies* $\Lambda[n, m+1) \underset{\Lambda}{\otimes} M=0$, *then* $M=0$.

Moreover using (1.1) an induction on degree shows

(1.3) *Let* M *be a connective* Λ*-module,* $s \geqq 0$ *and assume that* $\mathrm{Tor}_s^\Lambda(\Lambda[n, m+1), M)=0$. *Then also* $\mathrm{Tor}_s^\Lambda(\Lambda[n-1, m+1), M)=\mathrm{Tor}_s^\Lambda(\Lambda[n, m+2), M)=0$.

Lemma 1.1. *Let* M *be a connective* Λ*-module and* $s \geqq 0$.
i) *If* $\mathrm{Tor}_s^\Lambda(\Lambda[n, \infty), M)=0$, *then* $\mathrm{Tor}_{s+1}^\Lambda(\Lambda[n+1, \infty), M)=0$.
ii) *If* $\mathrm{Tor}_s^\Lambda(\Lambda[1, m+1), M)=0$, *then* $\mathrm{Tor}_{s+1}^\Lambda(\Lambda[1, m), M)=0$.

Proof. i) An iterated application of (1.3) yields that $\mathrm{Tor}_s^\Lambda(\Lambda[n-j, \infty), M)=0$ for all $j \geqq 0$. This then implies that $\mathrm{Tor}_{s+1}^\Lambda(\Lambda[1, \infty), M) \to \mathrm{Tor}_{s+1}^\Lambda(\Lambda[n+1, \infty), M)$ is epic, and so $\mathrm{Tor}_{s+1}^\Lambda(\Lambda[n+1, \infty), M)=0$.

ii) We use repetition of (1.3) to get $\mathrm{Tor}_s^\Lambda(\Lambda[1, m+j), M)=0$ for all $j \geqq 1$. Hence we see that $\mathrm{Tor}_{s+1}^\Lambda(\Lambda[1, m+j), M) \to \mathrm{Tor}_{s+1}^\Lambda(\Lambda[1, m), M)$ is epic for all $j \geqq 1$. However $\mathrm{Tor}_{s+1,t}^\Lambda(\Lambda[1, m+j), M)=0$ for sufficiently large j, and so $\mathrm{Tor}_{s+1}^\Lambda(\Lambda[1, m), M)=0$.

An iterated application of Lemma 1.1 ii) gives

Corollary 1.2. *Let* M *be a connective* Λ*-module and* $s \geqq 0$. *If* $\mathrm{Tor}_s^\Lambda(\Lambda[1, m+1), M)=0$, *then* $\mathrm{Tor}_{s+m}^\Lambda(k, M)=0$.

Making use of (1.3) and Lemma 1.1 we obtain

Proposition 1.3. *Let* M *be a connective* Λ*-module and* $s \geqq 0$.
i) $\mathrm{Tor}_s^\Lambda(\Lambda[n, \infty), M)=0$ *if and only if* $\mathrm{Tor}_{s+i}^\Lambda(\Lambda[n, \infty), M)=0$ *for all* $i \geqq 0$.
ii) $\mathrm{Tor}_s^\Lambda(\Lambda[1, m+1), M)=0$ *if and only if* $\mathrm{Tor}_{s+i}^\Lambda(\Lambda[1, m+1), M)=0$ *for all* $i \geqq 0$.

Let $f:M \to N$ be a homomorphism of connective Λ-modules for which $1 \underset{\Lambda}{\otimes} f: k \underset{\Lambda}{\otimes} M \to k \underset{\Lambda}{\otimes} N$ is epic. Since $k \underset{\Lambda}{\otimes} \mathrm{Coker}\, f=0$, (1.2) says that f is epic. We now add to f the assumption that $1 \underset{\Lambda}{\otimes} f: k \underset{\Lambda}{\otimes} M \xrightarrow{\cong} k \underset{\Lambda}{\otimes} N$ and $\mathrm{Tor}_1^\Lambda(1, f)$: $\mathrm{Tor}_1^\Lambda(k, M) \xrightarrow{\cong} \mathrm{Tor}_1^\Lambda(k, N)$. Then our hypothesis implies that $k \underset{\Lambda}{\otimes} \mathrm{Ker} f=0$ and so f is monic. Consequently we get

(1.4) *$f: M \to N$ is an isomorphism if and only if both $1 \underset{\Lambda}{\otimes} f: k \underset{\Lambda}{\otimes} M \longrightarrow k \underset{\Lambda}{\otimes} N$ and $\mathrm{Tor}_1^\Lambda(1, f): \mathrm{Tor}_1^\Lambda(k, M) \longrightarrow \mathrm{Tor}_1^\Lambda(k, N)$ are so.*

Lemma 1.4. *Let M be a connective Λ-module. M is Λ-free if and only if $k \underset{\Lambda}{\otimes} M$ is k-free and $\mathrm{Tor}_1^\Lambda(k, M)=0$.*

Proof. It is sufficient to show only the "if" part. Putting $F=\Lambda \underset{k}{\otimes} (k \underset{\Lambda}{\otimes} M)$ it is a connective free Λ-module. Obviously there exists a Λ-homomorphism $f: F \to M$ such that $1 \underset{\Lambda}{\otimes} f: k \underset{\Lambda}{\otimes} F \to k \underset{\Lambda}{\otimes} M$ coincides with the identity. By means of (1.4) we find that f is an isomorphism, and so M is Λ-free.

Define $\mathrm{hom\,dim}_\Lambda M$ and $\mathrm{w\,dim}_\Lambda M$ respectively to be the projective dimension and the weak dimension of M as Λ-module. Note that M is Λ-projective when $\mathrm{hom\,dim}_\Lambda M=0$, and that M is Λ-flat when $\mathrm{w\,dim}_\Lambda M=0$.

Proposition 1.5. *Let M be a connective Λ-module and assume that k is a field. The four conditions in* I) *are equivalent and the three in* II) *are so for each $m \geqq 0$:*

I) 0) *M is Λ-free,* i) *M is Λ-projective,* ii) *M is Λ-flat, and* iii) $\mathrm{Tor}_1^\Lambda(k, M)=0$.
II) $\mathrm{i})_m$ $\mathrm{hom\,dim}_\Lambda M \leqq m$, $\mathrm{ii})_m$ $\mathrm{w\,dim}_\Lambda M \leqq m$, *and* $\mathrm{iii})_m$ $\mathrm{Tor}_{m+1}^\Lambda(k, M)=0$.

Proof. I) is immediate from Lemma 1.4. II) is obtained by induction on m.

1.2. Let R be a subring of the rational numbers Q with unit. It is just the integers localized at l where l is the subset of primes which are not divisible in R, and it is frequently denoted by Z_l. We now restrict our interest in the case $k=R$, a subring of Q.

Lemma 1.6. *Let M be a connective Λ-module and assume that k is a subring R of Q. If $\Lambda[1, m+1) \underset{\Lambda}{\otimes} M$ is torsion free and $M \underset{R}{\otimes} Q$ is Λ-flat, then for each j, $1 \leqq j \leqq \infty$, $\Lambda[1, m+j) \underset{\Lambda}{\otimes} M$ is also torsion free and in addition* $\mathrm{Tor}_1^\Lambda(\Lambda[1, m), M)=0$.

Proof. Our hypotheses mean that $\Lambda[1, m+1) \underset{\Lambda}{\otimes} M \to \Lambda[1, m+1) \underset{\Lambda}{\otimes} M \underset{R}{\otimes} Q$

is monic and $\mathrm{Tor}_1^\Lambda(\Lambda[1, m+j), M\underset{R}{\otimes}Q)=0$ for all $j\geqq 0$. Since x_{m+1} has positive degree, an induction on degree shows that $\Lambda[1, m+2)\underset{\Lambda}{\otimes}M\to\Lambda[1, m+2)\underset{\Lambda}{\otimes}M\underset{R}{\otimes}Q$ is monic, thus $\Lambda[1, m+2)\underset{\Lambda}{\otimes}M$ is torsion free. Moreover we see easily that $\mathrm{Tor}_1^\Lambda(\Lambda[1, m+1), M)\to\mathrm{Tor}_1^\Lambda(\Lambda[1, m), M)$ is epic. A repetition of this argument shows that $\Lambda[1, m+j)\underset{\Lambda}{\otimes}M$ is torsion free and $\mathrm{Tor}_1^\Lambda(\Lambda[1, m+j), M)\to\mathrm{Tor}_1^\Lambda(\Lambda[1, m), M)$ is epic for every $j\geqq 1$. Notice that $M_t\cong(\Lambda[1, m+l)\underset{\Lambda}{\otimes}M)_t$ and $\mathrm{Tor}_{1,t}^\Lambda(\Lambda[1, m+l), M)=0$ for sufficiently large l. Then it follows immediately that M is also torsion free and $\mathrm{Tor}_1^\Lambda(\Lambda[1, m), M)=0$.

Lemma 1.7. *Let M be a connective Λ-module, $m\geqq 0$ and assume that k is a subring R of Q. Then* $\mathrm{w\,dim}_\Lambda M\underset{R}{\otimes}Q\leqq m$ *if and only if* $\mathrm{Tor}_{m+1}^\Lambda(Q, M)=0$.

Proof. The "only if" part is immediate because $\mathrm{Tor}_n^\Lambda(Q, M)\cong\mathrm{Tor}_n^\Lambda(R, M\underset{R}{\otimes}Q)$. Putting $F=\Lambda\underset{R}{\otimes}(Q\underset{\Lambda}{\otimes}M)$, it is a connective free $\Lambda\underset{R}{\otimes}Q$-module. We then obtain a Λ-homomorphism $f\colon F\to M\underset{R}{\otimes}Q$ for which $1\underset{\Lambda}{\otimes}f\colon R\underset{\Lambda}{\otimes}F\to R\underset{\Lambda}{\otimes}(M\underset{R}{\otimes}Q)\cong Q\underset{\Lambda}{\otimes}M$ is the identity. First we suppose that $\mathrm{Tor}_1^\Lambda(Q, M)\cong\mathrm{Tor}_1^\Lambda(R, M\underset{R}{\otimes}Q)=0$. Noting that $\Lambda\underset{R}{\otimes}Q$ is Λ-flat, we find by (1.4) that f is an isomorphism. Thus $M\underset{R}{\otimes}Q$ is $\Lambda\underset{R}{\otimes}Q$-free and so it is Λ-flat. The "if" part is easily shown by induction on m.

Proposition 1.8. *Let M be connective Λ-module and assume that k is a subring R of Q. The three conditions in* I) *are equivalent, the three in* II) *are so and also the four in* III) *are so for each $m\geqq 1$:*

I) 0) *M is Λ-free,* i) *M is Λ-projective, and* ii) *$R\underset{\Lambda}{\otimes}M$ is R-free and $M\underset{R}{\otimes}Q$ is Λ-flat.*

II) iii) *M is Λ-flat,* iv) $\mathrm{Tor}_1^\Lambda(Q/R, M)=0$ *and $M\underset{R}{\otimes}Q$ is Λ-flat, and* v) *$R\underset{\Lambda}{\otimes}M$ is R-flat (i.e., torsion free) and $M\underset{R}{\otimes}Q$ is Λ-flat.*

III) $\mathrm{i})_m$ $\mathrm{hom\,dim}_\Lambda M\leqq m$ *and* $\mathrm{w\,dim}_\Lambda M\underset{R}{\otimes}Q\leqq m-1$, $\mathrm{iii})_m$ $\mathrm{w\,dim}_\Lambda M\leqq m$ *and* $\mathrm{w\,dim}_\Lambda M\underset{R}{\otimes}Q\leqq m-1$, $\mathrm{iv})_m$ $\mathrm{Tor}_{m+1}^\Lambda(Q/R, M)=0$ *and* $\mathrm{Tor}_m^\Lambda(Q, M)=0$, *and* $\mathrm{v})_m$ $\mathrm{Tor}_m^\Lambda(R, M)=0$.

Proof. I) 0)→i)→ii) are obvious and we use Lemmas 1.4 and 1.6 to show the implication ii)→0).

III) Evidently $\mathrm{i})_m\to\mathrm{iii})_m$ and $\mathrm{iv})_m\to\mathrm{v})_m$, and $\mathrm{iii})_m\to\mathrm{iv})_m$ are immediate from Lemma 1.7. It remains for us to show the implication $\mathrm{v})_m\to\mathrm{i})_m$. Since $\mathrm{Tor}_m^\Lambda(Q, M)\cong\mathrm{Tor}_m^\Lambda(R, M)\underset{R}{\otimes}Q=0$, we note by Lemma 1.7 that $\mathrm{w\,dim}_\Lambda M\underset{R}{\otimes}Q\leqq m-1$. We first suppose that $\mathrm{Tor}_1^\Lambda(R, M)=0$ and choose a short exact sequence $0\to N\to F\to M\to 0$ of connective Λ-modules with F Λ-free. Then this yields short exact sequences $0\to R\underset{\Lambda}{\otimes}N\to R\underset{\Lambda}{\otimes}F\to R\underset{\Lambda}{\otimes}M\to 0$ and $0\to N\underset{R}{\otimes}Q\to F\underset{R}{\otimes}Q\to M\underset{R}{\otimes}Q\to 0$. So we see that $R\underset{\Lambda}{\otimes}N$ is R-free and $N\underset{R}{\otimes}Q$ is Λ-flat. Therefore I) says that N is Λ-free and so hom $\dim_{\Lambda}M\leqq 1$. By induction on m we get the result as desired.

II) Evidently iii)$\to$iv)$\to$v). We shall show the implication v)$\to$iii). By use of Lemma 1.6 we note that M is torsion free. Applying the short exact sequence $0\to M\to M\underset{R}{\otimes}Q\to M\underset{R}{\otimes}Q/R\to 0$ we obtain $\mathrm{Tor}_1^{\Lambda}(R, M\underset{R}{\otimes}Q/R)=O$. Then III) gives us that w $\dim_{\Lambda}M\underset{R}{\otimes}Q/R\leqq 1$ and hence M is Λ-flat.

2. Homology theories $BP[n, m+1)_*$ ( )

2.1. Let BP denote the Brown-Peterson spectrum at a fixed prime p. It is a ring spectrum which gives the multiplicative homology theory $BP_*(\ )$ defined on the category of CW-spectra. It has coefficient ring $BP_*=Z_{(p)}[v_1, \cdots, v_n, \cdots]$ where $Z_{(p)}$ is the ring of integers localized at the prime p and the polynomial generator v_n has degree $2(p^n-1)$. It is often convenient to put $v_0=p$. Using Baas-Sullivan theory of manifolds with singularities [1], we can kill the generators $v_0, v_1, \cdots, v_{n-1}, v_{m+1}, v_{m+2}, \cdots$ of BP_* to construct a homology theory $BP[n, m+1)_*(\ )$ which is represented by a certain CW-spectrum $BP[n, m+1)$, where $0\leqq n\leqq m+1\leqq\infty$. Its coefficient is

$$BP[n, m+1)_* = BP_*/(p, v_1, \cdots, v_{n-1}, v_{m+1}, \cdots) \cong \begin{cases} Z_{(p)}[v_1, \cdots, v_m] & \text{if } n=0 \\ Z_p[v_n, \cdots, v_m] & \text{if } n\geqq 1\,. \end{cases}$$

In particular we write

$$BP[n, \infty) = P(n),\quad BP[n, n+1) = k(n) \quad \text{for } 0\leqq n\leqq\infty\,, \text{ and}$$
$$BP[0, m+1) = BP\langle m\rangle \quad \text{for } -1\leqq m\leqq\infty \quad \text{(see [3] and [4])}$$

and note

$$BP[0, \infty) = P(0) = BP,\quad BP[0, 1) = k(0) = HZ_{(p)} \text{ and}$$
$$BP[n, n) = HZ_p \quad \text{for } 0\leqq n\leqq\infty\,.$$

Remark that $BP[n, m+1)^*(X)$ and $E^*(X\wedge BP[n, m+1))$ are always Hausdorff whenever $n\geqq 1$ and $\pi_*(E)\otimes Z_{(p)}$ is of finite type, and also $BP[0, m+1)^{2*}(X)$ is so if $\pi_{2*+1}(X)\otimes Q=0$ (use [12]). Since $BP[n, m+1)_*(\ )$ becomes a $BP_*(\ )$-module,

(2.1) *$BP[n, m+1)$ is a BP-module spectrum whose skeletons are finite where $0\leqq n\leqq m+1\leqq\infty$.*

For $0\leqq n\leqq k\leqq l+1\leqq m+1\leqq\infty$ we have a BP-module map $\mu_{n,m}^{k,l}$: $BP[n, m+1)\to BP[k, l+1)$ such that $\mu_{n,m}^{n,m}=id$ and $\mu_{k,l}^{q,r}\cdot\mu_{n,m}^{k,l}=\mu_{n,m}^{q,r}$. As a basic relation between BP-module spectra $BP[n, m+1)$ we obtain cofibering sequences

$$\text{(2.2)}\quad \text{i)}\ S^{2(p^n-1)}BP[n, m+1)\xrightarrow{\cdot v_n}BP[n, m+1)\to BP[n+1, m+1)$$

$$\text{ii)}\ S^{2(p^m-1)}BP[n, m+1)\xrightarrow{\cdot v_m}BP[n, m+1)\to BP[n, m)$$

where $\cdot v_i$, $i=n$ or m, is given by composition $S^{2(p^i-1)}BP[n, m+1)\xrightarrow{v_i\wedge 1}BP\wedge BP[n, m+1)\to BP[n, m+1)$. Because there exist natural exact sequences

$$\text{(2.2)}'\quad \cdots\to BP[n, m+1)_{*-2(p^n-1)}(X)\xrightarrow{\cdot v_n}BP[n, m+1)_*(X)\to BP[n+1, m+1)_*(X)\to$$

$$\cdots\to BP[n, m+1)_{*-2(p^m-1)}(X)\xrightarrow{\cdot v_m}BP[n, m+1)_*(X)\to BP[n, m)_*(X)\to$$

of $BP_*(\ )$-module homology theories in which $\cdot v_i$, $i=n$ or m, acts as multiplication by v_i. (Cf., [13, Appendix] for $n=0$). Notice that the cofiberings are of BP-module spectra except in the case ii) $n=0$.

According to a geometric observation of Morava's (see [4]),

(2.3) *$P(n)_*(X)$ is a module over $P(n)_*=BP_*/(p, v_1, \cdots, v_{n-1})$ for any CW-spectrum X.*

Then $\mu_n^{k,l}: P(n)\to BP[k, l+1)$ gives rise to a homomorphism $\tilde{\mu}_n^{k,l}$: $BP[k, l+1)_*\underset{P(n)_*}{\otimes}P(n)_*(X)\to BP[k, l+1)_*(X)$ when $0\leqq n\leqq k\leqq l+1\leqq\infty$.

2.2. By a *connective* CW-spectrum we mean one which is l-connected for some integer l. We may assume that a CW-spectrum X has no cells in dimension less than $l+1$ if it is l-connected.

Lemma 2.1. *Let X be a connective CW-spectrum and $n\geqq 0$. The following three conditions are equivalent:*
i) *$P(n)_*(X)$ is $P(n)_*$-flat,*
ii) *$\mu_n: P(n)_*(X)\to HZ_{p*}(X)$ is epic, and*
iii) *μ_n induces an isomorphism $\tilde{\mu}_n: Z_p\underset{P(n)_*}{\otimes}P(n)_*(X)\to HZ_{p*}(X)$.*

Proof. Trivially iii)→ii). If $P(n)_*(X)\to HZ_{p*}(X)$ is epic, then the Atiyah-Hirzebruch spectral sequence for $P(n)_*(X)$ collapses. Therefore ii)→i) and ii)→iii) are easily checked. So it is sufficient to show the implication i)→ii). The exact sequence (2.2)′ yields a short exact sequence

$$0\to P(n+1)_*\underset{P(n)_*}{\otimes}P(n)_*(X)\to P(n+1)_*(X)\to \mathrm{Tor}_1^{P(n)_*}(P(n+1)_*, P(n)_*(X))\to 0\,.$$

If $P(n)_*(X)$ is $P(n)_*$-flat, then we get

$$\mathrm{Tor}_s^{P(n+1)_*}(M, P(n+1)_*(X)) \cong \mathrm{Tor}_s^{P(n)_*}(M, P(n)_*(X)) = 0$$

for all $P(n+1)_*$-modules M and $s \geqq 1$. Thus $P(n+1)_*(X)$ is also $P(n+1)_*$-flat and $P(n)_*(X) \to P(n+1)_*(X)$ is epic. A repetition of this argument shows that $P(n)_*(X) \to P(m)_*(X)$ is epic for every $m \geqq n$. Observing that $P(m)_t(X) \cong (HZ_p)_t(X)$ for sufficiently large m, we obtain the required implication.

Notice that $BPQ_*(X) \cong HQ_*(X) \otimes BP_*$ and it is BP_*-flat. Putting Proposition 1.8 and Lemma 2.1 together we have

Corollary 2.2. *Let X be a connective CW-spectrum. The conditions* I) *and* II) *are respectively equivalent* (cf., [11, Proposition 6]):
I) 0) *$BP_*(X)$ is BP_*-free,* i) *$BP_*(X)$ is BP_*-projective, and* ii) *$HZ_{(p)*}(X)$ is $Z_{(p)}$-free.*
II) iii) *$BP_*(X)$ is BP_*-flat,* iv) $\mathrm{Tor}_1^{BP_*}(Z_p, BP_*(X))=0$, *and* v) *$HZ_{(p)*}(X)$ is $Z_{(p)}$-flat (i.e., torsion free).*

From Proposition 1.5, Lemma 2.1 and Corollary 2.2 it follows immediately

(2.4) *$P(n)_*(X)$ is $P(n)_*$-free (or $P(n)_*$-flat) if and only if so are $P(n)_*(X^s)$ for all s where X^s denotes the s-skeleton of a connective CW-spectrum X.*

The cofibering (2.2) gives short exact sequences

(2.5)
$$0 \to P(m)_*(P(n)) \to P(m)_*(P(n+1)) \to P(m)_{*-2p^n+1}(P(n) \to 0$$
$$0 \to P(m)^{*-2p^n+1}(P(n)) \to P(m)^*(P(n+1)) \to P(m)^*(P(n)) \to 0$$

for $m > n \geqq 0$ [4, Lemma 2.8]. Using these exact sequences an induction on n shows

(2.6) *$P(m)_*(P(n)^s)$ and $P(m)_*(D(P(n)^s)) \cong P(m)^{-*}(P(n)^s)$ are $P(m)_*$-free for $m \geqq n \geqq 0$ where $D(P(n)^s)$ is the Spanier-Whitehead dual of the s-skeleton $P(n)^s$.*

By a *partial $P(n)_*$-resolution* of a connective CW-spectrum X, $n \geqq 0$, we mean a cofibering sequence $W \xrightarrow{f} X \subset Y$ of connective CW-spectra such that $P(n)_*(W)$ is $P(n)_*$-free and the induced map $f_*: P(n)_*(W) \to P(n)_*(X)$ is epic.

The existence of partial $P(n)_*$-resolutions was implicitly given by Johnson-Wilson [4, Lemma 4.10].

Proposition 2.3. *Let X be a l-connected CW-spectrum and $n \geqq 0$. Then there exists a partial $P(n)_*$-resolution $W \to X \subset Y$ of X consisting of l-connected CW-spectra.*

Proof. All generators x of $P(n)_*$-module $P(n)_*(X)$ are realized in forms $S^* \xrightarrow{x'} X\wedge P(n)^{*-l-1}\subset X\wedge P(n)$. The duals of x' give a map $f: W=\bigvee_x S^*D(P(n)^{*-l-1})\to X$ for which $f_*: P(n)_*(W)\to P(n)_*(X)$ is epic. W is evidently l-connected and $P(n)_*(W)$ is $P(n)_*$-free by (2.6).

Proposition 2.4. *Let X be a connective CW-spectrum and $0\leqq n\leqq k\leqq l+1\leqq\infty$. Then there exists a natural spectral sequence*

$$E^2_{s,*} = \mathrm{Tor}^{P(n)_*}_{s,*}(BP[k, l+1)_*, P(n)_*(X)) \Rightarrow BP[k, l+1)_*(X)$$

whose edge homomorphism coincides with the reduced map

$$\tilde{\mu}^{k,l}_n: BP[k, l+1)_* \underset{P(n)_*}{\otimes} P(n)_*(X)\to BP[k, l+1)_*(X)\,.$$

Proof. Making use of Proposition 2.3 we establish a diagram

$$\begin{array}{l} X = X_0\subset X_1\subset\cdots\subset X_s\subset X_{s+1}\subset\cdots \\ \quad\ \ \nwarrow\quad \nwarrow\qquad\quad \nwarrow \\ \quad\ \ W_0\ \ W_1\qquad\ \ W_s \end{array}$$

such that $W_s\to X_s\subset X_{s+1}$ is a partial $P(n)_*$-resolution of X_s for any $s\geqq 0$. Note that W_s, X_s and the union $X_\infty=\cup X_s$ are all connective, and in addition that $P(n)_*(X_\infty)=0$. By means of Lemma 2.1 we see that $HZ_{p*}(X_\infty)=0$ and so $BP[k, l+1)_*(X_\infty)=0$. We now observe the spectral sequence for $BP[k, l+1)_*(X)$ arising from the increasing filtration $\{X_{s+1}/X\}$. By definition and use of Lemma 2.1 we obtain an exact couple with

$$\begin{aligned} D^1_{s,*} &= BP[k, l+1)_{s+*+1}(X_{s+1}/X) \quad \text{and} \\ E^1_{s,*} &\cong BP[k, l+1)_{s+*}(W_s) \cong BP[k, l+1)_* \underset{P(n)_*}{\otimes} P(n)_*(W_s)\,. \end{aligned}$$

Since the above diagram gives rise to a free $P(n)_*$-resolution

$$\to P(n)_{*+s}(W_s)\to\cdots\to P(n)_{*+1}(W_1)\to P(n)_*(W_0)\to P(n)_*(X)\to 0$$

of $P(n)_*(X)$, a standard argument shows that this spectral sequence is a satisfactory one.

2.3. Let X be a connective CW-spectrum and $0\leqq n\leqq k\leqq l+1\leqq m+1\leqq\infty$. Suppose that $BP[n, m+1)_i(X)\to BP[k, l+1)_i(X)$ is epic for any $i\leqq t$. Making use of the exact sequences (2.2)′ an induction on degree shows

(2.7) i) *$BP[n, m+1)_i(X)\to BP[k-1, l+1)_i(X)$ is epic for any $i\leqq t$ when $n\leqq k-1$ and $k\geqq 2$, and also*

ii) *$BP[n, m+1)_i(X)\to BP[k, l+2)_i(X)$ is so when $1\leqq l+1\leqq m$.*

On the other hand, we note that when $BP[0, l+1)_i(X)$ is torsion free for any $i\leqq t$, then so is $BP[0, r+1)_i(X)$ for $r\geqq l$. This then implies that $BP[0, r+1)_j(X)\to BP[0, r)_j(X)$ is epic for $j\leqq t+2p^r-1$. Hence we get

(2.8) *$BP_j(X)\to BP[0, r)_j(X)$ is epic for each $j\leqq t+2p^r-1$ and $r\geqq l$ if $BP[0, l+1)_i(X)$ is torsion free for $i\leqq t$.*

By iterated applications of (2.7) and use of (2.8) we have

Lemma 2.5. *Let X be a connective CW-spectrum and $0\leqq n\leqq k\leqq l+1\leqq m+1\leqq\infty$. If $BP[n, m+1)_i(X)\to BP[k, l+1)_i(X)$ is epic for any $i\leqq t$, then $BP[n, m+1)_i(X)\to BP[q, r+1)_i(X)$ is so where $n\leqq q\leqq k$ and $m\geqq r\geqq l$.*

We next apply the above result to obtain a useful tool (cf., [8, Corollary 4]).

Lemma 2.6. *Let X be a connective CW-spectrum, $W\to X\subset Y$ a partial $P(n)_*$-resolution of X and $0\leqq n\leqq k<l+1\leqq\infty$. If $P(n)_i(X)\to BP[k, l+1)_i(X)$ is epic for each $i\leqq t$, then $P(n)_j(Y)\to BP[k, l)_j(Y)$ and $P(n)_j(Y)\to BP[k+1, l+1)_j(Y)$ are both epic for $j\leqq t+2p^n-l$.*

Proof. We first remark by use of Lemma 2.5 that $P(n)_i(X)\to BP[q, r+1)_i(X)$ is epic for $i\leqq t$, where $n\leqq q\leqq k$ and $l\leqq r\leqq\infty$. Hence $BP[q, r+1)_i(Y)\to BP[q, r+1)_{i-1}(W)$ becomes monic. Therefore, for $q\leqq s\leqq r$ the multiplication $\cdot v_s\colon BP[q, r+1)_i(Y)\to BP[q, r+1)_{i+2(p^s-1)}(Y)$ is monic. In particular this means that $BP[q, r+1)_j(Y)\to BP[q, r)_j(Y)$ and $BP[q, r+1)_j(Y)\to BP[q+1, r+1)_j(Y)$ are epic for all $j\leqq t+2p^q-l$. Observing that $P(n)_j(Y)\cong BP[n, m+1)_j(Y)$ for sufficiently large m, we get the required result immediately.

Theorem 2.7. *Let X be a connective CW-spectrum and $n\leqq k\leqq l+1$. The following three conditions are equivalent:*

i) $\mathrm{Tor}_s^{P(n)_*}(BP[k, l+1)_*, P(n)_*(X))=0$ *for all* $s\geqq 1$,

ii) $\mu_n^{k,l}\colon P(n)_*(X)\to BP[k, l+1)_*(X)$ *is epic, and*

iii) $\mu_n^{k,l}$ *induces an isomorphism* $\tilde{\mu}_n^{k,l}\colon BP[k, l+1)_*\underset{P(n)_*}{\otimes} P(n)_*(X)\to BP[k, l+1)_*(X)$.

Proof. iii)$\to$ii) is trivial and i)$\to$iii) is an immediate consequence of Proposition 2.4.

ii)$\to$i): We first suppose that $P(n)_*(X)\to BP[n, l+1)_*(X)$ is epic. If $n=l+1$, then our hypothesis means that $P(n)_*(X)$ is $P(n)_*$-flat. So we assume that $n<l+1$ and proceed inductively. Choosing a partial $P(n)_*$-resolution $W\to X\subset Y$ of X, then Lemma 2.6 asserts that $P(n)_*(Y)\to BP[n, l)_*(Y)$ is epic and so $\mathrm{Tor}_s^{P(n)_*}(BP[n, l)_*, P(n)_*(Y))=0$ for all $s\geqq 1$. We now use (1.3) when $l\geqq 1$ and Corollary 2.2 when $l=0$ to compute that $\mathrm{Tor}_s^{P(n)_*}(BP[n, l+1)_*, P(n)_*(Y))=0$ for all $s\geqq 1$, and hence $BP[n, l+1)_*\underset{P(n)_*}{\otimes} P(n)_*(Y)\to BP[n, l+1)_*(Y)$

is an isomorphism. Then we see easily that $\mathrm{Tor}_1^{P(n)_*}(BP[n, l+1)_*, P(n)_*(X))=0$ because $BP[n, l+1)_*(Y)\to BP[n, l+1)_{*-1}(W)$ is monic. Moreover we note that $\mathrm{Tor}_{s+1}^{P(n)_*}(BP[n, l+1)_*, P(n)_*(X))\cong \mathrm{Tor}_s^{P(n)_*}(BP[n, l+1)_*, P(n)_*(Y))=0$ for all $s\geqq 1$.

We now apply induction on k ($\geqq n$). Assuming that $k>n$, $P(n)_*(X)\to BP[k-1, l+1)_*(X)$ becomes epic by Lemma 2.5. So induction hypothesis says that $\mathrm{Tor}_s^{P(n)_*}(BP[k-1, l+1)_*, P(n)_*(X))=0$ for all $s\geqq 1$ and so $BP[k-1, l+1)_* \otimes_{P(n)_*} P(n)_*(X)\to BP[k-1, l+1)_*(X)$ is an isomorphism. Since $BP[k-1, l+1)_*(X)\to BP[k, l+1)_*(X)$ is epic, it follows immediately that $\mathrm{Tor}_s^{P(n)_*}(BP[k, l+1)_*, P(n)_*(X))=0$ for all $s\geqq 1$.

3. The abelian category $\mathcal{P}(n)$ and Filtration theorem

3.1. Let $\mathcal{E}$ be the collection of all exponent sequences $E=(e_1, \cdots, e_i, \cdots)$ of non-negative integers with all but finitely many are zero, and $\mathcal{C}_n$ that of all exponent sequences $C=(c_0, \cdots, c_{n-1})$ consisting of zeros and ones. We put $|E|=\sum_{i\geqq 1} 2(p^i-1)e_i$ and $|C|=\sum_{0\leqq i\leqq n-1}(2p^i-1)c_i$. For each $E\in\mathcal{E}$ there exists a BP-operation $r_E\colon BP\to S^{|E|}BP$ with the following properties:

(3.1) i) r_0 *is the identity,*

ii) $r_E\cdot m_0=\sum_{E=F+G} m_0(r_F\wedge r_G)\in BP^{|E|}(BP\wedge BP)$

where $m_0\colon BP\wedge BP\to BP$ denotes the pairing of the ring spectrum BP, and

iii) $r_G\cdot r_F=\sum_E \lambda_E^{G,F} r_E\in BP^{|F|+|G|}(BP)$

in which $\lambda_E^{G,F}\in BP_{|E|-|F|-|G|}$ is zero when $|E|\geqq p/p-1(|F|+|G|)$, [9, Lemma 7.11].

Recall that $P(n)$ is a BP-module spectrum with the pairing $m_n\colon BP\wedge P(n)\to P(n)$, and that $S^{2(p^n-1)}P(n)\xrightarrow{\cdot v_n} P(n)\xrightarrow{g_n} P(n+1)\xrightarrow{h_n} S^{2p^n-1}P(n)$ is a BP-module cofibering. For $k\geqq 0$ we define a $P(k+1)$-operation $(Q_k)_{k+1}\colon P(k+1)\to S^{2p^k-1}p(k+1)$ by putting $(Q_k)_{k+1}=g_k\cdot h_k$. Obviously we have

(3.2) $(Q_k)_{k+1}\cdot m_{k+1}=m_{k+1}(1\wedge(Q_k)_{k+1})\in P(k+1)^{2p^k-1}(BP\wedge P(k+1))$.

Making use of (2.5) we construct (non-unique) $P(n)$-operations $(r_E)_n\colon P(n)\to S^{|E|}P(n)$ such that $(r_0)_n=id$, $(r_E)_0=r_E$ and $g_{n-1}\cdot(r_E)_{n-1}=(r_E)_n\cdot g_{n-1}$. Similarly $P(k+1)$-operations $(Q_k)_{k+1}$ gives rise to (non-unique) $P(n)$-operations $(Q_k)_n\colon P(n)\to S^{2p^k-1}P(n)$, $n>k+1$, so that $g_{n-1}\cdot(Q_k)_{n-1}=(Q_k)_n\cdot g_{n-1}$. To each $C=(c_0, \cdots, c_{n-1})\in\mathcal{C}_n$ we correspond a $P(n)$-operation $(Q^C)_n\colon P(n)\to S^{|C|}P(n)$ given by the composition $(Q^C)_n=(Q_0^{c_0})_n\cdots(Q_{n-1}^{c_{n-1}})_n$.

Fix such $P(n)$-operations $(r_E)_n$ and $(Q^C)_n$. By R we denote the (graded) free $Z_{(p)}$-module whose basis is given by all exponent sequences E in $\mathcal{E}$ with degree $|E|$, and by $E_n=E[\Delta_0, \cdots, \Delta_{n-1}]$ the (graded) exterior algebra over Z_p in n-variables Δ_i with degree $2p^i-1$. Identifying Δ_i with the exponent sequence $C=(c_0, \cdots, c_{n-1})$ with $c_i=1$ and $c_j=0$, $i\neq j$, all C in $\mathcal{C}_n$ form a basis of the Z_p-module E_n. Note that each element of $P(n)^*$ has non-positive degree as $P(n)^*\cong P(n)_{-*}$, but that $P(n)^*(P(n))$ is Hausdorff. We here obtain a $P(n)^*$-module isomorphism

(3.3) $\Phi_n\colon P(n)^*\hat{\otimes}R\otimes E_n\to P(n)^*(P(n))$

defined by $\Phi_n(\lambda\otimes E\otimes C)=\lambda(r_E)_n\cdot(Q^C)_n$ [4, Lemma 2.12]. This induces a $P(n+1)^*$-module isomorphism

$$\Phi_n'\colon P(n+1)^*\hat{\otimes}R\otimes E_n\to P(n+1)^*(P(n))$$

which is defined by $\Phi_n'(\lambda\otimes E\otimes C)=\lambda(r_E)_{n+1}\cdot(Q^C)_{n+1}\cdot g_n$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0\to & P(n+1)^*\hat{\otimes}R\otimes E_n & \to & P(n+1)^*\hat{\otimes}R\otimes E_{n+1} & \to & P(n+1)^*\hat{\otimes}R\otimes E_n & \to 0\\
 & \downarrow{\scriptstyle \Phi_n'} & & \downarrow{\scriptstyle \Phi_{n+1}} & & \downarrow{\scriptstyle \Phi_n'} & \\
0\to & P(n+1)^{*-2p^n+1}(P(n)) & \xrightarrow[h_n^*]{} & P(n+1)^*(P(n+1)) & \xrightarrow[g_n^*]{} & P(n+1)^*(P(n)) & \to 0
\end{array}$$

in which rows are exact and vertical arrows are all isomorphisms. We then remark

(3.4) *Any element in Image* h_n^* *is expressed in form* $\prod_{E,C}\lambda_{E,C}(r_E)_{n+1}\cdot(Q^C)_{n+1}\cdot(Q_n)_{n+1}$,

where the formal summation $\prod$ runs over all E in $\mathcal{E}$ and all C in $\mathcal{C}_n$ and $\lambda_{E,C}$ are elements of $P(n)^*$.

3.2. The pairing $m_n\colon BP\wedge P(n)\to P(n)$ gives us a homomorphism $\psi_n\colon P(n)^*(P(n))\to P(n)^*(BP\wedge P(n))\xleftarrow{\cong} BP^*(BP)\hat{\otimes}_{BP^*}P(n)^*(P(n))$ making $P(n)^*(P(n))$ into a $BP^*(BP)$-comodule. Note that the composition $(\iota^*\hat{\otimes}1)\cdot\psi_n$ is the identity where $\iota\colon S^0\to BP$ denotes the unit.

Lemma 3.1. *The coproduct actions on* $(r_E)_n$ *and* $(Q_j)_n$, $0\leqq j\leqq n-1$, *are uniquely given in forms of*

$$\psi_n(r_E)_n=\sum_{E=F+G}r_F\otimes(r_G)_n+\prod_{H\neq 0,K,C}\sum_{0\leqq i\leqq n-1}\lambda^{E,i}_{H,K,C}r_H\otimes(r_K)_n\cdot(Q^C)_n\cdot(Q_i)_n$$
$$\psi_n(Q_j)_n=1\otimes(Q_j)_n+\prod_{H\neq 0,K,C}\sum_{j<i\leqq n-1}\lambda^{j,i}_{H,K,C}r_H\otimes(r_K)_n\cdot(Q^C)_n\cdot(Q_i)_n$$

where the formal summations $\prod$ *run over all* $H\neq 0$, K *in* $\mathcal{E}$ *and all* C *in* $\mathcal{C}_i$, *and*

λ's are elements of $P(n)^*$ with suitable degrees.

Proof. The $n=0$ case is valid by (3.1) and (3.2). Assume inductively that $\psi_n(r_E)_n$ and $\psi_n(Q_j)_n$ are expressed as

$$\psi_n(r_E)_n = \sum_{E=F+G} r_F \otimes (r_G)_n + \prod_H \sum_{0\leqq i\leqq n-1} r_H \otimes (\nu_H^{E,i})_n \cdot (Q_i)_n$$
$$\psi_n(Q_j)_n = 1 \otimes (Q_j)_n + \prod_H \sum_{j<i\leqq n-1} r_H \otimes (\omega_H^{j,i})_n \cdot (Q_i)_n .$$

Then we can choose $P(n+1)$-operations $(\nu_H^{E,i})_{n+1}$ for $0\leqq i\leqq n-1$, and $(\omega_H^{j,i})_{n+1}$ for $j<i\leqq n-1$ which satisfy $g_n\cdot(\nu_H^{E,i})_n=(\nu_H^{E,i})_{n+1}\cdot g_n$ and $g_n\cdot(\omega_H^{j,i})_n=(\omega_H^{j,i})_{n+1}\cdot g_n$. Consider the short exact sequence

$$0\to BP^*(BP)\mathop{\hat{\otimes}}_{BP^*} P(n+1)^*(P(n)) \xrightarrow{1\hat{\otimes} h_n^*} BP^*(BP)\mathop{\hat{\otimes}}_{BP^*} P(n+1)^*(P(n+1)) \xrightarrow{1\hat{\otimes} g_n^*} BP^*(BP)\mathop{\hat{\otimes}}_{BP^*} P(n+1)^*(P(n))\to 0 .$$

A routine computation shows that $1\hat{\otimes}g_n^*(\psi_{n+1}(r_E)_{n+1})=1\hat{\otimes}g_n^*(\sum r_F\otimes(r_G)_{n+1}+\prod\sum r_H\otimes(\nu_H^{E,i})_{n+1}\cdot(Q_i)_{n+1})$. So we use (3.4) to gain the satisfactory expansion of $\psi_{n+1}(r_E)_{n+1}$. Similarly we get the required expression of $\psi_{n+1}(Q_j)_{n+1}$ for $0\leqq j\leqq n-1$. On the other hand, the $j=n$ case is immediate from (3.2). Finally we observe that $(\nu_0^{E,i})_n=(\omega_0^{j,i})_n=0$ because $(\iota^*\hat{\otimes}1)\cdot\psi_n=id$.

Lemma 3.2. *For each $F, G\in\mathcal{E}$ and $B, C\in C_n$ the products $(r_G)_n\cdot(Q^C)_n\cdot(r_F)_n\cdot(Q^B)_n$ are uniquely expanded to*

$$(r_G)_n\cdot(r_F)_n = \sum_E \lambda_E^{G,F}(r_E)_n + \prod_{K,D}\sum_{0\leqq i\leqq n-1}\lambda_{K,D,i}^{G,F}(r_K)_n\cdot(Q^D)_n\cdot(Q_i)_n$$

in the special case when $B=C=0$, and

$$(r_G)_n\cdot(Q^C)_n\cdot(r_F)_n\cdot(Q^B)_n = \prod_{K,D}\sum_{l\leqq i\leqq n-1}\lambda_{K,D,i}^{G,C,F,B}(r_K)_n\cdot(Q^D)_n\cdot(Q_i)_n$$

in which we denote by l the largest number j such that b_j or c_j is non-zero when $B=(b_0, \cdots, b_{n-1})\neq 0$ or $C=(c_0, \dots, c_{n-1})\neq 0$. Here the formal summations $\prod$ run over all K in $\mathcal{E}$ and all D in C_i, λ's are elements of $P(n)^*$ and particularly the elements $\lambda_E^{G,F}$ given in (3.1) *are viewed as those of $P(n)^*$.*

Proof. Making use of (3.1) iii) a similar discussion to Lemma 3.1 shows the first special case. In the second case we first note that $(r_G)_{l+1}\cdot(Q^C)_{l+1}\cdot(r_F)_{l+1}\cdot(Q^B)_{l+1}\cdot g_l=0$ because $(Q_l)_{l+1}\cdot g_l=0$. By induction on n ($\geqq l+1$) using (3.4) we see easily that the product $(r_G)_n\cdot(Q^C)_n\cdot(r_F)_n\cdot(Q^B)_n$ has the formal sum expansion as desired.

If $n<2(p-1)$, then $g_n^*: P(n+1)^{2(p-1)*}(P(n+1))\to P(n+1)^{2(p-1)*}(P(n))$ becomes an isomorphism. Therefore both $\psi_n(r_E)_n$ and $(r_G)_n\cdot(r_F)_n$ don't have supplementary terms. Thus

(3.5) $\psi_n(r_E)_n = \sum_{E=F+G} r_F \otimes (r_G)_n$ *and* $(r_G)_n \cdot (r_F)_n = \sum_E \lambda_E^{G,F}(r_E)_n$

when $n<2(p-1)$. (Cf., [4, Remark 2.14]).

3.3. By $\mathcal{P}(n)$ we denote the category of connective $P(n)_*$-modules M which are equipped with operations s_E on M of lower degree $|E|$ corresponding to all $E \in \mathcal{E}$ and Q_j of lower one $2p^j-1$ for $0 \leqq j \leqq n-1$, satisfying the following relations:

(3.6) i) *s_0 is the identity,*

ii) $s_E(\lambda x) = \sum_{E=F+G} r_F(\lambda) s_G(x) + \sum_{H \neq 0, K, C} \sum_{0 \leqq i \leqq n-1} \lambda_{H,K,C}^{E,i} r_H(\lambda) s_K \cdot Q^C \cdot Q_i(x)$

$Q_j(\lambda x) = \lambda Q_j(x) + \sum_{H \neq 0, K, C} \sum_{j<i \leqq n-1} \lambda_{H,K,C}^{j,i} r_H(\lambda) s_K \cdot Q^C \cdot Q_i(x)$,

iii) $s_G \cdot s_F(x) = \sum_E \lambda_E^{G,F} s_E(x) + \sum_{K,D} \sum_{l \leqq i \leqq n-1} \lambda_{K,D,i}^{G,F} s_K \cdot Q^D \cdot Q_i(x)$

$s_G \cdot Q^C \cdot s_F \cdot Q^B(x) = \sum_{K,D} \sum_{l \leqq i \leqq n-1} \lambda_{K,D,i}^{G,C,F,B} s_K \cdot Q^D \cdot Q_i(x)$

when $C \neq 0$ or $B \neq 0$, for any $\lambda \in P(n)_*$ and $x \in M$. Here λ's are the elements of $P(n)_*$ obtained in Lemmas 3.1 and 3.2, the non-negative integer l was defined in Lemma 3.2 and we write $Q^C = Q_0^{c_0} \cdots Q_{i-1}^{c_{i-1}}$ for each $C=(c_0, \cdots, c_{i-1})$, $1 \leqq i \leqq n$.

Let $\mathcal{P}(n)_0$ denote the full subcategory of $\mathcal{P}(n)$ consisting of finitely presented $P(n)_*$-modules. Notice that $\mathcal{P}(n)$ and $\mathcal{P}(n)_0$ are both abelian categories.

Let M be an object of $\mathcal{P}(n)$. Since $s_E(v_n x) = v_n s_E(x)$ and $Q_j(v_n x) = v_n Q_j(x)$ for all $x \in M$, the multiplication by v_n on M becomes a morphism in $\mathcal{P}(n)$. This implies that $P(n+1)_* \otimes_{P(n)_*} M$ is an object of $\mathcal{P}(n)$. Taking $Q_n=0$, the $P(n+1)_*$-module $P(n+1)_* \otimes_{P(n)_*} M$ is regarded as object of $\mathcal{P}(n+1)$. Therefore an iterated application shows

(3.7) *$P(m)_* \otimes_{P(n)_*} M$ lies in the category $\mathcal{P}(m)$, $m>n$, if M does in $\mathcal{P}(n)$.*

Every comodule M over $BP_*(BP)$ is provided with coaction map ψ: $M \to BP_*(BP) \otimes_{BP_*} M$ which is a BP_*-module homomorphism. This means that a connective comodule over $BP_*(BP)$ is just regarded as lying in $\mathcal{P}(0)$. Thus the category $\mathcal{P}(0)$ consists of all connective comodules over $BP_*(BP)$. Therefore a connective $P(n)_*$-module lies in $\mathcal{P}(n)$ whenever it admits a structure of comodule over $BP_*(BP)$. Conversely we remark by (3.5) that the category $\mathcal{P}(n)$ is just the full subcategory of $\mathcal{P}(0)$ consisting of $P(n)_*$-modules when $n<2(p-1)$.

Note that $r_H(\lambda)=0$ for $|H|>$degree λ and $(r_E)_n \cdot (Q^C)_n(x)=0$ provided $|E|+|C|>t$ for some positive integer t. By means of Lemmas 3.1 and 3.2 we have

(3.8) *$P(n)_*(X)$ lies in the category $\mathcal{P}(n)$ if X is a connective CW-spectrum.*

Recall that there exists a $P(n)^*$-module isomorphism $\Phi_n: P(n)^*\hat{\otimes} R\otimes E_n \to P(n)^*(P(n))$. Let $I^{(s)}$ be the submodule of $R\otimes E_n$ spanned by all $E\otimes C$ with degree $|E|+|C|>s$. Putting $F^{(s)}=P(n)^*(P(n))/\Phi_n(P(n)^*\hat{\otimes} I^{(s)})$, it is a finitely generated free $P(n)_*$-module which is $-(s+1)$-connected. Evidently $\Phi_n(P(n)^*\hat{\otimes} I^{(s)})$ is closed under the $P(n)$-operations $(r_E)_n$ and $(Q_j)_n$. As (3.8) we use Lemmas 3.1 and 3.2 to verify that $F^{(s)}$ lies in the category $\mathcal{P}(n)_0$.

We now show that the categories $\mathcal{P}(n)$ and $\mathcal{P}(n)_0$ have enough projectives.

Proposition 3.3. *For any object M of $\mathcal{P}(n)$ there exists an object F and a morphism $f: F\to M$ in this category so that F is $P(n)_*$-free and f is epic. If M is finitely generated, then F can be taken as so.*

Proof. The proof is essentially due to Landweber [7, Proposition 2.4]. We may assume that M is (-1)-connected. Take any element $x\in M$ with degree s. Note that $s_E\cdot Q^C(x)=0$ whenever $|E|+|C|>s$. We define a $P(n)_*$-module homomorphism $f_x': S^sP(n)^*(P(n))\to M$ by putting $f_x'(S^s\Phi_n(\lambda\otimes E\otimes C))=\lambda s_E\cdot Q^C(x)$. As is easily checked, f_x' is compatible with the operations, *i.e.*, $s_E\cdot f_x'=f_x'\cdot(r_E)_n$ and $Q_j\cdot f_x'=f_x'\cdot(Q_j)_n$. Therefore this induces a morphism $f_x: S^sF^{(s)}\to M$ in $\mathcal{P}(n)$ which has x in its image. When x runs over a set of generators of the $P(n)_*$-module M, we get a morphism $f: F\to M$ in $\mathcal{P}(n)$ so that F is $P(n)_*$-free and f is epic.

3.4. Since $P(n)_*$ is finitely presented as BP_*-module, we find

(3.9) *A $P(n)_*$-module M is finitely presented if and only if it is so as BP_*-module.*

Let M be an object of $\mathcal{P}(n)_0$. If $x\in M$ has lowest degree, then $s_E(\lambda x)=r_E(\lambda)x$ and $Q_j(\lambda x)=0$. Therefore $P(n)_*\cdot x$ is invariant under all operations of M, thus it is an invariant submodule of M. So it lies in $\mathcal{P}(n)_0$, and hence it is finitely presented as BP_*-module. Then the annihilator ideal $\mathrm{Ann}_{BP_*}(x)=\{\lambda\in BP_*;\lambda x=0\}$ becomes an invariant finitely generated ideal containing $I_n=(p, v_1, \cdots, v_{n-1})$.

We now show Filtration theorem in $\mathcal{P}(n)_0$, reducing it to Landwever's one [7, Theorem 2.3_{BP}]. In the following proof the idea was suggested by Yagita.

Theorem 3.4. *"Filtration theorem in $\mathcal{P}(n)_0$"*
Each object M of $\mathcal{P}(n)_0$ has a finite filtration

$$M=M_0\supset M_1\supset\cdots\supset M_s=0$$

in the category $\mathcal{P}(n)_0$ so that for $0\leqq i<s$, M_i/M_{i+1} is stably isomorphic to $P(k_i)_*$ in $\mathcal{P}(n)_0$ where $k_i\geqq n$.

Proof. Applying the method of Landweber [5, Lemma 3.3] we obtain a sequence

$$M=M_0\supset M_1\supset\cdots\supset M_t=0$$

by invariant submodules so that $M_i/M_{i+1}\cong BP_*/J_i$ in $\mathcal{P}(n)_0$ where J_i is an invariant finitely generated ideal containing $I_n=(p, v_1, \cdots, v_{n-1})$. Notice that the $P(n)_*$-module BP_*/J_i are coherent comodules over $BP_*(BP)$. Landweber filtration theorem combined with Invariant prime ideal theorem ([4] and [5]) asserts that there exists a filtration

$$BP_*/J_i=M_{i,0}\supset M_{i,1}\supset\cdots\supset M_{i,r}=0$$

by invariant submodules such that $M_{i,j}/M_{i,j+1}\cong P(k_{i,j})_*$ as comodule over $BP_*(BP)$ where $k_{i,j}\geqq n$. Consequently we get a satisfactory filtration of M.

REMARK. In the proof of our Filtration theorem we required only the assumption that the operations s_E and Q_j on M satisfy the relation ii) of (3.6), although Proposition 3.3 demanded that they satisfy all explicit relations i), ii) and iii) of (3.6). (Cf., [10]).

Let $\mathrm{Ass}_{BP_*}(M)$ denote the family of associated prime ideals of a BP_*-module M. Landweber [6, Theorem 3.1′] showed that $\mathrm{Ass}_{BP_*}(P(n)_*)=\{P_1, \cdots, P_r\}$ where P_i is one of the prime ideals $I_k=(p, v_1, \cdots, v_{k-1})$, $n\leqq k<\infty$. By virtue of [5, Lemma 3.2] our Filtration theorem implies

(3.10) $\mathrm{Ass}_{BP_*}(M)$ *consists of a finite number of prime ideals* $I_k=(p, v_1, \cdots, v_{k-1})$, $n\leqq k<\infty$, *for any object* M *of* $\mathcal{P}(n)_0$.

4. Exact functor theorem on $\mathcal{P}(n)$ and its applications

4.1. Let M be an object of $\mathcal{P}(n)$ and $\{M_\alpha\}$ the family of all invariant finitely generated submodules of M. Suppose that x is a non-zero element with lowest degree in $M-\bigcup_\alpha M_\alpha$. Then we can choose M_β so that the submodule $M_\beta(x)$ generated by M_β and x is closed under the operations s_E and Q_j. Evidently $M_\beta(x)=M_\alpha$ for some α, and so $x\in\bigcup_\alpha M_\alpha$. This is a contradiction. So we see

(4.1) *M is equal to the join of all M_α.*

A $P(n)_*$-module G is said to be *$\mathcal{P}(n)$-flat* when $\mathrm{Tor}_s^{P(n)_*}(G, M)=0$ for all $s\geqq 1$ and all $M\in\mathcal{P}(n)$. If a $P(n)_*$-module G is $\mathcal{P}(n)$-flat, then the functor $G\underset{P(n)_*}{\otimes}$ - on $\mathcal{P}(n)$ is exact.

Lemma 4.1. *Let G be a $P(n)_*$-module. If* $\mathrm{Tor}_1^{P(n)_*}(G, N)=0$ *for all* $N\in\mathcal{P}(n)_0$, *then G is $\mathcal{P}(n)$-flat.*

Proof. By virtue of Proposition 3.3 and (4.1) it is sufficient to show that $\mathrm{Tor}_1^{P(n)_*}(G, M)=0$ for any $M\in\mathcal{P}(n)$ which is finitely generated. Choose an exact sequence $0\to D\to F\to M\to 0$ in $\mathcal{P}(n)$ such that F is finitely generated and free. Let $\{D_\alpha\}$ be the family of all invariant finitely generated submodules of D. Since M is direct limit of $P(n)_*$-modules F/D_α lying in $\mathcal{P}(n)_0$, our hypothesis yields that $\mathrm{Tor}_1^{P(n)_*}(G, M)\cong \varinjlim \mathrm{Tor}_1^{P(n)_*}(G, F/D_\alpha)=0$.

We here give the $\mathcal{P}(n)$-version of Landweber exact functor theorem [7, Theorem 2.6_{BP}], applying our Filtration theorem.

Theorem 4.2. *"Exact functor theorem on $\mathcal{P}(n)$"*
Let G be a $P(n)_*$-module. The following three conditions are equivalent:
i) *G is $\mathcal{P}(n)$-flat,*
ii) $\mathrm{Tor}_1^{P(n)_*}(G, P(k)_*)=0$ *for all $k\geqq n$, and*
iii) *multiplications by v_k are monic on $G/(p, v_1, \cdots, v_{k-1})G$ for all $k\geqq n$.*

Proof. Evidently i)$\to$ii) and the converse follows from Theorem 3.4 and Lemma 4.1. ii)$\to$iii) is obvious and the converse is also valid because iii) implies that $\mathrm{Tor}_1^{P(n)_*}(G, P(n)_*)\to\mathrm{Tor}_1^{P(n)_*}(G, P(k)_*)$ is epic.

As consequence we get

Corollary 4.3. *If a $P(n)_*$-module G is $\mathcal{P}(n)$-flat, then $G\underset{P(n)_*}{\otimes} P(m)_*$ is $\mathcal{P}(m)$-flat for any $m\geqq n$.*

Proof. We observe that $\mathrm{Tor}_1^{P(m)_*}(G\underset{P(n)_*}{\otimes} P(m)_*, P(k)_*)\cong\mathrm{Tor}_1^{P(n)_*}(G,P(k)_*)=0$ for each $k\geqq m$.

4.2. If M lies in the category $\mathcal{P}(n)_0$, then Filtration theorem implies that $M[v_n^{-1}]$ is $P(n)_*[v_n^{-1}]$-free and so it is $P(n)_*$-flat. The same argument as before shows

$(4.2)_n$ *$M[v_n^{-1}]$ is $P(n)_*$-flat for each $M\in\mathcal{P}(n)$.*

Proposition 1.8 combined with $(4.2)_0$ shows

Proposition 4.4. *Let M be a connective comodule over $BP_*(BP)$. Then the following conditions in* I) *and* II) *are respectively equivalent* (cf., Corollary 2.2):
I) 0) *M is BP_*-free,* i) *M is BP_*-projective and* iii) *$Z_{(p)}\underset{BP_*}{\otimes} M$ is $Z_{(p)}$-free.*
II) iii) *M is BP_*-flat,* iv) $\mathrm{Tor}_1^{BP_*}(Z_p, M)=0$, *and* v) *$Z_{(p)}\underset{BP_*}{\otimes} M$ is $Z_{(p)}$-flat (i.e., torsion free).*

Taking $G=BP[n, m+1)_*[v_k^{-1}]$ for $n\leqq k\leqq m$ we apply our Exact functor theorem to obtain

(4.3) *$BP[n, m+1)_*[v_k^{-1}]$ is $\mathcal{P}(n)$-flat for $n\leqq k\leqq m$.*

We now have the following characterization of homological dimension in $\mathcal{P}(n)$ (cf., [7, Theorem 4.2]).

Theorem 4.5. *Let M be a $P(n)_*$-module lying in the category $\mathcal{P}(n)$.*
I) *Assume that $m \geqq 1$ when $n=0$ and that $m \geqq 0$ when $n \geqq 1$. Then* hom $\dim_{P(n)_*} M \leqq m$ *if and only if* w $\dim_{P(n)_*} M \leqq m$.
II) *The following four conditions are equivalent for $m \geqq n-1$:*
i) w $\dim_{P(n)_*} M \leqq m-n+1$, ii) $\mathrm{Tor}^{P(n)_*}_{m-n+2}(Z_p, M)=0$,
iii) $\mathrm{Tor}^{P(n)_*}_s(BP[n, m+1)_*, M)=0$ *for all $s \geqq 1$, and*
iv) $\mathrm{Tor}^{P(n)_*}_1(BP[n, m+1)_*, M)=0$.

Proof. I) and the equivalence of II) i) and ii) follow from Proposition 1.5 and Proposition 1.8 with $(4.2)_0$. The equivalence of iii) and iv) is immediate from Propositions 1.3 and 4.4, and iv)→ii) is obtained by Corollary 1.2 and Proposition 1.8 with $(4.2)_0$. We here apply Exact functor theorem to show the implication i)→iv). We proceed by induction on m ($\geqq n-1$), the $m=n-1$ case being trivial. Suppose that w $\dim_{P(n)_*} M \leqq m-n+1$ and then choose an exact sequence $0 \to N \to F \to M \to 0$ in $\mathcal{P}(n)$ with F $P(n)_*$-free. The induction hypothesis says that $\mathrm{Tor}^{P(n)_*}_2(BP[n, m)_*, M) \cong \mathrm{Tor}^{P(n)_*}_1(BP[n, m)_*, N)=0$. So we find that the multiplication by v_m is monic on $\mathrm{Tor}^{P(n)_*}_1(BP[n, m+1)_*, M)$. However (4.3) implies that $\mathrm{Tor}^{P(n)_*}_1(BP[n, m+1)_*, M)[v_m^{-1}]=0$ and hence $\mathrm{Tor}^{P(n)_*}_1(BP[n, m+1)_*, M)=0$ as desired.

4.3. Recall that $P(n)_*(X)$ lies in the category $\mathcal{P}(n)$ if X is connective. Since (4.3) means that the functor $BP[n, m+1)_*[v_k^{-1}] \underset{P(n)_*}{\otimes}$ - on $\mathcal{P}(n)$ is exact for $n \leqq k \leqq m$, we gain a new homology theory $BP[n, m+1)_* \underset{P(n)_*}{\otimes} P(n)_*(\ \ )[v_k^{-1}]$ on the category of CW-spectra.

Proposition 4.6. *Let X be a CW-spectrum and $n \leqq k \leqq m$. Then μ_n^m: $P(n) \to BP[n, m+1)$ induces an isomorphism*

$$\tilde{\mu}_n^m : BP[n, m+1)_* \underset{P(n)_*}{\otimes} P(n)_*(X)[v_k^{-1}] \to BP[n, m+1)_*(X)[v_k^{-1}]$$

of homology groups.

A $P(n)_*$-module M is said to be *v_k-torsion free, $k \geqq n$,* if multiplication by v_k on M is monic, *i.e.,* if the localization homomorphism $M \to M[v_k^{-1}]$ is monic.

Lemma 4.7. *Let X be a connective CW-spectrum and $n \leqq k \leqq m$. If $BP[n, m+1)_*(X)$ is v_k-torsion free, then $BP[n, m+j)_*(X)$ is so for each j, $1 \leqq i \leqq \infty$, and in addition $P(n)_*(X) \to BP[n, m)_*(X)$ is epic.*

Proof. Suppose that $BP[n, m+j)_*(X) \to BP[n, m+j)_*(X)[v_k^{-1}]$ is monic for $j\geqq 1$. As immediate consequence of Proposition 4.6 we have a short exact sequence $0\to BP[n, m+j+1)_*(X)[v_k^{-1}]\to BP[n, m+j+1)_*(X)[v_k^{-1}]\to BP[n, m+j)_*(X)[v_k^{-1}]\to 0$. Using this an induction on degree shows that $BP[n, m+j+1)_*(X)\to BP[n, m+j+1)_*(X)[v_k^{-1}]$ is monic. Moreover our assumption implies that $BP[n, m+j)_*(X)\to BP[n, m+j-1)_*(X)$ is epic. Noting that $P(n)_t(X)\cong BP[n, m+l)_t(X)$ for sufficiently large l, a repetition of this observation yields the desired result.

We now give our main result which is a characterization of homological dimension of $P(n)_*(X)$ (cf., [3, Theorem 1.1]).

Theorem 4.8. *Let X be a connective CW-spectrum.*
I) *Assume that $m\geqq 1$ when $n=0$ and that $m\geqq 0$ when $n\geqq 1$. Then* $\mathrm{hom\,dim}_{P(n)_*} P(n)_*(X)\leqq m$ *if and only if* $\mathrm{w\,dim}_{P(n)_*}P(n)_*(X)\leqq m$.
II) *The following seven conditions are equivalent for $m\geqq n-1$:*
  i) $\mathrm{w\,dim}_{P(n)_*}P(n)_*(X)\leqq m-n+1$,
  ii) $\mathrm{Tor}^{P(n)_*}_{m-n+2}(Z_p, P(n)_*(X))=0$,
  iii) $\mathrm{Tor}^{P(n)_*}_s(BP[n, m+1)_*, P(n)_*(X))=0$ *for all $s\geqq 1$,*
  iv) $\mathrm{Tor}^{P(n)_*}_1(BP[n, m+1)_*, P(n)_*(X))=0$
  v) $\mu_n^m: P(n)_*(X)\to BP[n, m+1)_*(X)$ *is epic,*
  vi) μ_n^m *induces an isomorphism* $\tilde{\mu}_n^m: BP[n, m+1)_*\underset{P(n)_*}{\otimes} P(n)_*(X)\to BP[n, m+1)_*(X)$, *and*
  vii) $\mu_{n,m+1}^{n,m}: BP[n, m+2)_*(X)\to BP[n, m+1)_*(X)$ *is epic.*

Proof. Since all $P(n)_*(X)$ are contained in the category $\mathcal{P}(n)$, I) and the equivalences of II) i), ii), iii) and iv) have already established in Theorem 4.5. On the other hand, the equivalences of iii), v) and vi) have done in Theorem 2.7. So it is sufficient to show the equivalence of v) and vii). Evidently v)$\to$vii) and for the converse we use Lemma 4.7 because vii) means that $BP[n, m+2)_*(X)$ is v_{m+1}-torsion free.

4.4. Let X be a connective CW-spectrum and X^s denote its s-skeleton. As is easily seen, we get

(4.4) i) $\cdot v_1^i: BP[0, 2)_{s+t}(X^s)\cong BP[0, 2)_{s+t+2(p-1)i}(X^s)$

for $-2(p-1)<t\leqq 0$ and $i\geqq 1$, and

ii) $\cdot v_n^i: BP[n, n+1)_{s+t}(X^s)\cong BP[n, n+1)_{s+t+2(p^n-1)i}(X^s)$

for $-2(p^n-1)<t\leqq 0$ and $i\geqq 1$ when $n\geqq 1$.
Making use of (4.4) we have

(4.5) $BP[0, m+1)_*(X)$ *is $Z_{(p)}$-free (or $Z_{(p)}$-flat) if and only if so are*

$BP[0, m+1)_*(X^s)$ *for all s, under the condition that* $m=0$ *or* 1.

Lemma 4.9. *Let X be a connective CW-spectrum and* $0\leqq n\leqq k\leqq l+1\leqq m+1\leqq\infty$. *Assume that* $l=-1, 0$ *or* 1 *when* $k=0$ *and that* $l=k-1$ *or k when* $k\geqq 1$. *Then* $BP[n, m+1)_i(X)\to BP[k, l+1)_i(X)$ *is epic for* $i\leqq s$ *if and only if so is* $BP[n, m+1)_*(X^s)\to BP[k, l+1)_*(X^s)$.

Proof. The "if' part is immediate.

The "only if" part: $BP[n, m+1)_i(X^s)\to BP[k, l+1)_i(X^s)$ is epic for any $i\leqq s$ as is easily checked. If $k=0$ and $l=-1$ or 0, or if $k\geqq 1$ and $l=k-1$, then $BP[k, l+1)_i(X^s)=0$ for $i\geqq s+1$. So we get the required result in these cases. On the other hand, we use (4.4) to get the required one in the case when $k=0$ and $l=1$, or when $k\geqq 1$ and $l=k$.

Combining Theorem 4.8 with Lemma 4.9 and Corollary 2.2 with (4.5) we obtain the $P(n)_*(\ )$-version of [2, Theorem 1].

Proposition 4.10. *Let X be a connective CW-spectrum and assume that* $m=0, 1$ *or* 2 *when* $n=0$ *and that* $m=0$ *or* 1 *when* $n\geqq 1$. *Then*

i) hom $\dim_{P(n)_*}P(n)_*(X)\leqq m$ *if and only if* hom $\dim_{P(n)_*}P(n)_*(X^s)\leqq m$ *for all s, and*

ii) w $\dim_{P(n)_*}P(n)_*(X)\leqq m$ *if and only if* w $\dim_{P(n)_*}P(n)_*(X^s)\leqq m$ *for all s.*

Proposition 4.11. *Let X be a connective CW-spectrum and* $0\leqq n\leqq m$. $P(n)_*(X)$ *is* v_n*-torsion free and* w $\dim_{P(n)_*}P(n)_*(X)\leqq m-n$ *if and only if* $BP[n, m+1)_*(X)$ *is* v_n*-torsion free.*

Proof. The "if" part is immediate from Lemma 4.7 and Theorem 4.8.

The "only if" part: Consider the short exact sequence $0\to P(n)_*(X)\xrightarrow{\cdot v_n} P(n)_*(X)\to P(n+1)_*(X)\to 0$. By means of (3.8) we note that $P(n+1)_*(X)$ lies in the category $\mathcal{P}(n)$, and also that w $\dim_{P(n)_*}P(n+1)_*(X)\leqq m-n+1$. Then Theorem 4.5 insists $\mathrm{Tor}_1^{P(n)_*}(BP[n, m+1)_*, P(n+1)_*(X))=0$. This yields that $BP[n, m+1)_*(X)$ is v_n-torsion free because $BP[n, m+1)_*\underset{P(n)_*}{\otimes} P(n)_*(X)\to BP[n, m+1)_*(X)$ is an isomorphism.

Finally we give another result of Johnson [2, Theorem 2] as corollary of Proposition 4.11.

Proposition 4.12. *Let X be a connective CW-spectrum. The following four conditions are equivalent*:

i) $BP[0, 2)_*(X)$ *is torsion free,*

ii) $BP[0, 2)_*(X^s)$ *is torsion free for every s,*

iii) $BP_*(X^s)$ *is torsion free for every s, and*

iv) $BP_*(X)$ *is torsion free and* hom $\dim_{BP_*}BP_*(X)\leqq 1$.

Proof. i)→ii)→iii) and iv)→i) follow from (4.5) and Proposition 4.11. It remains for us to show the implication iii)→iv). We see immediately that Ker $\{BP_*(X^s)\to BP_*(X)\}=$ Ker $\{BP_*(X^s)\to BP_*(X^{s+1})\}$ because Ker $\{BPQ_*(X^s)\to BPQ_*(X)\}=$ Ker $\{BPQ_*(X^s)\to BPQ_*(X^{s+1})\}$. This means that the Atiyah-Hirzebruch spectral sequence for $BP_*(X)$ collapses and so $BP_*(X)\to HZ_{(p)*}(X)$ is epic. Therefore hom $\dim_{BP_*}BP_*(X)\leqq 1$. On the other hand, it is trivial that $BP_*(X)$ is torsion free.

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