Hayakawa, K.
Osaka J. Math.
12 (1975), 767-776

ON THE LIMIT STATE OF SOLUTIONS OF SOME SEMILINEAR DIFFUSION EQUATIONS

KANTARO HAYAKAWA

(Received January 13, 1975)

Introduction. This paper is concerned with the behavior of solutions of the following Cauchy problem for the semilinear diffusion equation

$$(1) \qquad \partial_t u = \Delta u + f(u)\,, \quad u = u(t, x)\,, \quad t>0, \quad x \in R^N\,,$$
$$u(0, x) = u_0(x)\,, \quad x \in R^N\,,$$

where ∂_t and Δ denote $\partial/\partial t$ and $\sum_{j=1}^{N} \partial^2/\partial x_j^2$.

The type of phenomena that occur to solutions depends of course on the type of the nonlinear term $f(u)$ in the equation. For the function of type $u^{1+\alpha}$, H. Fujita [1] dealt with the problem of blowing-up of solutions in a finite time (see also H. Fujita [2], the present author [3] and S. Sugitani [4]). On the other hand A.M.Kolmogorov-I.G.Petrovsky-N.S.Piscounov [5] and Y.I. Kaneli [6] investigated the behavior of solution $u(t, x)$ of (1) as $t \to \infty$ in the case when the function $f(u)$ is $u(1-u)$ as the typical instance.

Here we deal with the problem (1) for the function of the type $u^{1+\alpha}(1-u)$ and investigate the limit state of the solution $u(t, x)$ as $t \to \infty$. Results may be roughly spoken as follows. Whether all nontrivial solutions tend to 1 or not depends on the degree α of the increase of f near 0. In the latter case solutions tend to 0 or 1 according to the magnitude of the initial value. This seems parallel to results in [1].

Precisely, our results are the followings. We assume that the function $f(r)$ satisfies next conditions (i), (ii) and (iii).

(i) $f(r)$ is of class C^1 on the closed interval [0, 1].
(ii) $f(r) > 0$ on the open interval (0,1) and $f(0)=f(1)=0$.
(iii) There exist positive constants C_0 and α, with which we have $f(r) \geqq C_0 r^{1+\alpha}$ for $0 \leqq r \leqq 1/2$.

Further, in Theorem 2, the assumption (iv) should be added.

(iv) $f(r) \leqq C_1 r^{1+\alpha}$ on [0, 1] for some constant $C_1 > 0$.

For the initial data $u_0(x)$ we only consider such functions that are compatible to $f(r)$, *i.e.*, $0 \leqq u_0(x) \leqq 1$, and that are continuous only for the sake of simplicity.

Then we have the following theorems.

Theorem 1[*]**.** *Let the constant α satisfy $N\alpha \leqq 2$. Assume that the function $f(r)$ satisfies conditions (i), (ii) and (iii). Then, for any nontrivial initial data u_0, i.e., $u_0(x) \not\equiv 0$, we have $\lim_{t\to\infty} u(t, x)=1$, in which the convergence is uniform on any bounded set of x in R^N.*

Theorem 2. *Let the constant α satisfy $N\alpha>2$. Assume that the function f satisfies conditions (i), (ii), (iii) and (iv). Take any real γ larger than $2^{\alpha}C_0^{-1}\alpha^{-1}$. Then there exist positive constants a_0 and a_1 having following properties:*
1) If $u_0(x)$ is less than the function $a_0H(\gamma, x)$ all over R^N then the solution $u(t, x)$ starting from u_0 goes to 0 uniformly on R^N.
2) If $u_0(x)$ is larger than $a_1H(\gamma, x)$ all over R^N. Then the solution goes to 1 uniformly on any bounded set of R^N. Here the function $H(t, x)$ denotes the fundamental solution $(2\pi t)^{-N/2} \exp[-|x|^2/4t]$ of the heat equation.

These theorems will be proved in 3. 1. is devoted to preliminary lemmas used in the following parts. In 2. we shall prove a key theorem (Theorem 3).

1. Preliminary lemmas. We consider the Cauchy problem for the quasilinear diffusion equation

$$(\mathrm{A}) \qquad \partial_t u = \Delta u + f(t, x, u), \quad u(0, x) = u_0(x).$$

Here, we assume that the function $f(t, x, r)$ is continuous in (t, x, r) and Lepschitz continuous in r (the Lipschitz constant is taken uniformly in (t, x, r)).

DEFINITION 1.1. For the bounded cuntinuous initial data $u_0(x)$ the function $u(t, x)$ is called the solution of problem (A) in $[0, T)\times R^N$ if it satisfies following conditions a), b) and c) ([1]).
a) For any $T'<T$ $u(t, x)$ is the bounded continuous function of (t, x) on $[0, T']\times R^N$.
b) Initial condition in (A) is satisfied in the usual sence.
c) The differential equation in (A) is satisfied in the sence of distribution in $(0, T)\times R^N$.

In proving our results, we apply the comparison theorem in the next form. For two functions $f_1(t, x, r)$ and $f_2(t, x, r)$ which satisfy the same conditions as above, we consider two Cauchy problems

$$(\mathrm{A}_1) \qquad \partial_t u^1 = \Delta u^1 + f_1(t, x, u^1), \quad u^1(0, x) = u_0^1(x),$$

*) Recently, T. Shirao, H. Tanaka and K. Kobayashi have reported to us that they got some generalized results based on our Theorems 1 and 2.

$(A_2)\qquad \partial_t u^2 = \Delta u^2 + f_2(t, x, u^2)\,,\quad u^2(0, x) = u_0^2(x)\,.$

We denote by $u^k(t, x)$ the solution of (A_k) in $[0, T^k)\times R^N$ for $k=1, 2$.

Lemma 1.2. (see [5]). *If we have $u_0^1(x)\leqq u_0^2(x)$ for all $x\in R^N$ and $f_1(t, x, r)\leqq f_2(t, x, r)$ for all (t, x, r), we have $u^1(t, x)\leqq u^2(t, x)$ on $[0, T')\times R^N$ $(T'\leqq T^1, T^2)$.*

By using the Previous lemma we can show that under the assumptions (i) and (ii) the Cauchy problem (1) admits the unique solution $u(t, x)$ in $[0, \infty)\times R^N$ for the compatible initial data, and this $u(t, x)$ satisfies $0\leqq u(t, x)\leqq 1$.

In the remaining part of this paper we assume that the function f satisfies conditions (i), (ii) and (iii).

Lemma 1.3. *For an arbitrary couple of real numbers (A, B) satisfying $0<A<B<1$, we can find such a positive real $\delta_0=\delta_0(A, B)$ that has following properties:*

(P1) *For two couples (A, B) and (A', B') satisfying $0<A'\leqq A<B\leqq B'<1$ we have $\delta_0(A', B')\geqq\delta_0(A, B)$.*

(P2) *If the initial data $u_0(x)$ is less than A all over R^N, then the solution $u(t, x)$ of (1) is less than B on $[0, \delta_0]\times R^N$.*

Proof of Lemma 1.3. Let $\varphi(t)$ be the solution of the ordinary defferential equation $d\varphi(t)/dt=f(\varphi(t))$, $\varphi(0)=A$. We define the constant $\delta_0(A, B)$ by $\varphi(\delta_0)=B$. Then $\varphi(t)$ is less than B for $0\leqq t\leqq\delta_0$. From Lemma 1.2, where we set $f_1=f_2=f$ and $u_0^1(x)\equiv A$ $u_0^2(x)=u_0(x)$, we have $u(t, x)\leqq\varphi(t)\leqq B$ for $0\leqq t\leqq\delta_0(A, B)$.

DEFINITION 1.4. We define the function $\Phi(r)$ by

$$\Phi(r) = \inf\,\{C_0\wedge(f(s)/s^{1+\alpha});\ 0<s\leqq r\}\,.$$

Here, $a\wedge b$ denotes $\min\{a, b\}$ for any real couple $\{a, b\}$ and C_0 is the constant of the condition (iii).

Then we have

Lemma 1.5. 1) *$\Phi(r)$ is continuous, nonnegative and non-increasing on the closed interval $[0, 1]$.*
2) *$\Phi(1)=0$ and $\Phi(r)$ goes to 0 if and only if r tends to 1.*

Simple calculation leads us to

Lemma 1.6. 1) *Let A, γ, C and h be positive constants satisfying*

$$(1.1)\qquad (1\wedge e^{1-(N\alpha/2)})\alpha C A^{\alpha}\gamma\geqq e^h\,.$$

Then we have

(1.2) $$\frac{|x|^2}{4\gamma^2}-\frac{N}{2\gamma}+CA^{\alpha}\exp[-\alpha|x|^2/4\gamma]\geqq h/\alpha\gamma \quad \textit{for all } x\in R^N .$$

(2) $$1-\sigma r\leqq(1+r)^{-\sigma}\leqq 1-\sigma r/3 \quad \textit{for} \quad \sigma>0,\ 0\leqq r\leqq(\sigma+1)^{-1} .$$

2. Key theorem. Before going to Theorems 1 and 2 we shall show a Key theorem.

DEFINITION 2.1. A continuous function $u(x)$ of x is said to be of class $G[A, \gamma]$ for some positive constants A and γ, if it satisfies

(2.1) $$1\geqq u(x)\geqq A\exp[-|x|^2/4\gamma] \quad \text{for all } x \text{ in } R^N .$$

DEFINITION 2.2. A couple of real constants $[A, \gamma]$ is said to satisfy the condition (*) if the next inequalities are valid:

(*) $$1>A>0, \quad \gamma>0, \quad (1\wedge e^{1-(N\alpha/2)})\alpha A^{\alpha}\Phi(A)\gamma>1 .$$

Theorem 3. *Let the initial data $u(x)$ be of class $G[A_0, \gamma_0]$ for some couple $[A_0, \gamma_0]$ satisfying (*). Then, for any constants A, γ with $0<A<1$, $\gamma>0$, we can find $T_0=T_0(A, \gamma; A_0, \gamma_0)>0$, so that the solution $u(t, x)$ of* (1) *exceeds the function $A\exp[-|x|^2/4\gamma]$ at any time $t\geqq T_0$.*

The proof of this theorem will be given in the last part of this paragraph. First we show the following poroposition.

Proposition 2.3. *Let $[A, \gamma]$ satisfy (*). If $u(t_0, x)$ is of class $G[A, \gamma]$ at some time $t_0\geqq 0$. Then there exist positive constants $\delta_1=\delta_1(A, \gamma)$ and $\varepsilon=\varepsilon(A, \gamma)$ such that*

(2.2) $$u(t_0+t, x)\geqq(1+\varepsilon t)A\exp[-|x|^2/4\gamma] \quad \textit{for } 0\leqq t\leqq\delta_1,\ x\in R^N .$$

Proof. As the equation in (1) is invarient under the translation of t, we can put $t_0=0$. Let $u^*(t, x)$ denote the solution of the Cauchy problem

(1*) $$\partial_t u^* = \Delta u^*+f(u^*), \quad u^*(0, x) = u_0^*(x) = A\exp[-|x|^2/4\gamma] .$$

By Lemma 1.2. we have $u(t, x)\geqq u^*(t, x)$. We shall prove (2.2) for $u^*(t, x)$. We define the constant $h>0$ by the relation $e^{2h}=(1\wedge e^{1-(N\alpha/2)})\alpha\Phi(A)A^{\alpha}\gamma$ and define A' by $e^h=(1\wedge e^{1-(N\alpha/2)})\alpha\Phi(A')A^{\alpha}\gamma$ $(A<A'<1)$. The definitions of the function Φ and the constant $\delta_0=\delta_0(A, A')$ lead us to

(2.3) $$f(u^*(t, x))\geqq\Phi(A')u^*(t, x)^{1+\alpha}, \quad 0\leqq t\leqq\delta_0, \quad x\in R^N .$$

Let $u^{**}(t, x)$ denote the solution of the problem

(1**) $$\partial_t u^{**} = \Delta u^{**}+\Phi(A')u^{**1+\alpha}, \quad u^{**}(0, x) = u_0^{**}(x) = u_0^*(x) .$$

Again by Lemma 1.2, we have $u^*(t, x)\geqq u^{**}(t, x)$ for $0\leqq t\leqq\delta_0$. The Cauchy problem (1**) is equivarent to the next problem (2**) in the integral form.

$$(2^{**})\qquad u^{**}(t,x)=\int_{R^N}H(t,x-y)u_0^{*}(y)\,\mathrm{d}y+$$
$$+\int_0^t\int_{R^N}H(t-\tau,x-y)\Phi(A')\{u^{**}(\tau,y)\}^{1+\alpha}\,dy\,d\tau\,.$$

From this equation we have

$$(2.4)\qquad u^{**}(\tau,y)\geqq\int_{R^N}H(\tau,y-z)u_0^{**}(z)\,dz=$$
$$=A\Big(1+\frac{\tau}{\gamma}\Big)^{-N/2}\exp\left[-|x|^2/(\tau+\gamma)\right].$$

Substituting (2.4) for the second term of the right hand side of (2**), we have

$$(2.5)\qquad \int_0^t\int_{R^N}H(t-\tau,x-y)\Phi(A')\{u^{**}(\tau,y)\}^{1+\alpha}\,dy\,d\tau$$
$$\geqq\Phi(A')A^{1+\alpha}\Big(1+\frac{(t-\tau)(1+\alpha)}{\tau+\gamma}\Big)^{-N/2}\Big(1+\frac{\tau}{\gamma}\Big)^{-(1+\alpha)N/2}$$
$$\exp\Big[-|x|^2/4\Big(\frac{\gamma+\tau}{1+\alpha}+t-\tau\Big)\Big]$$
$$\geqq\Phi(A')A^{1+\alpha}\Big(1+\frac{\tau}{\gamma}\Big)^{-(1+\alpha)N/2}\exp\left[-(1+\alpha)|x|^2/4(\gamma+\tau)\right]+$$
$$+\Phi(A')A^{1+\alpha}\Big\{\Big(1+\frac{(t-\tau)(1+\alpha)}{\tau+\gamma}\Big)^{-N/2}-1\Big\}\Big(1+\frac{\tau}{\gamma}\Big)^{-(1+\alpha)N/2}$$
$$\exp\left[-(1+\alpha)|x|^2/4(\gamma+t)\right].$$

Further, by the simple calculation, we have

$$(2.6)\qquad \int_{R^N}H(t,x-y)u_0^{**}(y)\,dy=u_0^{**}(x)+\int_0^t\frac{\partial}{\partial\tau}\int_{R^N}H(\tau,x-y)u_0^{**}(y)\,dy\,d\tau$$
$$=u_0^{**}(x)+\int_0^t A\Big(1+\frac{\tau}{\gamma}\Big)^{-N/2}\Big\{\frac{|x|^2}{4(\gamma+\tau)^2}-\frac{N}{2(\gamma+\tau)}\Big\}$$
$$\exp\left[-\alpha|x|^2/4(\gamma+\tau)\right]d\tau\,.$$

Substituting (2.5) and (2.6) in to the right hand side of (2**), we get

$$(2.7)\qquad u^{**}(t,x)\geqq u_0^{**}(x)+\int_0^t\Big\{\frac{|x|^2}{4(\gamma+\tau)^2}-\frac{N}{2(\gamma+\tau)}$$
$$+\Phi(A')A^{\alpha}\exp\left[-\alpha|x|^2/4(\gamma+\tau)\right]\Big\}AG(x,\tau;\gamma)\,d\tau+$$

$$+\int_0^t \Phi(A')A^{\alpha}\left\{\left(1+\frac{t(1+\alpha)}{\gamma+\tau}\right)^{-N/2}\left(1+\frac{\tau}{\gamma}\right)^{-N\alpha/2}-1\right\}$$
$$\exp[-\alpha|x|^2/4(\gamma+\tau)]AG(x,\tau;\gamma)d\tau,$$

where $G(x,\tau;\gamma)$ denotes $\left(1+\frac{\tau}{\gamma}\right)^{-N/2}\exp[-|x|^2/4(\tau+\gamma)]=(4\pi\gamma)^{N/2}H(x,\gamma+\tau)$. Applying 1) of Lemma 1.6 to the second term and 2) to the third term, we get

$$u^{**}(t,x)\geqq u_0^{**}(x)+\int_0^t\left\{\frac{h}{\alpha(\gamma+\tau)}-\frac{\Phi(A')A^{\alpha}N(1+2\alpha)}{\gamma+\tau}\right\}AG(x,\tau;\gamma)d\tau \tag{2.8}$$
$$\text{for}\quad t<\gamma(1+N)^{-1}(1+\alpha)^{-1}=d_1(\gamma).$$

Now we set $d_2(h,\gamma)=h/\{2\Phi(A')A^{\alpha}N(1+2\alpha)\}=he^{-h}(1\wedge e^{1-(N\alpha/2)})\gamma/N(1+2\alpha)$. For we have $AG(x,\tau;\gamma)\geqq\left(1+\frac{\tau}{\gamma}\right)^{-N/2}u_0^{**}(x)$, we get

$$u^{**}(t,x)\geqq\left(1+\frac{h}{4\alpha\gamma 2^{N/2}}t\right)u_0^{**}(x)\quad\text{for}\quad t\leqq d_1(\gamma)\wedge d_2(h,\gamma). \tag{2.9}$$

Thus we have (2.2) for the constants $\delta_1=\delta_0(A,A')\wedge d_1\wedge d_2$ and $\varepsilon=h/(4\alpha\gamma 2^{N/2})$.

Before finishing this proposition we make a remark on the constants $\delta_1(A,\gamma)$ and $\varepsilon(A,\gamma)$.

REMARK 2.4. Let constants B_0, B_1, B_2, satisfy $0<B_0<B_1<B_2<1$ and $(1\wedge e^{1-(N\alpha/2)})\alpha\Phi(B_2)B_0^{\alpha}\gamma>1$. Take a constant A between B_0 and B_1. Then we have following uniform estimates with respect to $\delta_1(A,\gamma)$ and $\varepsilon(A,\gamma)$.

$$\begin{aligned}\delta_1(A,\gamma)&\geqq\bar\delta_1(B_0,B_1,B_2,\gamma)=\delta_0(B_1,B_2)\wedge d_1(\gamma)\wedge d_2(h_1,\gamma),\\ \varepsilon(A,\gamma)&\geqq\bar\varepsilon(B_0,B_2,\gamma)=h_1/(4\alpha\gamma 2^{N/2}),\end{aligned} \tag{2.10}$$

where the constant h_1 denotes $1\wedge(1/2)\log\{(1\wedge e^{1-(N\alpha/2)})\alpha\Phi(B_2)B_0^{\alpha}\gamma$.

Proposition 2.5. *Let $[A_0,\gamma_0]$ satisfy the condition (*). If $u(t_0,x)$ is of class $G[A_0,\gamma_0]$ at some time $t_0\geqq 0$, the solution $u(t,x)$ remains in the class $G[A_0,\gamma_0]$ at all time after t_0.*

Proof. By Proposition 2.3 we have (2.2). Thus, at any time t between t_0 and $t_0+\delta_1(A_0,\gamma_0)$, we have

$$u(t,x)\geqq A_0\exp[-|x|^2/4\gamma_0]\quad\text{for all } x \text{ in } R^N. \tag{2.11}$$

By using this argument at the time $t_0+\delta_1(A_0,\gamma_0)$, we get (2.11) for $t\in[t_0+\delta_1(A_0,\gamma_0), t_0+2\delta_1(A_0,\gamma_0)]$. The repetition of this argument leads us to the same estimate (2.11) at any time t in $[t_0+n\delta_1(A_0,\gamma_0), t_0+(n+1)\delta_0(A_1^0,\gamma_0)]$ for $n=0,1,2,\cdots$. This proves our proposition.

Now we are going to prove Theorm 3. Chose constants A_0', h_0 just in the same way as A', h in the proof of Proposition 2.3. Take a real number $A_1 < A_0'$ sufficiently close to A_0', and put $t_0 = 0$. Then we get

Lemma 2.6. *Under the assumptions in Theorem* 3 *we have at some time* $t_0 + t_1''$

(2.12) $\quad u(t_0 + t'_1, x) \geqq A_1 \exp[-|x|^2/4\gamma_0]$ *for all* x *in* R^N.

Proof. When we have

(2.13) $\quad A_{0,1} = (1 + \varepsilon(A_0, \gamma_0)\delta_1(A_0, \gamma_0))A_0 \geqq A_1$,

we can take $t_1' = \delta_1(A_0, \gamma_0)$ by Proposition 2.3.

Else if (2.13) is false, we use Proposition 2.3 again, substituing $A_{0,1}$ for A_0. If the inequality (2.13) is true where A_0 is replaced by $A_{0,1}$, we can take $t_1' = \delta_1(A_0, \gamma_0) + \delta_1(A_{0,1}, \gamma_0)$. In the case when the inequality is false, we continue these steps defining $A_{0,k+1} = (1 + \varepsilon(A_{0,k}, \gamma_0)\, \delta_1(A_{0,k}, \gamma_0))A_{0,k}$, until the constant $A_{0,n}$ exceeds A_1. On account of REMARK 2.4 we can stop this iteration in a finite step. So, at the time $t_0 + t_1' = t_0 + \delta_1(A_0, \gamma_0) + \delta_1(A_{0,1}, \gamma_0) + \cdots + \delta_1(A_{0,n}, \gamma_0)$ the estimate (2.12) holds.

By using Lemma 1.2, where we set $f_1(r) = f(r)$, $f_2(r) = 0$ and $u_0^1(x) = A^* \exp[-|x|^2/4\gamma^*]$, we have

Lemma 2.7. *Let* A^*, γ^* *be some positive constants with* $A^* < 1$. *Let the solution* $u(t, x)$ *of* (1) *be larger than the function* $A^* \exp[-|x|^2/4\gamma^*]$ *at some time* $t^* > 0$. *Then we have*

(2.14) $\quad u(t + t^*, x) \geqq A^*\left(1 + \frac{t}{\gamma^*}\right)^{-N/2} \exp[-|x|^2/4(\gamma^* + t)]$ *for all* x *and* $t > 0$.

Using this lemma we have

(2.15) $\quad u(t_0 + t_1' + t_1'', x) \geqq A_0 \exp[-|x|^2/4(\gamma_0 + t_1'')]$ for all x in R^N,

where t_1'' is defined by $A_0 = \left(1 + \frac{t_1''}{\gamma_0}\right)^{-N/2} A_1$.

Now we put $t_1 = t_0 + t_1' + t_1''$, $\gamma_1 = \gamma_0 + t_1'' = (A_1/A_0)^{2/N}\gamma_0$.

By the fact that γ_1 is larger than γ_0, we can take the same process as the above argument, where t_0 and γ_0 are replaced by t_1 and γ_1. Thus we have constants $t_2 > t_1$ and $\gamma_2 = (A_1/A_0)^{2/N}\gamma_1$ such that

(2.16) $\quad u(t_2, x) \geqq A_0 \exp[-|x|^2/4\gamma_2]$ for all x in R^N.

Denote the constant $(A_1/A_0)^{2n/N}\gamma_0$ by γ_n for $n = 0, 1, 2, \cdots$. By repetition of these arguments we have, at some time t_n, (2.17) $u(t_n, x) \geqq A_0 \exp[-|x|^2/4\gamma_n]$ for all x in R^N.

We chose the integer n sufficiently large so that

(2.18) $\gamma_n > \gamma$,

(2.19) $(1 \wedge e^{1-(N\alpha/2)})\alpha A_0^\alpha \Phi(A)\gamma_n > 1$,

where A and γ are constants considered in the conclusion part of Theorem 3.

Because the function $\Phi(A)$ is continuous, we can find $B>A$, so that the ineqality (2.19) remains true for B changed in place of A. By Lemma 2.6 we can take a constant $T_0>0$ such that we have at the time $t=T_0$

(2.20) $u(t, x) \geqq A \exp[-|x|^2/4\gamma_n]$ for all x in R^N.

By the inequality $(1 \wedge e^{1-(N\alpha/2)})\alpha A^\alpha \Phi(A)\gamma_n > 1$ and Proposition 2.5 we have the same estimate (2.20) for any time after T_0. Because of the fact (2.18), this proves Theorem 3.

3. Proofs of Theorem 1 and 2. To prove Theorem 1, it is enough to prove it in the case of $N\alpha=2$.

Proposition 3.1. *Let α be equal to $2/N$. For any nontrivial solution $u(t, x)$ of* (1), *we can find real constants A_0 and γ_0, so that these constants satisfy* (*) *in Definition* 2.2, *and the solution $u(t, x)$ exceeds A_0 exp $[-|x|^2/4\gamma_0]$ at some time $t=t_0$.*

Proof. Lemma 2.7 shows that, at any time $t>0$, the solution $u(t, x)$ of (1) starting from a nontrivial initial data is positive for all x. Thus we can take a positive number ε, so that we have the estimate $u(1, x)>\varepsilon$ for $|x| \leqq 1$. Using this lemma again, we get

(3.1) $$u(t+1, x) \geqq \int_{R^N} H(t, x-y)u(1, y)\,dy \geqq \varepsilon \int_{|y|\leqq 1} H(t, x-y)\,dy\,.$$

The last term of the above inequality is larger than $E(t)H(2t, x)$ where $E(t)$ denote $\varepsilon 2^{N/2}\omega_N e^{-1/2t}$ (ω_N is the volume of the unit ball in R^N). So, we can assume that the initial data is larger than $C \exp[-|x|^2/4\beta]$ for some positive constants C and β with $C<1/2$. Now we define a function $v(t, x)$ by the integral equation

(3.2) $$v(t, x) = \int_{R^N} H(t, x-y)v_0(y)\,dy + \\ + \int_0^t \int_{R^N} H(t-\tau, x-y)\, C_0 \Big\{\int_{R^N} H(\tau, y-z)v_0(z)\,dz\Big\}^{1+\alpha} dy\,d\tau\,,$$

where $v_0(y)$ denotes the function C $\exp[-|x|^2/4\beta]$.

By some calculation we have

$$(3.3)\qquad v(t, x) \leqq C\left(1+\frac{t}{\beta}\right)^{-N/2}\left(1+KC^{\alpha}\log\left(1+\frac{t}{\beta}\right)\right), \quad K = C_0\beta(4\pi)^{N/2},$$

$$(3.4)\qquad v(t, x) \geqq A(t)\exp[-|x|^2/4\beta(t)],$$

where $A(t)=C_0C^{1+\alpha}\beta^{1+(N/2)}\left(t+\frac{\beta}{1+\alpha}\right)^{-N/2}\log\left(1+\frac{t}{\beta}\right)$ and $\gamma(t)=(\beta+t)/(1+\alpha)$.
By changing the constant C for the smaller one if necessary, we may assume that C^{α} is less than $N/2K$. From this assumption and the inequality (3.3) we have

$$(3.5)\qquad v(t, x) \leqq C < 1/2 \quad \text{for all } t \geqq 0 \text{ and all } x \text{ in } R^N.$$

Operating $\partial_t - \Delta$ to $v(t, x)$ of (3.2), we have

$$(3.6)\qquad \partial_t v - \Delta v = C_0\left\{\int_{R^N} H(t, x-y)v_0(y)\,dy\right\}^{1+\alpha}.$$

The right hand side of (3.6) is less than $C_0\{v(t, x)\}^{1+\alpha}$. So the condition (iii) for $f(r)$ and the inequality (3.5) lead us to

$$(3.7)\qquad \partial_t v(t, x) \leqq \Delta v(t, x) + f(v(t, x)) \quad \text{for} \quad t>0,\ x \in R^N.$$

On the other hand we have $u(0, x) \geqq v(0, x)$. This shows that the solution $u(t, x)$ is larger than $v(t, x)$ on $[0, \infty[\times R^N$.
Chose a constant t_0 large enough so that the quantity

$$\alpha\Phi(A(t_0))A(t_0)^{\alpha}\gamma(t_0) = \alpha C_0^{1+\alpha}C^{\alpha(1+\alpha)}\beta^{1+\alpha}\left(t_0+\frac{\beta}{1+\alpha}\right)^{-1} \left(\log\left(1+\frac{t_0}{\beta}\right)\right)^{\alpha}(\beta+t_0)/(1+\alpha)$$

is larger than 1 and denote $A(t_0)$, $\gamma(t_0)$ by A_0, γ_0.
Thus we have $u(t_0, x) \geqq v(t_0, x) \geqq A_0\exp[-|x|^2/4\gamma_0]$ where $[A_0, \gamma_0]$ satisfies (*).

Theorem 1 is an immediate consequence of Proposition 3.1 and Theorem 3.

Proof of Theorem 2. The existence of the constant a_1 is obvious because we can take $a_1=(1/2)(4\pi\gamma)^{-N/2}$ on account of the fact that γ is larger than $2^{\alpha}C_0^{-1}\alpha^{-1}$. Taking this a_1 and denoting $a_1(4\pi\gamma)^{N/2}=1/2$ by A, we have (*) with this $[A, \gamma]$. Theorem 3 shows that this a_1 has the property in Theorem 2. The existence of the constant a_0 will be proved by using the next proposition, which was proved in [1].

Proposition 3.2. (Theorem 2 in [1]). *Let the function $f(r)$ of nonlinear* *term satisfy (iv) in addition to (i), (ii) and (iii) and let the constant α be larger than $2/N$. Take any positive number γ. Then there exists a positive number a_0 with the following property; if the initial data $u_0(x)$ is less than the function $a_0H(\gamma, x)$, then the solution of (1) is subject to*

$$0 \leqq u(t, x) \leqq MH(t+\gamma, x), \quad t>0, \quad x \in R_N, \tag{3.8}$$

for some positive constant M.

OSAKA UNIVERSITY, COLLEGE OF GENERAL EDCATION

References

[1] H. Fujita: *On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$*, J. Fac. Sci. Univ. Tokyo Sect. I, **13** (1966), 109–124.

[2] ————: *On some nonexistence and nonuniqueness theorems for nonlinear parabolic equations*, Proc. Symposia in Pure Math. vol. 18, A.M.S., 105–113.

[3] K. Hayakawa: *On nonexistence of global solutions of some semilinear parabolic differential equations*, Proc. Japan. Acad. **49** (1973), 503–505.

[4] S. Sugitani: *On nonexistence of global solutions for some nonlinear integral equations*, Osaka J. Math. **12** (1975), 45–51.

[5] A.M. Kolmogorov, I.G. Petrovsky and N.S. Piscounov: *Étude de l'equation de la diffusion avec croissance de la quantité de matière et son application a un problème biologique*, Bull. de l'Univ. a Moscou, (1937).

[6] Y.I. Kaneli: *On the stability of solutions of the equation in the theory of burning for finite initial functions*, Mat. Sb. **65** (1964), 398–413. (Russian).