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ON CERTAIN LOCALLY FLAT HOMOGENEOUS MANIFOLDS OF SOLVABLE LIE GROUPS

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Introduction

Let M be a connected differentiable manifold with a locally flat linear connection D (A linear connection is locally flat, if its torsion and curvature tensors vanish identically). Then, for each point $p\in M$, there exists a local coordinate system $\{x^1, \cdots, x^n\}$ in a neighbourhood of p such that $D_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}=0$, which we call an *affine local coordinate system.* A Riemannian metric g on M is said to be *locally Hessian* with respect to D, if for, each point $p\in M$, there exists a real-valued function ϕ of class C^∞ on a neighbourhood of p such that

$$g = D^2\phi\,,$$

that is,

$$g = \sum_{i,j}\frac{\partial^2\phi}{\partial x^i\partial x^j}dx^i dx^j\,,$$

where $\{x^1, \cdots, x^n\}$ is an affine local coordinate system around p. If this condition is verified with a function ϕ defined over M, the metric g is called a *Hessian metric* on M. A locally flat manifold with a (locally) Hessian metric is called a *(locally) Hessian manifold.*

The following proposition is essentially due to S. Murakami and will be proved in §1.

Proposition. *Let M be a connected differentiable manifold with a locally flat linear connection D and a Riemannian metric g. Let γ be the cotangent bundle-valued 1-form on M defined by*

$$(\gamma(X))(Y) = g(X, Y)$$

for vector fields X, Y on M. The cotangent bundle being locally flat, we may consider the exterior differentiation $\underline{\underline{d}}$ for cotangent bundle-valued forms on M. Then the following conditions (1)~(4) are equivalent:

(1) *g is locally Hessian with respect to D.*

(2) *For each affine local coordinate system* $\{x^1, \cdots, x^n\}$, *the components* g_{ij} *of* g *satsify the relations*

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial g_{ik}}{\partial x^j} \qquad (1 \leqq i, j, k \leqq n).$$

(3) $(D_Z g)(X, Y) = (D_Y g)(X, Z)$ *for all differentiable vector fields* X, Y, Z *on* M.

(4) $\underline{\underline{d}}\gamma = 0$.

In addition to these equivalent conditions, assume further $H^1(M, \mathbf{R}) = \{0\}$ *and that* D *is flat. Then* g *is a Hessian metric.*

EXAMPLE 1. Let M be a locally flat Riemannian manifold, that is, the Riemannian connection ∇ determined by the Riemannian metric g on M is locally flat. By $\nabla g = 0$ and by Proposition (3), g is locally Hessian with respect to ∇.

EXAMPLE 2. Let M be a domain in the n-dimensional real affine space with an affine coordinate system $\{x^1, \cdots, x^n\}$ and let ϕ be a real valued function on M of class C^∞ such that the Hessian $g = \left[\frac{\partial^2 \phi}{\partial x^i \partial x^j}\right]$ of ϕ is positive definite on M. Then g defines a Hessian metric on M with respect to the natural flat linear connection D on M given by $D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$.

EXAMPLE 3. Let M be an affine homogeneous convex domain in the n-dimensional real affine space which does not contain any full straight line and let ϕ denote the characteristic function on M. Then it is well known the the Hessian $g = \left[\frac{\partial^2 \log \phi}{\partial x^i \partial x^j}\right]$ of $\log \phi$ is positive definite on M (cf. [3] [7]), and so g is a Hessian metric on M.

Now let M be a homogeneous manifold of a connected Lie group G. Assume that M admits a locally flat linear connection D and a volume element ω which are invariant under G. If ω has an expression

$$\omega = K\, dx^1 \wedge \cdots \wedge dx^n$$

in an affine local coordinate system $\{x^1, \cdots, x^n\}$, then the forms

$$\alpha = \sum_i \frac{\partial \log K}{\partial x^i} dx^i ,$$

$$D\alpha = \sum_{i,j} \frac{\partial^2 \log K}{\partial x^i \partial x^j} dx^i dx^j ,$$

are called the *Koszul form* and the *canonical bilinear form* respectively [3].

Koszul proved the following fundamental theorem concerning the form $D\alpha$ [3]: *Let M be a homogeneous manifold with an invariant flat linear connection and an invariant volume element. Then the canonical bilinear form $D\alpha$ is positive definite if and only if M is an affine homogeneous convex domain not containing any full straight line.*

Several authors have pointed out an intimate connection and an analogy between affine homogeneous convex domains and homogeneous bounded domains (cf. [3] [7]). Now recall that a hermitian metric g on a complex manifold is said to be Kählerian if its local components $g_{i\bar{j}}$ with respect to a holomorphic local coordinate system $\{z^1, \cdots, z^n\}$ satisfy one of the following conditions

$$(1) \qquad g_{i\bar{j}} = \frac{\partial^2 \psi}{\partial z^i \partial \bar{z}^j} \qquad (1 \leqq i, j \leqq n),$$

$$(2) \qquad \frac{\partial g_{i\bar{j}}}{\partial z^k} = \frac{\partial g_{i\bar{k}}}{\partial z^j} \qquad (1 \leqq i, j, k \leqq n),$$

where ψ is a real valued function in the coordinate neighbourhood. It seems to the author that homogeneous locally Hessian manifolds have, in a way, analogous properties as homogeneous Kähler manifolds. The aim of this paper is to establish the following theorem analogous to that in [5].

Theorem. *Let G be a connected solvable Lie group and M an orientable differentiable manifold on which G acts simply transitively. Suppose that M admits a locally flat linear connection D and a locally Hessian metric g with respect to D, which are invariant under G. Let ω be the volume element defined by g. If the canonical bilinear form $D\alpha$ determined by ω is non-degenerate, then $D\alpha$ is positive definite.*

Combined with the Koszul's theorem recalled above, we get immediately.

Corollary. *Under the same assumptions as in Theorem, assume further that D is flat. Then, M is an affine homogeneous convex domain not containing any full straight line.*

1. Preliminaries

We shall first prove Proposition in the introduction. It is trivial that (1) implies (2) and that (2) is equivalent to (3). The form γ defined in Proposition can be locally expressed as $\gamma = \sum_i (\sum_j g_{ij} dx^j) dx^i$, where $\{x^1, \cdots, x^n\}$ an affine local coordinate system and g_{ij} the components of g. We have then

$$\underline{\underline{d}}\gamma = \sum_i (\sum_j dg_{ij} \wedge dx^j) dx^i$$

$$= \sum_i \left\{ \sum_{j<k} \left(\frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j} \right) dx^k \wedge dx^j \right\} dx^i .$$

It follows immediately that the conditions (2) and (4) are equivalent. It remains to show that (2) implies (1).

We shall first prove the last part of Proposition. Suppose that D is flat. Then there exist 1-forms $\omega^1, \cdots, \omega^n$ such that $D\omega^i = 0$ and that, for each point p in M the values of these forms at p form a basis of the cotangent space at p. Thus the cotangent bundle $T(M)^*$ is a trivial bundle and the $T(M)^*$-valued de Rham cohomology group of M is isomorphic to the $\mathbf{R}^n$-valued de Rham cohomology group of M. If $H^1(M, \mathbf{R}) = \{0\}$ and if the condition (2) is satisfied, there exists a cross section β of $T(M)^*$ such that

$$\gamma = \underline{d}\beta .$$

If the 1-form β on M has a local expression $\beta = \sum \beta_i dx^i$, then it follows $d\beta_i = \sum_j g_{ij} dx^j$ and $\frac{\partial \beta_i}{\partial x^j} = g_{ij}$. Since $g_{ij} = g_{ji}$, we have

$$d\beta = 0 .$$

Again by $H^1(M, \mathbf{R}) = \{0\}$, there exists a function ϕ on M of class C^∞ such that

$$\beta = d\phi .$$

Thus we have $g_{ij} = \frac{\partial \beta_i}{\partial x^j} = \frac{\partial^2 \phi}{\partial x^j \partial x^i}$ and hence $g = D^2\phi$, which completes the proof for the last part of Proposition. Now, by a same argument and applying the Poincaré's lemma, we see that (2) implies (1). Thus the proof of Proposition is completed.

We retain the notation and assumptions settled in Theorem in the introduction.

Let $\mathfrak{g}$ be the Lie algebra of the Lie group G. For $X \in \mathfrak{g}$ we denote by X^* the vector field on M induced by the 1-parameter group of transformations $\exp(-tX)$. We put $A_{X^*} = L_{X^*} - D_{X^*}$ where L_{X^*}, D_{X^*} are the Lie derivative and the covariant derivative for D by X^* respectively. Then A_{X^*} is a derivation of the algebra of tensor fields on M, which maps every function into zero. Since D is locally flat, we have for $X, Y \in \mathfrak{g}$ (cf. [2])

$$A_{X^*} Y^* = -D_{Y^*} X^* , \tag{1.1}$$

$$A_{X^*} Y^* - A_{Y^*} X^* = [X^*, Y^*] , \tag{1.2}$$

$$[A_{X^*}, A_{Y^*}] = A_{[X^*, Y^*]} . \tag{1.3}$$

We fix a point $o \in M$. Let V be the tangent space of M at o and let $f(X)$, $q(X)$

denote the values of A_{X^*}, X^* at o respectively. From (1.2) (1.3), it follows immediately

Lemma 1.1. *For $X, Y \in \mathfrak{g}$ we have*

$$(1) \qquad f([X, Y]) = [f(X), f(Y)],$$

$$(2) \qquad q([X, Y]) = f(X)q(Y) - f(Y)q(X).$$

Lemma 1.2. *Let α_o, $D\alpha_o$ denote the values of α, $D\alpha$ at o respectively. Then, for $X, Y \in \mathfrak{g}$ we have*

$$(1) \qquad \alpha_o(q(X)) = \operatorname{Tr} f(X),$$

$$(2) \qquad (D\alpha_o)(q(X), q(Y)) = \alpha_o(f(X)q(Y)).$$

Proof. Let $\{x^1, \cdots, x^n\}$ be an affine local coordinate system in a neighborhood of o. We write $X^* = \sum_i \xi^i \frac{\partial}{\partial x^i}$ and $\omega = K\, dx^1 \wedge \cdots \wedge dx^n$. Then we see

$$\begin{aligned} L_{X^*}\omega &= (L_{X^*}K)dx^1 \wedge \cdots \wedge dx^n + \sum_j K\, dx^1 \wedge \cdots \wedge L_{X^*}dx^j \wedge \cdots \wedge dx^n \\ &= \left\{X^*K + \left(\sum_j \frac{\partial \xi^j}{\partial x^j}\right)K\right\} dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

Since the volume element ω is invariant by G, we have

$$(1.4) \qquad X^* \log K = -\sum_j \frac{\partial \xi^j}{\partial x^j}.$$

By (1.1) we get $(D_{\frac{\partial}{\partial x^i}}(A_{X^*}))\left(\frac{\partial}{\partial x^j}\right) = D_{\frac{\partial}{\partial x^i}}\left(A_{X^*}\left(\frac{\partial}{\partial x^j}\right)\right) - A_{X^*}\left(D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right) = -D_{\frac{\partial}{\partial x^i}} D_{\frac{\partial}{\partial x^j}} \left(\sum_k \xi^k \frac{\partial}{\partial x^k}\right) = -\sum_k \frac{\partial^2 \xi^k}{\partial x^i \partial x^j} \frac{\partial}{\partial x^k}$. On the other hand, since D is locally flat and since X^* is an infinitesimal affine transformation with respect to D, we know $D_{\frac{\partial}{\partial x^i}}(A_{X^*}) = 0$ (cf. [2]). Hence we get $\frac{\partial^2 \xi^k}{\partial x^i \partial x^j} = 0$. From this and (1.4) it follows $L_{X^*}\alpha = L_{X^*}D \log K = D\, L_{X^*} \log K = -D\left(\sum_j \frac{\partial \xi^j}{\partial x^j}\right) = -\sum \frac{\partial^2 \xi^j}{\partial x^i \partial x^j} dx^i = 0$. Thus we have

$$(1.5) \qquad L_{X^*}\alpha = 0, \qquad \text{for all } X \in \mathfrak{g}.$$

By (1.4) we see $\alpha(X^*) = (D \log K)(X^*) = D_{X^*} \log K = -\sum_j \frac{\partial \xi^j}{\partial x^j}$. By (1.1) we get $A_{X^*}\left(\frac{\partial}{\partial x^j}\right) = -D_{\frac{\partial}{\partial x^j}} X^* = -\sum_i \frac{\partial \xi^i}{\partial x^j} \frac{\partial}{\partial x^i}$. Hence we have $\operatorname{Tr} f(X) = -\sum_i \frac{\partial \xi^i}{\partial x^i}(o) = \alpha_o(X_o^*) = \alpha_o(q(X))$, which implies (1). Using (1.5) and the fact that A_{Y^*} is a derivation of the algebra of tensor fields which maps every function into zero, we obtain $(D\alpha)(X^*, Y^*)=(D_{Y^*}\alpha)(X^*)=-(A_{Y^*}\alpha)(X^*)=-A_{Y^*}(\alpha(X^*))+\alpha(A_{Y^*}X^*)=\alpha(A_{Y^*}X^*)$. This means $D\alpha_o(q(X), q(Y))=\alpha_o(f(Y)q(X))$. Q.E.D.

Lemma 1.3. *Let* $\langle\ ,\ \rangle$ *denote the value of* g *at* o. *Then, for* $X, Y\in\mathfrak{g}$ *we have*

$$\text{(C)}\qquad \begin{aligned}&\langle f(X)q(Y), q(Z)\rangle+\langle q(Y), f(X)q(Z)\rangle\\ &= \langle f(Y)q(X), q(Z)\rangle+\langle q(X), f(Y)q(Z)\rangle .\end{aligned}$$

Proof. Since A_{X^*} is a derivation of the algebra of tensor fields and maps every function into zero, we see $(A_{X^*}g)(Y^*,Z^*)=A_{X^*}(g(Y^*,Z^*))-g(A_{X^*}Y^*,Z^*)-g(Y^*, A_{X^*}Z^*)=-g(A_{X^*}Y^*, Z^*)-g(Y^*, A_{X^*}Z^*)$. Since X^* is an infinitesimal isometry, we have $L_{X^*}g=0$ and hence $(A_{X^*}g)(Y^*, Z^*)=-(D_{X^*}g)(Y^*, Z^*)$. Thus we have $(D_{X^*}g)(Y^*, Z^*)=g(A_{X^*}Y^*, Z^*)+g(Y^*, A_{X^*}Z^*)$. Since g is locally Hessian it follows $(D_{X^*}g)(Y^*, Z^*)=(D_{Y^*}g)(X^*, Z^*)$. This shows

$$g(A_{X^*}Y^*, Z^*)+g(Y^*, A_{X^*}Z^*) = g(A_{Y^*}X^*, Z^*)+g(X^*, A_{Y^*}Z^*) ,$$

which implies (C). Q.E.D.

Since q is a linear isomorphism of $\mathfrak{g}$ onto V, for each $v\in V$ there exists a unique $X_v\in\mathfrak{g}$ such that

$$(1.6)\qquad q(X_v) = v .$$

We now define an operation of multiplication in V by the formula

$$(1.7)\qquad u\cdot v = f(X_u)v \qquad \text{for} \quad u, v\in V .$$

We use the following notation

$$\begin{aligned}L_u v &= u\cdot v,\ \ R_u v = v\cdot u ,\\ [u\cdot v\cdot w] &= u\cdot(v\cdot w)-(u\cdot v)\cdot w .\end{aligned}$$

From Lemma 1.1, it follows

$$(1.8)\qquad [L_u, L_v] = L_{u\cdot v-v\cdot u} ,$$

$$(1.9)\qquad [u\cdot v\cdot w] = [v\cdot u\cdot w] ,$$

$$(1.10)\qquad [L_u, R_v] = R_{u\cdot v}-R_v R_u ,$$

and these conditions are mutually equivalent. An algebra satisfying one of the above conditions (1.8)~(1.10) is said to be *left symmetric* [7].

The condition (C) and the formula in Lemma 1.2 are reduced to

$$(\text{C}')\qquad \langle u\cdot v, w\rangle+\langle v, u\cdot w\rangle = \langle v\cdot u, w\rangle+\langle u, v\cdot w\rangle ,$$

$$\alpha_o(v) = \operatorname{Tr} L_v \,, \tag{1.11}$$

$$(D\alpha_o)(u, v) = \alpha_o(u\cdot v) \,. \tag{1.12}$$

Using (1.9) and (1.12), we have

$$\begin{aligned}(D\alpha_o)(u\cdot v, w)&+(D\alpha_o)(v, u\cdot w)\\ &= (D\alpha_o)(v\cdot u, w)+(D\alpha_o)(u, v\cdot w)\,.\end{aligned} \tag{1.13}$$

Lemma 1.4. *Let V be a left symmetric algebra endowed with an inner product satisfying the condition* (C′), *and let U be a subalgebra of V. For a fixed element $u\in U$ we put $P=\{p\in U;\ p\cdot u=0\}$. Suppose $L_uP\subset P$. Then we have*

$$L_u(p\cdot q) = (L_up)\cdot q+p\cdot(L_uq)\,, \tag{1}$$

$$\exp tL_u(p\cdot q) = (\exp tL_up)\cdot(\exp tL_uq)\,, \tag{2}$$

$$\begin{aligned}&\frac{d}{dt}\langle \exp tL_up, \exp tL_uq\rangle\\ &= \langle u, \exp tL_u(p\cdot q)\rangle\,,\end{aligned} \tag{3}$$

for $p, q\in P$.

Proof. (1) follows immediately from (1.9), and (2) is a consequence of (1). By the condition (C′) and (2), we have

$$\begin{aligned}&\frac{d}{dt}\langle \exp tL_up, \exp tL_uq\rangle\\ &= \langle L_u \exp tL_up, \exp tL_uq\rangle+\langle \exp tL_up, L_u \exp tL_uq\rangle\\ &= \langle(\exp tL_up)\cdot u, \exp tL_uq\rangle+\langle u, (\exp tL_up)\cdot(\exp tL_uq)\rangle\\ &= \langle u, \exp tL_u(p\cdot q)\rangle \qquad \text{for } p, q\in P\,.\end{aligned}$$

Q.E.D.

A left symmetric algebra V is called *elementary,* if V satisfies the following conditions:

$$V = \{u\}+P \qquad \text{(direct sum of vector spaces)}, \tag{E.1}$$

$$u\cdot u = u, \quad u\neq 0\,, \tag{E.2}$$

$$u\cdot P\subset P, \quad P\cdot u = \{0\}\,, \tag{E.3}$$

$$p\cdot q = \Phi(p, q)u \qquad \text{for } p, q\in P\,, \tag{E.4}$$

where Φ is a symmetric bilinear form on P.

Proposition 1.5. *Let Ω be a homogeneous domain in V containing* 0, *on which an affine Lie group G acts simply transitively. Suppose that the left symmetric algebra V of Ω at* 0 *is elementary,* i.e. *$V=\{u\}+P$ satisfies the above conditions*

(E.1)~(E.4). *Then we have*

$$\Omega=\left\{au+p;\ a-\frac{1}{2}\Phi(p,p)>-1\ \ for\ \ a\in\mathbf{R},\ p\in P\right\}.$$

In particular, if Φ is a positive definite symmetric bilinear form on P, then Ω is the interior of a paraboloid.

Proof. From $X_v x=L_v x+v$ for $x, v\in V$, it follows $\exp X_v x=x+\sum_{k=0}^{\infty}\frac{L_v^k}{(k+1)!}(L_v x+v)$. Using this formula, for $t\in\mathbf{R}$, $q\in P$ we get

$$\exp tX_u(au+p)=(ae^t+e^t-1)u+\exp tL_u p\,,$$
$$\exp X_q(au+p)=\left(a+\Phi(p,q)+\frac{1}{2}\Phi(q,q)\right)u+p+q\,,$$

where $a\in\mathbf{R}$, $p\in P$. We show first that $\exp tX_u$ and $\exp X_q$ leave $\Omega'=\{au+p;\ a-\frac{1}{2}\Phi(p,p)>-1$ *for* $a\in\mathbf{R}, p\in P\}$ invariant. Let $au+p\in\Omega'$. Then we have $(ae^t+e^t-1)-\frac{1}{2}\Phi(\exp tL_u p, \exp tL_u p)=(ae^t+e^t-1)-\frac{1}{2}\Phi(p,p)e^t=e^t(a-\frac{1}{2}\Phi(p,p)+1)-1>-1$, by Lemma 1.4. Therefore $(\exp tX_u)(au+p)\in\Omega'$. On the other hand, $\exp X_q(au+p)\in\Omega'$, since $a+\Phi(p,q)+\frac{1}{2}\Phi(q,q)-\frac{1}{2}\Phi(p+q,p+q)=a-\frac{1}{2}\Phi(p,p)>-1$. For any $au+p\in\Omega'$ we have $\exp t_0X_u \exp X_{-p}(au+p)=0$, where $t_0=-\log(a-\frac{1}{2}\Phi(p,p)+1)$. These show that G acts transitively on Ω'. Since G acts transitively on Ω and Ω' and since $\Omega\cap\Omega'\ni 0$, we conclude $\Omega=\Omega'$.

Q.E.D.

Now assume that V is decomposed into a direct sum of vector spaces

$$V=\sum_{k=1}^{m-1}(\{u_k\}+P_k)+V^m \tag{A.0}$$

with the following properties:

(A.1) *$V_k=\{u_k\}+P_k$ is an elementary left symmetric algebra such that the real parts of the eigenvalues of L_{u_k} on P_k are equal to $\frac{1}{2}$ and that the symmetric bilinear form $\Phi_k(p,q)$ is positive definite on P_k, where $p\cdot q=\Phi_k(p,q)u_k$ for $p,q\in P_k$.*

(A.2) *If we set $V^{k+1}=\sum_{l=k+1}^{m-1}V_l+V^m$, then V^{k+1} is a left symmetric subalgebra of V such that*

$$u_k\cdot V^{k+1}\subset V^{k+1},\quad V^{k+1}\cdot u_k=\{0\}\,,$$
$$P_k\cdot V^{k+1}\subset P_k,\quad V^{k+1}\cdot P_k\subset P_k\,,$$

and the real parts of the eigenvalues of L_{u_k} on V^{k+1} are equal to 0.

(A.3) *The factors of the decomposition $V=\sum_{k=1}^{m-1}(\{u_k\}+P_k)+V^m$ are mutually orthogonal with respect to $D\alpha_o$ and $D\alpha_o$ is positive definite on $\sum_{k=1}^{m-1}(\{u_k\}+P_k)$ and non-degenerate on V^m.*

2. Proof of Theorem: Existence of u_m in V^m

The main purpose of this section is to prove the following.

Proposition 2.1. *Let $V=\sum_{k=1}^{m-1}(\{u_k\}+P_k)+V^m$ be the decomposition given in* (A.1)∼(A.3). *Then there exists a non-zero element u_m in V^m such that*

$$u_m \cdot u_m = u_m\,, \tag{1}$$

$$V^m \cdot u_m \subset \{u_m\}\,. \tag{2}$$

We set $\mathfrak{g}^m=\{X_v\in\mathfrak{g};\ v\in V^m\}$. Since V^m is a left symmetric subalgebra of V, $\mathfrak{g}^m$ is a Lie subalgebra of $\mathfrak{g}$. Since $\mathfrak{g}^m$ is solvable, by Lie's theorem there exist elements $u\neq 0$, $v\in V^m$ such that

$$f(X)\,(u+iv) = (\tilde{\lambda}(X)-i\tilde{\mu}(X))\,(u+iv) \qquad \text{for } X\in\mathfrak{g}^m\,,$$

where $i^2=-1$, and $\tilde{\lambda}$, $\tilde{\mu}$ are real linear functions on $\mathfrak{g}^m$. Hence we have

$$\begin{aligned} x\cdot u &= \lambda(x)u+\mu(x)v \\ x\cdot v &= -\mu(x)u+\lambda(x)v \end{aligned} \tag{2.1}$$

for $x\in V^m$, where $\lambda=\tilde{\lambda}\circ q^{-1}$, $\mu=\tilde{\mu}\circ q^{-1}$. We shall now prove that u and v are linearly dependent and so $V^m\cdot\{u\}\subset\{u\}$.

Suppose that u and v be linearly independent. Let W be the subspace of V^m spanned by the elements $\{u, v\}$. Then we have

Lemma 2.2. *Let $x\in V^m$. If $(D\alpha_o)(x, w)=0$ for all $w\in W$, then $x\cdot w=0$ for all $w\in W$.*

Proof. We first remark $\alpha_o\neq 0$ on W. Indeed, if $\alpha_o=0$ on W, we have $(D\alpha_o)(y, u)=\alpha_o(y\cdot u)=0$ for all $y\in V^m$. Since $D\alpha_o$ is non-degenerate on V^m (cf. (A.3)), we have $u=0$, which is a contradiction. From the assumption $(D\alpha_o)(x, u)=(D\alpha_o)(x, v)=0$, we get $\lambda(x)\alpha_o(u)+\mu(x)\alpha_o(v)=0$, $-\mu(x)\alpha_o(u)+\lambda(x)\alpha_o(v)=0$. Since $\alpha_o(u)\neq 0$ or $\alpha_o(v)\neq 0$ as remarked above, we get $\lambda(x)=\mu(x)=0$ and hence $x\cdot u=x\cdot v=0$. Thus $x\cdot w=0$ for all $w\in W$. Q.E.D.

Consider now the subspace $W_0=\{w_0\in W;\ (D\alpha_o)(w_0, w)=0 \text{ for all } w\in W\}$ of W. We shall first show that $W_0\neq\{0\}$. Suppose that $W_0=\{0\}$. Then $D\alpha_o$ is non-degenerate on W and hence there exists a non-zero element $z_1\in W$ such that $(D\alpha_o)(z_1, w)=\alpha_o(w)$ for all $w\in W$. When $z_1=au+bv$ ($a, b\in\mathbf{R}$), put $z_2=-bu+av$. Then $\{z_1, z_2\}$ is a basis of W such that

$$\begin{aligned} x\cdot z_1 &= \lambda'(x)z_1+\mu'(x)z_2\,, \\ x\cdot z_2 &= -\mu'(x)z_1+\lambda'(x)z_2\,, \end{aligned} \tag{2.1'}$$

for $x\in V^m$, where λ' and μ' are linear functions on V^m. By (1.13) we have

$$\begin{aligned}(D\alpha_o)(w, z_1\cdot z_1) &= (D\alpha_o)(w\cdot z_1, z_1)+(D\alpha_o)(z_1, w\cdot z_1)-(D\alpha_o)(z_1\cdot w, z_1)\\ &= \alpha_o(w\cdot z_1)+\alpha_o(w\cdot z_1)-\alpha_o(z_1\cdot w)\\ &= (D\alpha_o)(w, z_1)\\ &= \alpha_o(w)\end{aligned}$$

for all $w\in W$. This implies $z_1\cdot z_1=z_1$ and by (2.1′) $z_1\cdot z_2=z_2$. Put $z_2\cdot z_1=\lambda_0 z_1+\mu_0 z_2$ and $z_2\cdot z_2=-\mu_0 z_1+\lambda_0 z_2$. Then we have

$$(z_1\cdot z_2)\cdot z_1-z_1\cdot(z_2\cdot z_1) = (z_2\cdot z_1)\cdot z_1-z_2\cdot(z_1\cdot z_1)$$

by (1.9) and so

$$0 = \lambda_0\mu_0 z_1+(\mu_0^2-\mu_0)z_2\,.$$

Therefore $\mu_0=0$, or $\mu_0=1$ and $\lambda_0=0$. In the case $\mu_0=0$, we put $x=\alpha_o(z_2)z_1-\alpha_o(z_1)z_2$. Then

$$\begin{aligned}&(D\alpha_o)(z_1, x) = \alpha_o(z_1\cdot x) = \alpha_o(x) = \alpha_o(z_2)\alpha_o(z_1)-\alpha_o(z_1)\alpha_o(z_2) = 0\,,\\ &(D\alpha_o)(z_2, x) = \alpha_o(z_2\cdot x) = \lambda_0\alpha_o(x) = 0\,,\end{aligned}$$

which imply $x=0$ and $\alpha_o(z_1)=\alpha_o(z_2)=0$. This contradicts $\alpha_o\neq 0$ on W. If $\mu_0=1$ and $\lambda_0=0$, then it follows from (C′)

$$\langle z_1\cdot z_2, z_2\rangle+\langle z_2, z_1\cdot z_2\rangle = \langle z_2\cdot z_1, z_2\rangle+\langle z_1, z_2\cdot z_2\rangle$$

and hence $\langle z_1, z_1\rangle+\langle z_2, z_2\rangle=0$, which is a contradiction. Thus we have shown that $W_0\neq\{0\}$.

Now, we show $\dim W_0>1$. Suppose $\dim W_0=1$. Then W_0 is spanned by a non-zero element $z_1=au+bv$ ($a, b\in\mathbf{R}$). If we set $z_2=-bu+av$, then $\{z_1, z_2\}$ is a basis of W such that

$$\begin{aligned}x\cdot z_1 &= \lambda''(x)z_1+\mu''(x)z_2\,,\\ x\cdot z_2 &= -\mu''(x)z_1+\lambda''(x)z_2\,,\end{aligned}\tag{2.1″}$$

for $x\in V^m$, where λ'' and μ'' are linear functions on V^m. Since $(D\alpha_o)(z_1, w)=0$ for all $w\in W$, it follows from Lemma 2 that $z_1\cdot z_1=z_1\cdot z_2=0$. Using this and (1.13) we get

$$\begin{aligned}&(D\alpha_o)(z_2\cdot z_1, z_1) = 0\,,\\ &(D\alpha_o)(z_2\cdot z_1, z_2) = (D\alpha_o)(z_1\cdot z_2, z_2)+(D\alpha_o)(z_2, z_1\cdot z_2)-(D\alpha_o)(z_1, z_2\cdot z_2) = 0\,,\end{aligned}$$

and hence we can write $z_2\cdot z_1=\lambda_0 z_1$, $z_2\cdot z_2=\lambda_0 z_2$. We have from (C′)

$$\langle z_2\cdot z_1, z_1\rangle+\langle z_1, z_2\cdot z_1\rangle = \langle z_1\cdot z_2, z_1\rangle+\langle z_2, z_1\cdot z_1\rangle$$

and so

$$2\lambda_0\langle z_1, z_1\rangle = 0\,.$$

Therefore, we obtain $\lambda_0=0$ and $(D\alpha_o)(z_2, z_1)=(D\alpha_o)(z_2, z_2)=0$. This means $z_2\in W_0$ and dim $W_0=2$, which is a contradiction. Thus dim $W_0=1$ does not occur.

Finally suppose dim $W_0=2$. Since $D\alpha_o=0$ on W, we have by Lemma 2.2.

$$W\cdot W = \{0\}\,. \tag{2.2}$$

In this case we first prove:

(2.3) *Let $P=\{p\in V^m; p\cdot u=0\}$. Then $L_u V^m\subset P$ and the real parts of the eigenvalues of L_u on V^m are equal to* 0.

Proof of (2.3). By (1.9), (2.1) and (2.2), we have $(u\cdot x)\cdot u=u\cdot(x\cdot u)+(x\cdot u)\cdot u-x\cdot(u\cdot u)=0$ for all $x\in V^m$. Hence it follows $L_u V^m\subset P$. Let $(V^m)^c$ be the complexification of V^m and let P^c be the complex subspace of $(V^m)^c$ spanned by P. Then the inner product $\langle\ ,\ \rangle$ on V^m can be extended to a complex symmetric bilinear form on $(V^m)^c$, which is denoted also by $\langle\ ,\ \rangle$. Let $\lambda+i\mu$ $(\lambda, \mu\in\mathbf{R})$ be an eigenvalue of L_u on P^c and let $p+iq$ $(p, q\in P)$ be an eigenvector corresponding to $\lambda+i\mu$, i.e. $L_u(p+iq)=(\lambda+i\mu)(p+iq)$. Then we have $\langle\exp tL_u(p+iq), \exp tL_u(p-iq)\rangle=\langle e^{(\lambda+i\mu)t}(p+iq), e^{(\lambda-i\mu)t}(p-iq)\rangle$ $=e^{2\lambda t}(\langle p, p\rangle+\langle q, q\rangle)$. On the other hand, it follows from Lemma 1.4, (C′) and (2.2) that

$$\begin{aligned}
&\frac{d^3}{dt^3}\langle\exp tL_u(p+iq), \exp tL_u(p-iq)\rangle\\
&=\frac{d^2}{dt^2}\langle u, \exp tL_u((p+iq)\cdot(p-iq))\rangle\\
&=\langle u, u\cdot p'\rangle\\
&=\langle p'\cdot u, u\rangle+\langle u, p'\cdot u\rangle-\langle u\cdot u, p'\rangle = 0\,,
\end{aligned}$$

where $p'=L_u \exp tL_u((p+iq)\cdot(p-iq))\in P^c$. Therefore we get $(2\lambda)^3(\langle p, p\rangle+\langle q, q\rangle)e^{2\lambda t}=0$. Since $\langle p, p\rangle+\langle q, q\rangle>0$, we have $\lambda=0$. According to this and $L_u V^m\subset P$, we see that the real parts of the eigenvalues of L_u on V^m are equal to 0. Thus the proof of (2.3) is completed.

We shall next show:

$$\mathrm{Tr}_{P_k} L_u = 0\,. \tag{2.4}$$

Proof of (2.4). We have $L_u P_k\subset P_k$ by (A.2). Let $p, q\in P_k$. Then it follows from (A.1), (A.3) and (1.13) that

$$(D\alpha_o)(L_u p, q)+(D\alpha_o)(p, L_u q) = (D\alpha_o)(R_u p, q)+(D\alpha_o)(u, p\cdot q)$$

$$= (D\alpha_o)(R_u p, q) + (D\alpha_o)(u, \phi_k(p, q)u_k)$$
$$= (D\alpha_o)(R_u p, q)\,.$$

Since $D\alpha_0$ is positive definite on P_k by (A.3), denoting by ${}^tL_u$ the transpose of L_u on P_k with respect to $D\alpha_0$, we have

$$R_u = L_u + {}^tL_u \quad \text{on} \quad P_k$$

From (1.10) and (2.2), we have $[L_u, R_u] = R_{u\cdot u} - R_u^2 = -R_u^2$ and hence $\mathrm{Tr}_{P_k} R_u {}^tR_u = \mathrm{Tr}_{P_k} R_u^2 = -\mathrm{Tr}_{P_k}[L_u, R_u] = 0$. This means $R_u = 0$ and ${}^tL_u = -L_u$ on P_k. Therefore we obtain $\mathrm{Tr}_{P_k} L_u = 0$.

Using (A.2), (2.3) and (2.4), we get

$$\alpha_o(u) = \sum_{k=1}^{m-1} \mathrm{Tr}_{\{u_k\}} L_u + \sum_{k=1}^{m-1} \mathrm{Tr}_{P_k} L_u + \mathrm{Tr}_{V^m} L_u = 0\,.$$

Taking v for u, we have similarly

$$\alpha_o(v) = 0\,.$$

Hence $\alpha_o = 0$ on W. As remarked in the proof of Lemma 2, this contradicts the assumption (A.3).

Thus we conclude that $\dim W = 1$, and this proves that u and v are linearly dependent which contradicts the assumption that u and v are linearly independent and that $V^m \cdot u \subset \{u\}$.

Suppose now $u \cdot u = 0$. Then, by the same argument as above, we get $\alpha_o = 0$ on W and this contradicts (A.3). Therefore we have $u \cdot u = \lambda_0 u$, where $\lambda_0 \neq 0 \in \mathbf{R}$. Putting $u_m = \dfrac{1}{\lambda_0} u$ we get

$$u_m \cdot u_m = u_m\,. \tag{2.5}$$

This completes the proof of Proposition 2.1.

3. Proof of Theorem (continued): Decomposition of V^m

Proposition 3.1 *Let u_m be the element in Proposition 2.1. We set $P = \{p \in V^m;\ p \cdot u_m = 0\}$. Then $L_{u_m} P \subset P$ and the real parts of the eigenvalues of L_{u_m} on P are equal to 0 or $\frac{1}{2}$. Let P_m and V^{m+1} denote the largest subspaces of P on which the real parts of the eigenvalues of L_{u_m} are equal to $\frac{1}{2}$ and 0, respectively. Then we get the decomposition*

$$V = \sum_{k=1}^{m} (\{u_k\} + P_k) + V^{m+1}$$

of V and each factor of the decomposition has the properties stated in (A.1)$\sim$(A.3).

Proof. For simplicity, we write u for the element u_m. First we have

$$L_u P \subset P, \tag{3.1}$$

$$V^m = \{u\} + P \qquad \text{(direct sum)}. \tag{3.2}$$

In fact, for each $p \in P$ we have $(u \cdot p) \cdot u = (p \cdot u) \cdot u - p \cdot (u \cdot u) + u \cdot (p \cdot u) = 0$ by (1.9), which shows (3.1). The relation (3.2) follows from $x - \lambda(x)u \in P$ for all $x \in V^m$, where λ is a linear function on V^m such that $x \cdot u = \lambda(x)u$.

Let P^c denote the complexification of P and let $P^c{}_{[\lambda]}$ denote the larget subspace of P^c on which the real parts of the eigenvalues of L_u on P^c are equal to λ, i.e. $P^c{}_{[\lambda]} = \{p \in P^c;\ (L_u - (\lambda + i\mu))^r p = 0$ for some $\mu \in \mathbf{R}$ and sufficiently large $r\}$.

Lemma 3.2. *The real parts of the eigenvalues of L_u on P are equal to* 0 *or* $\frac{1}{2}$, *i.e.* $P^c = P^c{}_{[\frac{1}{2}]} + P^c{}_{[0]}$.

Proof. For $p \in P^c$, we have

$$\frac{d}{dt}\langle \text{ext } tL_u p,\ \exp tL_u u\rangle = \langle u,\ \exp tL_u(p \cdot u)\rangle = 0\,.$$

by Lemma 1.4. Since $\exp tL_u u = e^t u$, it follows

$$\langle \text{ext } tL_u p,\ u\rangle = ae^{-t} \tag{3.3}$$

where a is a constant determined by p, not depending on t. Therefore, for each $x = cu + p \in (V^m)^c$ $(c \in \mathbf{C},\ p \in P^c)$ we have

$$\begin{aligned} \langle u,\ \exp tL_u x\rangle &= \langle u,\ ce^t u + \exp tL_u p\rangle \\ &= c\langle u,\ u\rangle e^t + \langle u,\ \exp tL_u p\rangle \\ &= ae^{-t} + be^t\,, \end{aligned} \tag{3.4}$$

where a, b are constants determined by x, not depending on t. Let $\lambda + i\mu$ $(\lambda, \mu \in \mathbf{R})$ be an eigenvalue of L_u on P^c and let $p + iq$ $(p, q \in P)$ be an eigenvector corresponding to $\lambda + i\mu$. Then we have

$$\begin{aligned} &\frac{d}{dt}\langle \exp tL_u(p+iq),\ \exp tL_u(p-iq)\rangle \\ &= \frac{d}{dt}\langle e^{(\lambda+i\mu)t}(p+iq),\ e^{(\lambda-i\mu)t}(p-iq)\rangle \\ &= \frac{d}{dt}e^{2\lambda t}(\langle p, p\rangle + \langle q, q\rangle) \\ &= 2\lambda(\langle p, p\rangle + \langle q, q\rangle)\, e^{2\lambda t} \end{aligned}$$

On the other hand, we get from Lemma 1.4. and (3.4)

$$\frac{d}{dt}\langle \exp tL_u(p+iq),\ \exp tL_u(p-iq)\rangle$$

$$= \langle u, \exp tL_u((p+iq)\cdot(p-iq))\rangle$$
$$= ae^{-t}+be^t .$$

Thus we have

$$(3.5) \qquad 2\lambda(\langle p, p\rangle+\langle q, q\rangle)e^{2\lambda t} = ae^{-t}+be^t .$$

This implies $\lambda=0$, $\frac{1}{2}$ or $-\frac{1}{2}$. Let $p_\lambda \in P^c{}_{[\lambda]}$ be such that $(L_u-(\lambda+i\mu))^r p_\lambda=0$ for some μ and r. Then

$$\exp tL_u p_\lambda = e^{(\lambda+i\mu)t}\sum_{l=0}^{r-1}\frac{t^l}{l!}(L_u-(\lambda+i\mu))^l p_\lambda ,$$

and

$$\langle \exp tL_u p_\lambda, u\rangle = e^{(\lambda+i\mu)t}h(t) ,$$

where $h(t)$ is a polynomial of degree $r-1$ at most. From this and (3.3), we obtain

$$ae^{-(1+\lambda)t} = h(t)e^{i\mu t} .$$

Assume now $a\neq 0$. Since $1+\lambda>0$ and since $h(t)$ is a polynomial of degree $\leqq r-1$, we have

$$\lim_{t\to-\infty}\left|\frac{ae^{-(1+\lambda)t}}{t^r}\right| = \infty \text{ and } \lim_{t\to-\infty}\left|\frac{h(t)e^{i\mu t}}{t^r}\right| = 0 ,$$

which is a contradiction. Hence we have $a=0$. Thus it follows

$$(3.3) \qquad \langle \exp tL_u p, u\rangle = 0 \qquad \text{for all } p\in P^c ,$$

$$(3.4) \qquad \langle \exp tL_u x, u\rangle = be^t \qquad \text{for } x\in (V^m)^c$$

and

$$(3.5) \qquad 2\lambda(\langle p, p\rangle+\langle q, q\rangle)e^{2\lambda t} = be^t .$$

This implies $\lambda=0$ or $\frac{1}{2}$, which proves Lemma 3. Q.E.D.

Lemma 3.3. *Let $P_{[\lambda]}$ denote the largest subspace of P on which the real parts of the eigenvalues of L_u are equal to λ. Then the factors of the decomposition $V^m=\{u\}+P_{[\frac{1}{2}]}+P_{[0]}$ are mutually orthogonal with respect to $D\alpha_o$ and satisfy the following relations*

$$P_{[0]}\cdot P_{[\frac{1}{2}]}\subset P_{[\frac{1}{2}]},\ \ P_{[\frac{1}{2}]}\cdot P_{[0]}\subset P_{[\frac{1}{2}]} ,$$
$$P_{[0]}\cdot P_{[0]}\subset P_{[0]},\ \ P_{[\frac{1}{2}]}\cdot P_{[\frac{1}{2}]}\subset \{u\} .$$

Proof. For $\lambda\in\mathbf{R}$ we put $(V^m)^c{}_{[\lambda]}=\{x\in(V^m)^c;\ (L_u-(\lambda+i\mu))^r x=0$ for some $\mu\in\mathbf{R}$ and sufficiently large $r\}$. Then $V^m{}_{[1]}=\{u\}$, $(V^m)^c{}_{[\frac{1}{2}]}=P^c{}_{[\frac{1}{2}]}$ and

$(V^m)^c{}_{[0]}=P^c{}_{[0]}$. Let $p_\lambda\in P^c{}_{[\lambda]}$ and $p_{\lambda'}\in P^c{}_{[\lambda']}$ such that $(L_u-(\lambda+i\mu))^r p_\lambda=0$, $(L_u-(\lambda'+i\mu'))^{r'}p_{\lambda'}=0$ respectively. For $s\geqq r+r'$ we have by Lemma 1

$$\begin{aligned}&(L_u-(\lambda+\lambda'+i(\mu+\mu')))^s(p_\lambda\cdot p_{\lambda'})\\&=\sum_{t=0}^{s}\frac{s!}{t!(s-t)!}(L_u-(\lambda+i\mu))^t p_\lambda\cdot(L_u-(\lambda'+i\mu'))^{s-t}p_{\lambda'}\,,\\&=0\,.\end{aligned}$$

This implies $P^c{}_{[\lambda]}\cdot P^c{}_{[\lambda']}\subset(V^m)^c{}_{[\lambda+\lambda']}$. From $(D\alpha_o)(p,u)=\alpha_o(p\cdot u)=0$ for $p\in P$, it follows that $\{u\}$ and P are orthogonal with respect to $D\alpha_o$. Let $p\in P_{[\frac12]}$ and $q\in P_{[0]}$. Since $p\cdot q\in P_{[\frac12]}$ and since L_u is non-degenerate on $P_{[\frac12]}$, there exists an element $p'\in P_{[\frac12]}$ such that $u\cdot p'=p\cdot q$. Hence we have $(D\alpha_o)(p,q)=\alpha_o(p\cdot q)=\alpha_o(u\cdot p')=(D\alpha_o)(u,p)=0$. This shows that $P_{[\frac12]}$ and $P_{[0]}$ are orthogonal with respect to $D\alpha_o$. Q.E.D.

Lemma 3.4. $\alpha_o(u)>0$.

Proof. We have $[L_u, R_u]=R_{u\cdot u}-R_u^2=R_u-R_u^2$ by (1.10), and $R_u=L_u+{}^tL_u$ on P_k as in the proof of (2.4). Hence it follows

$$\begin{aligned}\mathrm{Tr}_{P_k}L_u&=\frac{1}{2}\mathrm{Tr}_{P_k}R_u\\&=\frac{1}{2}\mathrm{Tr}_{P_k}(R_u^2+[L_u, R_u])\\&=\frac{1}{2}\mathrm{Tr}_{P_k}R_u{}^tR_u\geqq 0\,.\end{aligned}$$

According to (A.2) and Lemma 3.2, we have $\mathrm{Tr}_{\{u_k\}}L_u=0$, $\mathrm{Tr}_{V^m}L_u=1+\frac{1}{2}\dim P_{[\frac12]}>0$. Thus we obtain

$$\begin{aligned}\alpha_o(u)&=\mathrm{Tr}\,L_u\\&=\sum_{k=1}^{m-1}\mathrm{Tr}_{\{u_k\}}L_u+\sum_{k=1}^{m-1}\mathrm{Tr}_{P_k}L_u+\mathrm{Tr}_{V^m}L_u>0\,.\end{aligned}$$

Q.E.D.

Lemma 3.5. *$D\alpha_o$ is positive definite on $P_{[\frac12]}$.*

Proof. For $\mu\geqq 0$ we put $P^c{}_{(\frac12\pm i\mu)}=\{p\in P^c;\ (L_u-(\frac12\pm i\mu))^r p=0$ for sufficiently large $r\}$ and $P_{(\frac12+i\mu)}=\{p+\bar{p};\ p\in P^c{}_{(\frac12+i\mu)}\}$. We shall then prove that the decomposition $P_{[\frac12]}=\sum_{\mu\geqq 0}P_{(\frac12+i\mu)}$ is orthogonal with respect to $D\alpha_o$. For $\mu,\mu'\geqq 0$, let $p\in P^c{}_{(\frac12+i\mu)}$ and $p'\in P^c{}_{(\frac12+i\mu')}$ such that $(L_u-(\frac12+i\mu))^r p=0$, $(L_u-(\frac12+i\mu'))^{r'}p'=0$. Then we have $\exp tL_u p=e^{(\frac12+i\mu)t}p(t)$ and $\exp tL_u p'=e^{(\frac12+i\mu')t}p'(t)$, where $p(t)=\sum_{l=0}^{r-1}\frac{t^l}{l!}(L_u-(\frac12+i\mu))^l p$ and $p'(t)=\sum_{l=0}^{r'-1}\frac{t^l}{l!}(L_u-(\frac12+i\mu'))^l p'$, respectively.

Therefore we get

$$\frac{d}{dt}\langle \exp tL_u p, \overline{\exp tL_u p'}\rangle = \frac{d}{dt}\langle e^{(\frac{1}{2}+i\mu)t}p(t), e^{(\frac{1}{2}-i\mu')t}\overline{p'(t)}\rangle \tag{3.6}$$

$$= \frac{d}{dt} e^{(1+i(\mu-\mu'))t}\langle p(t), \overline{p'(t)}\rangle$$

$$= e^{(1+i(\mu-\mu'))t} g(t)\,,$$

where $g(t)=\frac{d}{dt}\langle p(t), \overline{p'(t)}\rangle+(1+i(\mu-\mu'))\langle p(t), \overline{p'(t)}\rangle$. On the other hand, it follows from Lemma 1.4 and 3.3

$$\frac{d}{dt}\langle \exp tL_u p, \overline{\exp tL_u p'}\rangle = \langle u, \exp tL_u(p\cdot \overline{p'})\rangle \tag{3.7}$$

$$= \langle u, \lambda e^t u\rangle$$

$$= \lambda\langle u, u\rangle e^t\,,$$

where we put $p\cdot\overline{p'}=\lambda u$ ($\lambda\in\mathbf{C}$). Thus we get $g(t)=\lambda\langle u, u\rangle e^{i(\mu'-\mu)t}$. Assume $\mu\neq\mu'$. Since $g(t)$ is a polynomial and since $\mu'-\mu\neq 0$, we get $\lambda=0$ and $p\cdot\overline{p'}=0$. Hence we have

$$P^c_{(\frac{1}{2}+i\mu)}\cdot P^c_{(\frac{1}{2}-i\mu')} = \{0\}, \qquad \text{if } \mu\neq\mu'\,. \tag{3.8}$$

Similarly, using

$$\frac{d}{dt}\langle \exp tL_u p, \exp tL_u p'\rangle = \langle u, \exp tL_u(p\cdot p')\rangle\,,$$

we obtain $h(t)=\nu\langle u, u\rangle e^{i(\mu'+\mu)t}$, where $h(t)=\frac{d}{dt}\langle p(t), p'(t)\rangle+(1+i(\mu'+\mu))\langle p(t), p'(t)\rangle$ and $p\cdot p'=\nu u$ ($\nu\in\mathbf{C}$). If $\mu+\mu'>0$, then we see $\nu=0$ and $p\cdot p'=0$. Thus we have

$$P^c_{(\frac{1}{2}+i\mu)}\cdot P^c_{(\frac{1}{2}+i\mu')} = \{0\} \quad \text{when } \mu+\mu'>0\,. \tag{3.9}$$

If $\mu\neq\mu'$, then it follows from (3.8) (3.9) that $P_{(\frac{1}{2}+i\mu)}\cdot P_{(\frac{1}{2}+i\mu')}=\{0\}$ and hence $P_{(\frac{1}{2}+i\mu)}$, $P_{(\frac{1}{2}+i\mu')}$ are orthogonal with respect to $D\alpha_o$. Now, let $p\in P^c_{(\frac{1}{2}+i\mu)}$ be a non-zero element such that $(L_u-(\frac{1}{2}+i\mu))^r p=0$. Then we have by (3.6) (3.7)

$$\frac{dk(t)}{dt}+k(t) = \lambda\langle u, u\rangle\,,$$

where $k(t)=\langle p(t), \overline{p(t)}\rangle$, $p(t)=\sum_{l=0}^{r-1}\frac{t^l}{l!}(L_u-(\frac{1}{2}+i\mu))^l p$, and $p\cdot\bar{p}=\lambda u$ ($\lambda\in\mathbf{C}$). The solution of this equation is $k(t)=ce^{-t}+\lambda\langle u, u\rangle$ where c is an arbitrary constant. Since $k(t)$ is a polynomial, we get $c=0$ and $k(t)=\lambda\langle u, u\rangle$. Thus we have

$$\lambda = \frac{k(0)}{\langle u, u\rangle} = \frac{\langle p, \bar{p}\rangle}{\langle u, u\rangle} > 0 \tag{3.10}$$

In the case $\mu>0$, by (3.8) (3.9) we obtain $(p+\bar{p})\cdot(p+\bar{p})=p\cdot\bar{p}+\bar{p}\cdot p=2\lambda u$. Therefore it follows from Lemma 3.4 and (3.10)

$$(D\alpha_o)(p+\bar{p}, p+\bar{p}) = \alpha_o((p+\bar{p})\cdot(p+\bar{p})) = 2\lambda\alpha_o(u)>0\,,$$

for $p\neq 0\in P^c{}_{(\frac{1}{2}+i\mu)}$. In the case $\mu=0$ we have

$$(D\alpha_o)(p, p) = \alpha_o(p\cdot p) = \lambda\alpha_o(u)>0\,,$$

for $p\neq 0\in P_{(\frac{1}{2})}$. Since $P_{(\frac{1}{2}+i\mu)}$ and $P_{(\frac{1}{2}+i\mu')}(\mu\neq\mu')$ are orthogonal with respect to $D\alpha_o$, it follows that $D\alpha_o$ is positive definite on $P_{[\frac{1}{2}]}=\sum_{\mu\geq 0}P_{(\frac{1}{2}+i\mu)}$. Q.E.D.

This completes the proof of Proposition 3.1.

Applying Proposition 2.1 and 3.1 successively, our theorem follows by induction on m.

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