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K-GROUPS OF SYMMETRIC SPACES II

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1. Introduction

Let $M=G/K$ be a symmetric homogeneous space such that G is a simply connected compact Lie group. In [I] the author showed that the unitary K-group of M is isomorphic to the tensor product of $R(K)\underset{R(G)}{\otimes}Z$ and an exterior algebra E over Z, where $R(G)$ and $R(K)$ are the complex representation rigns of G and K respectively, and in particular described the generators of E as an exterior algebra explicitly.

The purpose of this paper is to present a structure of $R(K)\underset{R(G)}{\otimes}Z$ as a group in the following nine cases:

$$\text{Type of } M = AIII,\ BDI(a)(Spin(2p+2q+2)/Spin(2p+1)\cdot Spin(2q+1)),$$
$$BDII(b)(Spin(2n+1)/Spin(2n)),\ DIII,\ CII,\ EI,\ FI,\ FII \text{ or } G.$$

Now let us denote by $n(L)$ the order of the Weyl group of a compact connected Lie group L. We know that if U is a closed connected subgroup of G of maximal rank then $R(U)\underset{R(G)}{\otimes}Z$ is a free module of rank $n(G)/n(U)$ and is isomorphic to $K^*(G/U)$ [12]. Throughout this paper we shall identify $R(U)\underset{R(G)}{\otimes}Z$ with the K-group of G/U in the above situation and denote by the same letter ρ the element of $K^*(G/U)$ defined by an element ρ of $R(U)$ in the natural way. Furthermore we shall denote by $Z(g)$ the free module generated by an element g.

2. Representation rings

In this section we recall the structure of the complex representation rings of classical groups.

Write ρ_n for the canonical representations $SU(n)\to U(n)$, $U(n)\to U(n)$, $Sp(n)\to U(2n)$ and $Spin(n)\to SO(n)\to U(n)$ for each n, and write $\lambda^i\rho_n$ for the i-th exterior product of ρ_n. According to [10] we have

$$
(2.1)\quad
\begin{aligned}
R(SU(n)) &= Z[\lambda^1\rho_n, \cdots, \lambda^{n-1}\rho_n]\,,\\
R(U(n)) &= Z[\lambda^1\rho_n, \cdots, \lambda^n\rho_n, (\lambda^n\rho_n)^{-1}]\,,\\
R(Sp(n)) &= Z[\lambda^1\rho_n, \cdots, \lambda^n\rho_n] = Z[\sigma_1, \cdots, \sigma_n]\,,\\
R(Spin(2n+1)) &= Z[\lambda^1\rho_{2n+1}, \cdots, \lambda^{n-1}\rho_{2n+1}, \Delta_{2n+1}]\,,\\
R(Spin(2n)) &= Z[\lambda^1\rho_{2n}, \cdots, \lambda^{n-2}\rho_{2n}, \Delta^+_{2n}, \Delta^-_{2n}]\,.
\end{aligned}
$$

Here we denote by $\sigma_1, \cdots, \sigma_n$ the elementary symmetric functions in the n variables $t_1+t_1^{-1}, \cdots, t_n+t_n^{-1}$ when we set $R(T)=Z[t_1, t_1^{-1}, \cdots, t_n, t_n^{-1}]$ for a maximal torus T of $Sp(n)$, and denote by $\Delta^+_{2n}, \Delta^-_{2n}$ and Δ_{2n+1} the half-spin representations of $Spin(2n)$ and the spin representation of $Spin(2n+1)$, respectively.

Proposition 2.1 (See [17], p. 120). *If G is a compact Lie group, N is a finite normal subgroup of G and π: $G \to G/N$ is the canonical map, then there is a homomorphism of $R(G/N)$-modules π_*: $R(G) \to R(G/N)$ such that $\pi_*(1)=1$.*

Proof. It is easy to see that the correspondence $V \to V^N$, where V is a G-module and V^N the N-invariant submodule of V, defines the homomorphism π_*, as desired. q.e.d.

Using Proposition 2.1 we can calculate the representation rings of some quotient groups. For example,

$$
(2.2)\quad R(PSp(4)) = Z[\lambda^2\rho_4, \lambda^4\rho_4, (\rho_4)^2, (\lambda^3\rho_4)^2, \rho_4\lambda^3\rho_4]
$$

as a subring of $R(Sp(4))$ and

$$
R(Sp(3) \underset{Z_2}{\times} SU(2)) = Z[\lambda^2\rho_3, (\rho_3)^2, (\lambda^3\rho_3)^2, \rho_3\lambda^3\rho_3, (\rho_2)^2, \rho_2\rho_3, \rho_2\lambda^3\rho_3]
$$

as a subring of $R(Sp(3) \times SU(2))$ where Z_2 is the intersection of the centers of $Sp(3)$ and $SU(2)$.

Using the relations of [10], §13, Theorem 10.3 we get

$$
(2.3)\quad
\begin{aligned}
&R(Spin(2m+1) \underset{Z_2}{\times} Spin(2n+1))\\
&\quad = Z[\lambda^1\rho_{2m+1}, \cdots, \lambda^m\rho_{2m+1}, \lambda^1\rho_{2n+1}, \cdots, \lambda^n\rho_{2n+1}, \Delta_{2m+1}\Delta_{2n+1}]/I
\end{aligned}
$$

as a subalgebra of $R(Spin(2m+1) \times Spin(2n+1))$, where Z_2 is the intersection of the centers of $Spin(2m+1)$ and $Spin(2n+1)$, and I is the ideal generated by the element

$$
(\Delta_{2m+1}\Delta_{2n+1})^2 - (\lambda^m\rho_{2m+1} + \cdots + \lambda^1\rho_{2m+1} + 1)(\lambda^n\rho_{2n+1} + \cdots + \lambda^1\rho_{2n+1} + 1)\,.
$$

3. AIII, BDI(a), BDII(b) and CII

Type AIII($U(m+n)/U(m) \times U(n)$). Let $T = S^1_1 \times \cdots \times S^1_n$ be the canonical maximal torus of $U(n)$ where S_i^1, $1 \leqq i \leqq n$, are the circle groups, and set $R(S_i^1) = Z[t_i, t_i^{-1}]$ for each i where t_i is a standard 1-dimensional non-trivial representation of S_i^1. Moreover let us define F_k to be the free module generated by $1, t_k, \cdots, t_k^{m+k-1}$ for $k=1, \cdots, n$.

Lemma 3.1. *$R(U(m)\times T)$ is a free $R(U(m+n))$-module (by restriction) generated by $t_1^{a_1}\cdots t_n^{a_n}$ $(0\leqq a_k \leqq m+k-1)$. Namely,*

$$R(U(m)\times T) \cong R(U(m+n))\otimes F_1 \otimes \cdots \otimes F_n$$

with the above notation.

Proof. $R(U(m)\times U(1))$ is freely generated as an $R(U(m+1))$-module by $1, t, \cdots, t^m$, when we put $R(U(1)) = Z[t, t^{-1}]$ ([9], Lemma 7.3). Let

$$U_k = U(m+k)\times S_{k+1}^1 \times \cdots \times S_n^1 \qquad \text{for } k = 0, \cdots, n-1$$

and $\qquad U_n = U(m+n)$.

Then we have

$$R(U_k) \cong R(U_{k+1})\otimes F_{k+1}$$

for $k=0, \cdots, n-1$ and this implies Lemma 3.1.

Theorem 3.2.

$$K^*(U(m+n)/U(m)\times U(n)) \cong \bigoplus_{\substack{p_1 \geqq 0, \cdots, p_n \geqq 0 \\ p_1+\cdots+p_n \leqq m}} Z((\lambda^1 \rho_n)^{p_1} \cdots (\lambda^n \rho_n)^{p_n})$$

for $m, n \geqq 1$.

Proof. Put

$$G_k = U(m+n)/U(m)\times U(k)\times S_{k+1}^1 \times \cdots \times S_n^1 \qquad \text{for } k = 1, \cdots, n-1,$$
$$G_n = U(m+n)/U(m)\times U(n),$$

and

$$W_k = \bigoplus_{\substack{p_1 \geqq 0, \cdots, p_k \geqq 0 \\ p_1+\cdots+p_k \leqq m}} Z((\lambda^1 \rho_k)^{p_1} \cdots (\lambda^k \rho_k)^{p_k})$$

for $k=1, \cdots, n$.

$K^*(G_k)$ is a free module of rank $(m+n)!/m!k!$ and identified with $R(U(m) \times U(k)\times S_{k+1}^1 \times \cdots \times S_n^1) \underset{R(U(m+n))}{\otimes} Z$ for each k. In particular, from Lemma 3.1 we have

$$K^*(G_1) = F_1 \otimes \cdots \otimes F_n .$$

Therefore we see that $K^*(G_k)$ contains $W_k \otimes F_{k+1} \otimes \cdots \otimes F_n$ as a free subgroup by considering the injective homomorphism $K^*(G_k) \rightarrow K^*(G_1)$ for each k ([I], Proposition 7.1).

We have

$$\begin{aligned}\text{rank}\ \ W_k\otimes F_{k+1}\otimes\cdots\otimes F_n &= (\textstyle\sum_{s=0}^{m} {}_kH_s)(m+k+1)\cdots(m+n)\\ &= \tbinom{m+k}{k}(m+k+1)\cdots(m+n)\\ &= (m+n)!/m!\,k!\end{aligned}$$

where ${}_kH_s=\binom{k+s-1}{k-1}$ is the number of the repeated combination. This proves

(a) $K^*(G_k)\otimes Q = W_k\otimes F_{k+1}\otimes\cdots\otimes F_n\otimes Q \quad$ for $k=1,\cdots,n-1$,
$K^*(G_n)\otimes Q = W_n\otimes Q$.

Next we shall prove by induction on k

(b) $K^*(G_k) = W_k\otimes F_{k+1}\otimes\cdots\otimes F_n \quad$ for $k=1,\cdots,n-1$,
$K^*(G_n) = W_n$.

Since $W_1=F_1$, (b) in case of $k=1$ follows by Lemma 3.1. Suppose that (b) is true when $k=l$. For any element $x\in K^*(G_{l+1})$ there is an integer $N>0$ such that

$$Nx = \sum_{\substack{p_1\geqq 0,\cdots,p_{l+1}\geqq 0\\ p_1+\cdots+p_{l+1}\leqq m}} a_{p_1\cdots p_{l+1}}(\lambda^1\rho_{l+1})^{p_1}\cdots(\lambda^{l+1}\rho_{l+1})^{p_{l+1}}$$

where $a_{p_1\cdots p_{l+1}}\in F_{l+2}\otimes\cdots\otimes F_n$ by (a). Let

$$i^*\colon K^*(G_{l+1})\to K^*(G_l)$$

be the natural injective homomorphism. Since

$$i^*(\lambda^i\rho_{l+1}) = \lambda^i\rho_l+(\lambda^{i-1}\rho_l)t_{l+1} \qquad \text{for } i=1,\cdots,l$$

and
$$i^*(\lambda^{l+1}\rho_{l+1}) = (\lambda^l\rho_l)t_{l+1},$$

we have

$$\begin{aligned}&i^*((\lambda^1\rho_{l+1})^{p_1}\cdots(\lambda^{l+1}\rho_{l+1})^{p_{l+1}})\\ &=(\lambda^1\rho_l)^{p_2}\cdots(\lambda^l\rho_l)^{p_{l+1}}t_{l+1}^{p_1+\cdots+p_{l+1}}+\text{ lower monomials}\end{aligned}$$

where the lower monomial implies a monomial whose degree with respect to the variable t_{l+1} is lower than $p_1+\cdots+p_{l+1}$. Observe the image of Nx by i^* then we see by the inductive hypothesis that $a_{p_1\cdots p_{l+1}}$ is divisible by N. Thus we have $x\in W_{l+1}\otimes F_{l+2}\otimes\cdots\otimes F_n$. This completes the induction. q.e.d.

Type CII$(Sp(m+n)/Sp(m)\times Sp(n))$. Let $Sp_1(1)\times\cdots\times Sp_n(n)$, where $Sp_i(1)=Sp(1)\ (1\leqq i\leqq n)$, be the subgroup of $Sp(n)$ embedded diagonally, and put $R(Sp_i(1))=Z[\theta_i]$ for each i where $\theta_i=t_i+t_i^{-1}$ and t_i is the standard 1-dimensional non-trivial representation of a maximal torus of $Sp_i(1)$.

By replacing S_k^1 and t_k in case of Type AIII by $Sp_k(1)$ and θ_k for

$k=1, \cdots, n$ respectively, we obtain analogously the following results.

Lemma 3.3. *Let E_k be the free module generated by $1, \theta_k, \cdots, \theta_k^{m+k-1}$ for $k=1, \cdots, n$. Then we have an isomorphism*

$$R(Sp(m)\times Sp_1(1)\times\cdots\times Sp_n(1))\cong R(Sp(m+n))\otimes E_1\otimes\cdots\otimes E_n$$

with the above notation.

Theroem 3.4.

$$K^*(Sp(m+n)/Sp(m)\times Sp(n))\cong \bigoplus_{\substack{p_1\geqq 0,\cdots,p_n\geqq 0\\ p_1+\cdots+p_n\leqq m}} Z(\sigma_1^{p_1}\cdots\sigma_n^{p_n})$$
$$= \bigoplus_{\substack{p_1\geqq 0,\cdots,p_n\geqq 0\\ p_1+\cdots+p_n\leqq m}} Z((\lambda^1\rho_n)^{p_1}\cdots(\lambda^n\rho_n)^{p_n})$$

for $m, n\geqq 1$.

The equality in Theorem 3.4 is obtained immediately by the formula

$$\lambda^k\rho_n = \sigma_k+\textstyle\sum_{l<k}a_l\sigma_l$$

for $a_l\in Z$ and $k=1, \cdots, n$ ([10], 13, Proposition 5.4).

Type BDI(a)(Spin(2m+2n+2)/Spin(2m+1)·Spin(2n+1)). From the relations of [10], §13, Theorem 10.3 and (2.3) we see that

$$R(Spin(2m+1)\cdot Spin(2n+1))\underset{R(Spin(2m+2n+2))}{\otimes} Z$$
$$= Z[\lambda^1\rho_{2m+1}, \cdots, \lambda^m\rho_{2m+1}, \lambda^1\rho_{2n+1}, \cdots, \lambda^n\rho_{2n+1}]/I$$

where I is the ideal generated by the elements

$$\textstyle\sum_{i+j=l}(\lambda^i\rho_{2m+1})(\lambda^j\rho_{2n+1})-\binom{2m+2n+2}{l}$$

for all l.

On the other hand, when we put $\lambda_i'=\lambda^i\rho_m+\lambda^{i-1}\rho_m\,(1\leqq i\leqq m)$ and $\lambda_j=\lambda^j\rho_n+\lambda^{j-1}\rho_n\,(1\leqq j\leqq n)$

$$R(Sp(m)\times Sp(n))\underset{R(Sp(m+n))}{\otimes} Z = Z[\lambda_1', \cdots, \lambda_m', \lambda_1, \cdots, \lambda_n]/J$$

where J is the ideal generated by the elements

$$\textstyle\sum_{i+j=l}\lambda_i'\lambda_j-\binom{2m+2n+2}{l}$$

for all l.

Hence we see that the correspondences $\lambda_i'\to\lambda^i\rho_{2m+1}$ and $\lambda_j\to\lambda^j\rho_{2n+1}$ $(1\leqq i\leqq m, 1\leqq j\leqq n)$ induce an isomorphism of algebras $R(Sp(m)\times Sp(n))\underset{R(Sp(m+n))}{\otimes} Z$ and $R(Spin(2m+1)\cdot Spin(2n+1))\underset{R(Spin(2m+2n+2))}{\otimes} Z$. Thus we have by Theorem 3.4

$$\begin{aligned} &R(Spin(2m+1)\cdot Spin(2n+1)) \underset{R(Spin(2m+2n+2))}{\otimes} Z \\ &\cong \bigoplus_{\substack{p_1\geqq 0,\cdots,p_n\geqq 0 \\ p_1+\cdots+p_n\leqq m}} Z(\lambda_1^{p_1}\cdots\lambda_n^{p_n}) \\ &\cong \bigoplus_{\substack{p_1\geqq 0,\cdots,p_n\geqq 0 \\ p_1+\cdots+p_n\leqq m}} Z((\lambda^1\rho_{2n+1})^{p_1}\cdots(\lambda^n\rho_{2n+1})^{p_n}). \end{aligned}$$

This and [I], Proposition 7.1 prove the following

Theorem 3.5.

$$\begin{aligned} &K^*(Spin(2m+2n+2)/Spin(2m+1)\underset{Z_2}{\times} Spin(2n+1)) \\ &= \{\bigoplus_{\substack{p_1\geqq 0,\cdots,p_n\geqq 0 \\ p_1+\cdots+p_n\leqq m}} Z((\lambda^1\rho_{2n+1})^{p_1}\cdots(\lambda^n\rho_{2n+1})^{p_n})\}\otimes\wedge(\beta(\Delta^+_{2m+2n+2}-\Delta^-_{2m+2n+2})). \end{aligned}$$

for $m, n\geqq 0$.

Type $BDII(b)$ $(Spin(2n+1)/Spin(2n))$. The following is an immediate result of [10], §13, Theorem 10.3.

Theorem 3.6. $K^*(Spin(2n+1)/Spin(2n))\cong\wedge(\tilde{\Delta}^+_{2n})$ *for* $n\geqq 1$ *where* $\tilde{\Delta}^+_{2n}=\Delta^+_{2n}-2^{n-1}$.

4. DIII

We regard $U(n)$ as a subgroup of $SO(2n)$ by the map

$$A=((a_{ij}))\rightarrow A'=\left(\begin{pmatrix} x_{2i-1,2j-1} & -x_{2i,2j} \\ x_{2i,2j} & x_{2i-1,2j-1}\end{pmatrix}\right)$$

where $a_{ij}=x_{2i-1,2j-1}+\sqrt{-1}\,x_{2i,2j}$ $(1\leqq i,j\leqq n)$.

We see that the canonical inclusion map of $SO(2n-1)$ to $SO(2n)$ induces a homeomorphism

$$SO(2n-1)/U(n-1)\approx SO(2n)/U(n) \tag{4.1}$$

bacause of $SO(2n-1)\cap U(n)=U(n-1)$ and $SO(2n)=U(n)\cdot SO(2n-1)$. Let π: $Spin(2n)\rightarrow SO(2n)$ denote the two fold covering map of $SO(2n)$ and define $\tilde{U}(n)$ (resp. $\tilde{U}(n-1)$) to be the inverse image of $U(n)$ (resp. $U(n-1)$) by π. By (4.1) we have homeomorphisms

$$Spin(2n-1)/\tilde{U}(n-1)\approx Spin(2n)/\tilde{U}(n) \tag{4.2}$$

and

$$SO(2n)/U(n)\approx Spin(2n)/\tilde{U}(n).$$

Next we shall consider the complex representation ring of $\tilde{U}(n)$. Let T be the standard maximal torus of $U(n)$ and put $\tilde{T}=\pi^{-1}(T)$, which becomes a maximal torus of $\tilde{U}(n)$. Here, using the notation of [10], §13 we define the homomorphism

$$f\colon R(T)[u_n]/(u_n^2-(\alpha_1\cdots\alpha_n)^{-1}) \to R(\tilde{T})$$

by $f(x+yu_n)=\pi^*(x)+\pi^*(y)(\alpha_1\cdots\alpha_n)^{-1/2}$ $x, y\in R(T)$. Then we can easily check that f is isomorphic and compatible with the actions of the Weyl groups of $U(n)$ and $\tilde{U}(n)$, and so we have

(4.3) *$R(\tilde{U}(n))$ is isomorphic to the algebra*

$$Z[\lambda^1\rho_n, \cdots, \lambda^n\rho_n, (\lambda^n\rho_n)^{-1}, u_n]/I$$

where I is the ideal generated by the elements

$$(\lambda^n\rho_n)(\lambda^n\rho_n)^{-1}-1 \text{ and } u_n^2-(\lambda^n\rho_n)^{-1}.$$

Theorem 4.1. *With the above notation*

$$K^*(Spin(2n)/\tilde{U}(n))\cong \bigoplus_{\substack{\varepsilon_k=0,1\\ 0\leqq k\leqq n-2}} Z(u_n^{\varepsilon_0}g_1^{\varepsilon_1}\cdots g_{n-2}^{\varepsilon_{n-2}})$$

for $n\geqq 2$ where

$$g_k = u_n\{\sum_{s_1\geqq 1,\cdots,s_k\geqq 1}\sum_{i=0}^{k}(-1)^i\tbinom{k}{i}g(n, 2s_1+\cdots+2s_k-k+i+1)\}$$

for $k=1, \cdots, n-2$ and

$$g(n, i) = \lambda^{n-i}\rho_n+\lambda^{n-i-2}\rho_n+\cdots$$

for $i=0, \cdots, n$.

Proof. Denote by i_n: $Spin(2n-1)/\tilde{U}(n-1)\to Spin(2n)/\tilde{U}(n)$ the homeomorphism of (4.2) and put

$$R(\tilde{T}) = Z[\alpha_1, \alpha_1^{-1}, \cdots, \alpha_n, \alpha_n^{-1}, (\alpha_1\cdots\alpha_n)^{-1/2}]$$

using the notation of [10], §13, Proposition 8.3. We proceed by induction on n.

The homomorphism i_2^*: $K^*(Spin(4)/\tilde{U}(2))\to K^*(Spin(3)/\tilde{U}(1))$ is isomorphic, and we have

$$R(U(1))\underset{R(Spin(3))}{\otimes} Z = Z[\alpha^{-1/2}]/((\alpha^{-1/2}-1)^2),$$
$$i_2^*(u_2) = \alpha^{-1/2}$$

when we put $R(U(1))=Z[\alpha^{1/2}, \alpha^{-1/2}]$. Therefore we get the statement when $n=2$.

Put $E=Spin(2n+1)/\tilde{U}(n)$, $F=Spin(2n)/\tilde{U}(n)$ and denote the inclusions $(F, \phi)\to(E, \phi)\to(E, F)$ by i and j respectively. Then there is a short exact sequence

$$0\to K^*(E, F)\xrightarrow{j^*} K^*(E)\xrightarrow{i^*} K^*(F)\to 0.$$

Moreover we denote the projection $E \to Spin(2n+1)/Spin(2n)$ by p. Then we have an isomorphism

$$\varphi\colon K^*(F)\otimes \tilde{K}^*(Spin(2n+1)/Spin(2n)) \to K^*(E, F)$$

defined by $j^*\varphi(x\otimes\Delta_{2n}^{+})=yp^*(\Delta_{2n}^{+})$ $x\in K(F)$ where y is an element of $K^*(E)$ such that $i^*(y)=x$.

Here suppose that the assertion for $K^*(Spin(2n)/\tilde{U}(n))$ is true. By Theorem 3.6 we may assume that $K^*(Spin(2n+1)/Spin(2n))=\wedge(\Delta_{2n}^{-}-2^{n-1})$. Consider the element $i_{n+1}^{*-1}p^*(\Delta_{2n}^{-}-2^{n-1})$ of $K^*(Spin(2n+2)/\tilde{U}(n+1))$. By the definition of Δ_{2n}^{-}

$$p^*(\Delta_{2n}^{-}-2^{n-1}) = u_n(\lambda^{n-1}\rho_n+\lambda^{n-3}\rho_n+\cdots)-2^{n-1}\,.$$

Hence

$$i_{n+1}^{*-1}p^*(\Delta_{2n}^{-}-2^{n-1}) = u_{n+1}\{\textstyle\sum_{s\geqq 1}(g(n+1, 2s)-g(n+1, 2s+1))\}-2^{n-1}$$

because of $i_{n+1}^{*}(g(n+1, i)-g(n+1, i+1))=\lambda^{n-i+1}\rho_n$.

For the completion of the induction it is sufficient to prove that

$$(i_{n+1}\tilde{i})^*(u_{n+1}\{\sum_{s_1\geqq 1,\cdots,s_{k+1}\geqq 1}\textstyle\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}g(n+1, 2s_1+\cdots+2s_{k+1}-k+i)\}) = g_k$$

for $k=2, \cdots, n-1$. This follows from the following equalities:

$$\begin{aligned}
&u_{n+1}(\sum_{s_1\geqq 1,\cdots,s_{k+1}\geqq 1}\textstyle\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}g(n+1, 2s_1+\cdots+2s_{k+1}-k+i))\\
&= u_{n+1}\{\sum_{s_1\geqq 1,\cdots,s_{k+1}\geqq 1}(g(n+1, 2s_1+\cdots+2s_{k+1}-k)+\textstyle\sum_{i=1}^{k}(-1)^i(\binom{k}{i}+\binom{k}{i-1})g(n+1, 2s_1\\
&\quad +\cdots+2s_{k+1}-k+i)+(-1)^{k+1}g(n+1, 2s_1+\cdots+2s_{k+1}+1))\}\\
&= u_{n+1}\{\sum_{s_1\geqq 1,\cdots,s_k\geqq 1}\textstyle\sum_{i=0}^{k}(-1)^i\binom{k}{i}(\sum_{s_{k+1}\geqq 1}(g(n+1, 2s_1+\cdots+2s_k+2s_{k+1}-k+i)\\
&\quad -g(n+1, 2s_1+\cdots+2s_k+2s_{k+1}-k+i+1)))\}
\end{aligned}$$

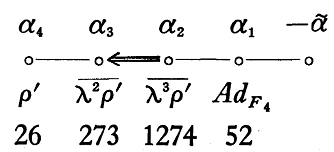
and $g(n, j)=(i_{n+1}\tilde{i})^*(\sum_{k\geqq 1}(g(n+1, j+2k-1)-g(n+1, j+2k)))$ for $j\geqq 0$.

5. EI and FI (1)

In this section we discuss the symmetric spaces $E_6/PSp(4)$ and $F_4/Sp(3)\underset{Z_2}{\times} SU(2)(=F_4/Sp(3)\cdot SU(2))$([11], p. 131).

We reproduce the Dynkin diagram of F_4 in [I] added the maximal root $\tilde{\alpha}$ and the simple roots $\alpha_1, \cdots, \alpha_4$ corresponding to the vertexes.

(5.1)



Then the Dynkin diagram of $Sp(3)\underset{Z_2}{\times}SU(2)$ is obtained by omitting the vertex with the symbol α_1.

$$\begin{array}{cccccc} \beta_1 & \beta_2 & \beta_3 & & \beta \\ \circ\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!- & \circ \Leftarrow & \circ & & \circ \\ \rho_3 & \overline{\lambda^2\rho_3} & \overline{\lambda^3\rho_3} & & \rho_2 \\ 6 & 14 & 14 & & 2 \end{array} \tag{5.2}$$

where the explanation of the symbols and the numbers is quite similar to that of the above diagram.

According to [16], Tables I, III and VIII, the fundamental weights of F_4 and $Sp(3)\cdot SU(2)$ determined by the above fundamental root systems are as follows:

$$\begin{aligned} w_1 &= 2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4 = \tilde{\alpha}\,, \\ w_2 &= 3\alpha_1+6\alpha_2+8\alpha_3+4\alpha_4\,, \\ w_3 &= 2\alpha_1+4\alpha_2+6\alpha_3+3\alpha_4\,, \\ w_4 &= \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4\,, \\ \overline{w}_1 &= \beta_1+\beta_2+\frac{1}{2}\beta_3\,, \\ \overline{w}_2 &= \beta_1+2\beta_2+\beta_3\,, \\ \overline{w}_3 &= \beta_1+2\beta_2+\frac{3}{2}\beta_3\,, \\ \overline{w} &= \frac{1}{2}\beta\,. \end{aligned} \tag{5.3}$$

Hereafter, for simplicity we denote the weights $m_1\alpha_1+\cdots+m_4\alpha_4$, $n_1\beta_1+\cdots+n_3\beta_3$, $n_1\beta_1+\cdots+n_3\beta_3+n\beta$ by $(m_1\cdots m_4)$, $(n_1\cdots n_3)$ and $(n_1\cdots n_3,\, n)$ respectively.

Since ρ_3 is the irreducible representation of $Sp(3)$ with $\left(11\frac{1}{2}\right)$ as the highest weight, by acting the elements of the Weyl group on it we get the all weights of ρ_3:

$$\left(11\frac{1}{2}\right)\left(01\frac{1}{2}\right)\left(00\frac{1}{2}\right)\left(00-\frac{1}{2}\right)\left(0-1-\frac{1}{2}\right)\left(-1-1-\frac{1}{2}\right). \tag{5.4}$$

Let i: $Sp(3)\cdot SU(2)\to F_4$ be the inclusion of $Sp(3)\cdot SU(2)$ and $i^*(w)$ denote the reduction of a weight w of F_4 to $Sp(3)\cdot SU(2)$. Then we have

$$i^*(-\tilde{\alpha}) = \beta,\ i^*(\alpha_2) = \beta_3,\ i^*(\alpha_3) = \beta_2 \ \text{ and } \ i^*(\alpha_4) = \beta_1$$

and so using the first formula of (5.3)

$$\begin{aligned} i^*(1000) &= \left(-1-2-\frac{3}{2},\ -\frac{1}{2}\right), \\ i^*(0100) &= (001,0)\,, \\ i^*(0010) &= (010,0)\,, \\ i^*(0001) &= (100,0)\,. \end{aligned} \tag{5.5}$$

Proposition 5.1. *With the notations of* (5.1) *and* (5.2) *we have in* $R(Sp(3)\cdot SU(2))$

(i) $i^*(\rho') = \lambda^2\rho_3+\rho_2\rho_3-1$,

(ii) $i^*(Ad_{F_4}) = \rho_2\lambda^3\rho_3-\lambda^2\rho_3+\rho_2^2+\rho_3^2-\rho_2\rho_3-1$.

Proof. By restricting all the weights of the adjoint representation of E_6 to F_4 we obtain those of ρ', which are listed at the end of this section, since we know all the roots of F_4 ([16], Table VIII). It follows obviously that the weights of ρ_2 are $\frac{1}{2}\beta$ and $-\frac{1}{2}\beta$.

(i) When we observe the restrictions of the weights of ρ' to $Sp(3)\cdot SU(2)$ making use of (5.5) we get (i).

(ii) Considering that

$$Ad_{Sp(3)} = \rho_3^2-\lambda^2\rho_3 \quad \text{and} \quad Ad_{SU(2)} = \rho_2^2-1$$

we get (ii) similarly. q.e.d.

Lemma 5.2. *In* $R(Sp(3)\cdot SU(2)) \underset{R(F_4)}{\otimes} Z$ *we have*

(i) $\lambda^2\rho_3 = -\rho_2\rho_3+27$,

(ii) $\rho_2\lambda^3\rho_3 = -\rho_2^2-\rho_3^2+80$,

(iii) $\rho_3\lambda^3\rho_3 = \rho_2^2\rho_3^2+\rho_2^3\rho_3-\rho_3^2-27\rho_2^2-30\rho_2\rho_3+432$,

(iv) $(\lambda^3\rho_3)^2 = \rho_2^4+\rho_3^4-\rho_2^4\rho_3^2+54\rho_2^3\rho_3+2\rho_2\rho_3^3+6\rho_2^2\rho_3^2$
$\qquad -216\rho_2\rho_3-136\rho_3^2-812\rho_2^2+6080$.

Proof. (i) and (ii) These are immediate results of Proposition 5.1.

(iii) From (i) of Proposition 5.1 we get

$$i^*(\lambda^2\rho'+\rho') = \lambda^2(\rho_2\rho_3)+(\rho_2\rho_3)\lambda^2\rho_3+\lambda^2(\lambda^2\rho_3)$$

and by the direct calculation we have

$$\begin{cases}\lambda^2(\rho_2\rho_3) = (\rho_2^2-2)\lambda^2\rho_3+\rho_3^2\\ \lambda^2(\lambda^2\rho_3) = \rho_3\lambda^3\rho_3-\lambda^2\rho_3 .\end{cases}$$

Therefore,

$$i^*(\lambda^2\rho') = \rho_3\lambda^3\rho_3+(\rho_2^2+\rho_2\rho_3-4)\lambda^2\rho_3+(\rho_3^2-\rho_2\rho_3+1)$$

and so from (i), (iii) follows.

(iv) By the direct caluculation we get

$$\begin{cases}\lambda^2(\rho_2^2) = 2\rho_2^2-2\\ \lambda^2(\rho_3^2) = -2(\lambda^2\rho_3)^2+2\rho_3^2\lambda^2\rho_3\\ \lambda^2(\lambda^3\rho_3) = (\lambda^2\rho_3)^2-\rho_3^2+1\end{cases}$$

and from (ii) we have

$$(\lambda^3\rho_3)^2+(\rho_2^2-2)\lambda^2(\lambda^3\rho_3)+\lambda^2(\rho_2^2)+\lambda^2(\rho_3^2)$$
$$+\rho_2^3\lambda^3\rho_3+\rho_2\rho_3^2\lambda^3\rho_3+\rho_2^2\rho_3^2=3160\,.$$

Therefore, making use of the above formulas, (i) and (ii) we have (iv). q.e.d.

Theorem 5.3. *With the notation of* [I], *Proposition* 7.3

$$K^*(E_6/PSp(4))\cong\wedge(\beta(\rho_1-\rho_2),\ \beta(\lambda^2\rho_1-\lambda^2\rho_2))\otimes Z[\rho_4^2]/((\rho_4^2-64)^3)\,.$$

Proof. Let j: $Sp(3)\cdot SU(2)\to PSp(4)$ be the inclusion map of $Sp(3)\cdot SU(2)$. Then we have

$$\begin{cases} j^*(\lambda^2\rho_4)=\lambda^2\rho_3+\rho_2\rho_3+1 \\ j^*(\lambda^4\rho_4)=\rho_2\lambda^3\rho_3+2\lambda^2\rho_3 \\ j^*(\rho_4^2)=(\rho_2+\rho_3)^2 \\ j^*((\lambda^3\rho_4)^2)=(\lambda^3\rho_3+\rho_2\lambda^2\rho_3+\rho_3)^2 \\ j^*(\rho_4(\lambda^3\rho_4))=(\rho_2+\rho_3)\,(\lambda^3\rho_3+\rho_2\lambda^2\rho_3+\rho_3) \end{cases}$$

and from Lemma 5.2 we have in $R(PSp(4))\underset{R(E_6)}{\otimes}Z$

$$\begin{cases} \lambda^2\rho_4=28 \\ \lambda^4\rho_4=-\rho_4^2+134 \\ \rho_4\lambda^3\rho_4=-\rho_4^2+512 \\ (\lambda^3\rho_4)^2=\rho_4^4-191\rho_4^2+11264\,. \end{cases}$$

This and (2.2) show that

$$R(PSp(4))\underset{R(E_6)}{\otimes}Z=Z[\rho_4^2]/((\rho_4^2-64)^3)$$

and so Theorem 5.3 follows from [I], Proposition 7.3. q.e.d.

(5.6) The weights of ρ' and the positive roots of F_4 are as follows respectively:

1 2 3 2		2 3 4 2		
1 2 3 1		1 3 4 2		
1 2 2 1		1 2 4 2		
1 1 2 1		1 2 3 2		
1 1 1 1	0 1 2 1	1 2 3 1	1 2 2 2	
1 1 1 0	0 1 1 1	1 2 2 1	1 1 2 2	
0 1 1 0	0 0 1 1	1 2 2 0	0 1 2 2	1 1 2 1
0 0 1 0	0 0 0 1	1 1 2 0	0 1 2 1	1 1 1 1
0 0 0 0	0 0 0 0	1 1 1 0	0 1 2 0	0 1 1 1

0 0-1 0	0 0 0-1	1 1 0 0	0 1 1 0	0 0 1 1	
0-1-1 0	0 0-1-1	1 0 0 0	0 1 0 0	0 0 1 0	0 0 0 1
-1-1-1 0	0-1-1-1				
-1-1-1-1	0-1-2-1				
-1-1-2-1					
-1-2-2-1					
-1-2-3-1					
-1-2-3-2					

where the sequence of integers $m_1 \cdots m_4$ indicates a weight $m_1\alpha_1 + \cdots + m_4\alpha_4$.

6. EI and FI (2)

This section is a continuation of the section 5.

Put

$$x = \rho_2^2,\ y = (\rho_2+\rho_3)^2 \ \textit{and}\ \ w = \rho_2\rho_3 + x\,. \tag{6.1}$$

Then

$$w^2 = xy \tag{6.2}$$

obviously. We obtain from Lemma 5.2

$$\begin{aligned} &\text{(i)} && \rho_3^2 = x+y-2w\,,\\ &\text{(ii)} && \lambda^2\rho_3 = x-w+27\,,\\ &\text{(iii)} && \rho_2\lambda^3\rho_3 = -2x+(2w-y+80)\,,\\ &\text{(iv)} && \rho_3\lambda^3\rho_3 = (2-w)x+(w^2-28w-y+432)\,,\\ &\text{(v)} && (\lambda^3\rho_3)^2 = -x^3+(2w-48)x^2+(-w^2+44w-732)x\\ &&& \qquad +(6w^2+56w-2yw+y^2-136y+6080) \end{aligned} \tag{6.3}$$

and from Theorem 5.3

$$(y-64)^3 = 0\,. \tag{6.4}$$

From (6.2), (iii) and (iv) of (6.3) we have

$$\begin{aligned} &\text{(i)} && wx^2+(-w^2+24w-512)x+(4w^2-yw+80w) = 0\,,\\ &\text{(ii)} && -wx^2+(2w^2-24w+512)x+(-w^3+20w^2+5yw-592w-y^2\\ &&& +80y) = 0\,. \end{aligned} \tag{6.5}$$

From (6.2) and (6.5) we have

$$-w^2x+w^3-24w^2-4yw+512w+y^2-80y = 0\,. \tag{6.6}$$

From (6.2), (6.4) and (6.6) we have

(6.7) $$w^4-yw^3+24yw^2+4y^2w-512yw-112y^2+12288y-262144=0\,.$$

From (6.2), (iii) and (v) of (6.3) we have

(6.8) $$\begin{aligned}&x^4+(48-2w)x^3+(w^2-44w+736)x^2+(-6w^2-64w-6400)x\\&+(2w^3+144w^2+320w-yw^2-4yw+y^2-160y+6400)=0\,.\end{aligned}$$

(2.2), (6.1) and (6.3) show that $R(Sp(3)\cdot SU(2))\underset{R(F_4)}{\otimes}Z$ is generated by the elements x, y and w as an algebra and moreover (6.4), (6.7) and (6.8) imply

Lemma 6.1. *$R(Sp(3)\cdot SU(2))\underset{R(F_4)}{\otimes}Z$ is generated by the elements $x^a y^b w^c$ for $a, c=0, 1, 2, 3$ and $b=0, 1, 2$, as a module.*

Let M denote the submodule of $R(Sp(3)\cdot SU(2))\underset{R(F_4)}{\otimes}Z$ generated by the elements:

$$1, x, x^2, x^3, y, y^2, w, w^2, w^3, xw,$$
$$yw, y^2w, yw^2, y^2w^2, yw^3, y^2w^3\,.$$

From (6.4) and (6.7) we have

(6.9) $$y^iw^j\in M \quad for\ \ i, j\geqq 0\,.$$

Hence, from (6.6) we have

(6.10) $$xw^{j+2}\in M \quad for\ \ j\geqq 0\,.$$

From (i) of (6.5), (6.9) and (6.10) we have

(6.11) $$x^2w^j\in M \quad for\ \ j\geqq 0\,.$$

From (i) of (6.5) and (6.6) we get

(6.12) $$x^2w=w^3-28w^2+432w-3yw-24xw+512x+y^2-80y$$

and so we see that x^2w, x^3w, x^3w^2 and x^3w^3 are contained in M from (6.9), (6.10) and (6.11). Thus we obtain

Lemma 6.2. *With the above notation*

$$R(Sp(3)\cdot SU(2))\underset{R(F_4)}{\otimes}Z=M\,.$$

Theorem 6.3. *With the notation of* (6.1) *$K^*(F_4/Sp(3)\cdot SU(2))$ is a free module generated by the elements*

$$1, x, x^2, x^3, y, y^2, w, w^2, w^3, xw, yw, yw^2\,.$$

Proof. Let N be the submodule of $K^*(F_4/Sp(3)\cdot SU(2))$ generated by the elements mentioned in the theorem.

From (iii), (iv) and (v) of (6.3) we have

$$-x^4+(3w-48)x^3+(-3w^2+94w-736)x^2$$
$$+(w^3-42w^2+768w+5376)x+(-5w^3-168w^2-7456w+2yw^2$$
$$-y^2w+162yw+y^2-512y+34560)=0\,.$$

From this equality and (6.8) we have

$$wx^3+(-2w^2+50w)x^2+(w^3-48w^2+704w-1024)x$$
$$+(-3w^3-24w^2-7136w+yw^2-y^2w+158yw+2y^2-672y+40960)=0\,.$$

Moreover, from this equality, (6.5) and (6.6) we have

(6.13) $\quad y^2w=512x^2-512xw+12288x-10240w+192yw-512y+40960\,.$

This shows

(6.14) $$y^2w\in N\,.$$

From (6.4) we have

(6.15) $$y^2w^2=192yw^2-12288w^2+262144x$$

using (6.2) and so

(6.16) $$y^2w^2\in N\,.$$

From (6.2) and (6.13)

$$yw^3=x(y^2w)$$
$$=512x^3-512x^2w+12288x^2-10240xw+192w^3-512w^2+40960x\,.$$

and so we have

(6.17) $$yw^3\in N$$

since $x^2w\in N$ by (6.12). From (6.15) and (6.17) we have

(6.18) $$y^2w^3\in N\,.$$

(6.14), (6.16), (6.17) and (6.18) imply Theorem 6.3 since $K^*(F_4/Sp(3)\cdot SU(2))$ is a free module of rank 12. q.e.d.

7. FII and G

Type FII$(F_4/Spin(9))$. According to [15], Theorem 15.1 we have in $R(Spin(9))\underset{R(F_4)}{\otimes} Z$

$$\begin{cases} \lambda^1\rho_9 = -\Delta_9+25 \\ \lambda^2\rho_9 = -\Delta_9+52 \\ \lambda^3\rho_9 = \Delta_9^2-23\Delta_9+196 \\ (\Delta_9-16)^3 = 0 . \end{cases}$$

This proves

Theorem 7.1. $K^*(F_4/Spin(9)) \cong Z[\Delta_9]/((\Delta_9-16)^3)$.

Type $G(G_2/SU(2)\cdot SU(2))$. We observe the extended Dynkin diagram of G_2 with the irreducible representations corresponding to the vertices and their dimensions written next to vertices ([16], Table 30):

$$\begin{array}{ccc} \alpha_1 & \alpha_2 & -\tilde{\alpha} \\ \circ \Lleftarrow & \circ \text{———} & \circ \\ \rho & Ad_{G_2} & \\ 7 & 14 & \end{array} \tag{7.1}$$

where α_1, α_2 are the simple roots and $\tilde{\alpha}$ is the maximal root.

Let us denote by σ the involutive automorphism of G_2 for the symmetric space of type G ([11], Theorem 3.1). Then we see that the subgroup consisting of fixed points of σ is $SU(2)\underset{Z_2}{\times}SU(2)(=SU(2)\cdot SU(2))$ where Z_2 is the intersection of the centers of the two groups $SU(2)$, and its Dynkin diagram is obtained by omitting the vertex with the symbol α_2.

$$\begin{array}{cc} \beta_1 & \beta_2 \\ \circ & \circ \\ \rho_2 & \rho_2' \\ 2 & 2 \end{array} \tag{7.2}$$

in which the explanation of the symbols and the numbers are as in the diagram of G_2.

If we denote the fundamental weights of G_2 and $SU(2)\cdot SU(2)$ by w_k and $\bar{w}_k$ for $k=1, 2$ respectively, then we have from [16], Tables I and IX

$$\begin{aligned} w_1 &= 2\alpha_1+\alpha_2 , \\ w_2 &= 3\alpha_1+2\alpha_2 = \tilde{\alpha} , \\ \bar{w}_1 &= \frac{1}{2}\beta_1 , \\ \bar{w}_2 &= \frac{1}{2}\beta_2 . \end{aligned} \tag{7.3}$$

Let i: $SU(2)\cdot SU(2) \to G_2$ be the inclusion of $SU(2)\cdot SU(2)$ and $i^*(w)$ be the reduction of a weight w of G_2 to $SU(2)\cdot SU(2)$. Since

$$i^*(\alpha_1)=\beta_1 \text{ and } i^*(-\tilde{\alpha})=\beta_2\,,$$

we have by (7.3)

$$(7.4)\qquad \begin{aligned} i^*(\alpha_1) &= \beta_1\,, \\ i^*(\alpha_2) &= -\frac{3}{2}\beta_1-\frac{1}{2}\beta_2\,. \end{aligned}$$

Proposition 7.2. *With the notation of* (7.1)

(i) $i^*(\rho)=\rho_2^2+\rho_2\rho_2'-1$,

(ii) $i^*(Ad_{G_2})=\rho_2^2+\rho_2'^2+\rho_2^3\rho_2'-2\rho_2\rho_2'-2$

where $i^*: R(G_2)\to R(SU(2)\cdot SU(2))$ *is the restriction.*

Proof. Denote the weights $m_1\alpha_1+m_2\alpha_2$ and $n_1\beta_1+n_2\beta_2$ by $(m_1\ m_2)$ and $(n_1,\ n_2)$ respectively.

(i) Since ρ is the irreducible representation of G_2 with (2 1) as the highest weight, by operating the elements of the Weyl group on it we see that the weights of ρ is as follows:

$$(2\ 1)\ (1\ 1)\ (1\ 0)\ (0\ 0)\ (-1\ 0)\ (-1\ -1)\ (-2\ -1)\,.$$

Consider the restrictions of the weights of ρ to $SU(2)\cdot SU(2)$ using (7.4) then (i) follows because the weights of ρ_2 and ρ_2' are

$$\left(\frac{1}{2},\ 0\right)\left(-\frac{1}{2},\ 0\right) \text{ and } \left(0,\ \frac{1}{2}\right)\left(0,\ -\frac{1}{2}\right)$$

respectively.

(ii) From [16], Table IX the weights of Ad_{G_2} are as follows:

$$\begin{gathered}(3\ 2)\ (3\ 1)\ (2\ 1)\ (1\ 1)\ (1\ 0)\ (0\ 1)\ (0\ 0)\ (-3\ -2)\\ (-3\ -1)\ (-2\ -1)\ (-1\ -1)\ (-1\ 0)\ (0-1)\ (0\ 0)\,.\end{gathered}$$

By observing the reduction of these weights to $SU(2)\cdot SU(2)$ we obtain (ii) analogously. q.e.d.

Theorem 7.3. $K^*(G_2/SU(2)\underset{Z_2}{\times}SU(2))\cong Z[\rho_2^2]/((\rho_2^2-4)^3)$ *with the notation of* (7.2).

Proof. By Proposition 7.2 we get in $R(SU(2)\cdot SU(2))\underset{R(G_2)}{\otimes}Z$

$$\rho_2^2+\rho_2\rho_2'=8 \text{ and } \rho_2^2+\rho_2'^2+\rho_2^3\rho_2'-2\rho_2\rho_2'=16\,.$$

From these equalities we have

$$\begin{cases} \rho_2\rho_2' = 8-\rho_2^2 \\ \rho_2'^2 = \rho_2^4-11\rho_2^2+32 \\ (\rho_2^2-4)^3 = 0\,. \end{cases}$$

Therefore the theorem is proved because $R(SU(2)\cdot SU(2))$ equals the ring $Z[\rho_2^2, \rho_2\rho_2', \rho_2'^2]$. q.e.d.

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