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# PRODUCTS OF TORSION THEORIES AND APPLICATIONS TO COALGEBRAS

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## 1. Introduction

Throughout this note $R$ is a ring with 1. We shall write $I \leqslant R$ if $I$ is a right ideal of $R$. A non-empty set of right ideals $\Gamma$ of $R$ is called a Gabriel filter if it satisfies

T1. If $I \in \Gamma$ and $r \in R$, then $(I:r) \in \Gamma$.

T2. If $I$ is a right ideal and there exists $J \in \Gamma$ such that $(I:r) \in \Gamma$ for every $r \in J$, then $I \in \Gamma$.

It is well-known [4] that there is a one to one correspondence between Gabriel filters of $R$ and hereditary torsion theories for the category of right $R$-modules. W. Schelter [3] investigated products of torsion theories or equivalently of Gabriel filters that for a family of pairs $\{(R_i, \Gamma_i), \Gamma_i$: Gabriel filter of $R_i\}$, $\Gamma_0 = \{D \leqslant \pi R_i \mid D \supseteq \sum_{\oplus} D_i, D_i \in \Gamma_i\}$ is a Gabriel filter of the product ring $\pi R_i$, furthermore the ring of right quotient of $\pi R_i$ with respect to $\Gamma_0$ is isomorphic to the product of rings of right quoteint of $R_i$ with respect to $\Gamma_i$: $(\pi R_i)_{\Gamma_0} \cong \pi (R_i)_{\Gamma_i}$ This result generalizes one of Y. Utumi theorems [6]. In this paper these two sets $\Gamma_1 = \{D \leqslant \pi R_i \mid D \supseteq \pi D_i, D_i \in \Gamma_i\}$ and $\Gamma_2 = \{D \leqslant \pi R_i \mid D \supseteq \pi D_i D_i \in \Gamma_i$ and almost all $D_i = R_i\}$ will be studied. $\Gamma_1$ does not always satisfy T2. A necessary and sufficient condition for $\Gamma_1$ to be a Gabriel filter is given. It follows that $\Gamma_1$ is a better notion of products of perfect torsion theories. However $\Gamma_2$ is a Gabriel filter of $\pi R_i$, and we use this fact to prove that over an algebraically closed field, cocommutative coalgebra has a torsion rat functor if and only if each space of primitives of its irreducible components is finitedimensional.

For a coalgebra $(C, \Delta, \varepsilon)$ over a field $K$, there exists a natural algebra structure on its dual space $C^* = \mathrm{Hom}_K (C, K)$ induced by the diagonal map $\Delta$ and every left comodule $(M, \phi_M)$ over C can be defined as a right $C^*$-module by $mc^* = (c^* \otimes 1)\phi_M(m), m \in M, c^* \in C^*$. Moroever a right $C^*$-module $M$ is called a rational module if it is a left comodule $(M, \phi_M)$ over $C$ and its right $C^*$-module structure is derived in the way described above. With these observations we can embed the category of left $C$-comodules ${}^C\mathcal{M}$, as a full subcategory, into the category of right $C^*$-modules $\mathcal{M}_{C^*}$. A subspace $I$ of $C^*$ is callecd cofinite closed if $I=V^{\perp}$ for some finite-dimensional subspace $V$ of $C$.

We assume the reader is familiar with torsion theories of modules and elementary coalgebra theories. The terminology and notation are those of Stenstrom [4] and Sweedler [5].

## 2. Some properties

In this section we derive some properties of $\Gamma_1$ and $\Gamma_2$. For convenience, we write a pair $(R_i, \Gamma_i)$ as $\Gamma_i$ is a Gabriel filter of $R_i$. The following are easily proved.

**Lemma 1.** *If $I$ is a right ideal of $R$ and there exists $J\in\Gamma$ such that $(I:r)\in\Gamma$ for $r$ runs through a family of generators of $J$, then $I\in\Gamma$.*

**Lemma 2.** $\Gamma_1$, $\Gamma_2$ *satisfy* T1.

**Proposition 1.** If $\{(R_i, \Gamma_i)_{i\in I}\}$ *is a family of pairs and each $\Gamma_i$ has a cofinal family of n-generated right ideals (for a fixed integer n), then $\Gamma_1=\{D\leqslant\pi_I R_i \mid D\supseteq\pi_I D_i,\ D_i\in\Gamma_i,\ \text{all } i\in I\}$ is a Gabriel filter of $\pi_I R_i$. Moreover $(\pi R_i)_{\Gamma_1}\cong\pi(R_i)_{\Gamma_i}$.*

Proof. It only has to check T2 for $\Gamma_1$. Let $T\leqslant\pi R_i$ and $D\in\Gamma_1$ such that $(T: d)\in\Gamma_1$ for every $d\in D$. We can assume $D=\pi D_i$, $D_i\in\Gamma_i$ and each $D_i$ has $n$ generators; $x_i^1, \cdots, x_i^n$. Construct $n$ elements of $\pi D_i$ as $x^1=(x_i^1), \cdots, x^n=(x_i^n)$, then we have $(T: x^j)\in\Gamma_1$. Therefore for each $j=1, \cdots, n$, there is $\pi D_i^{(j)}$ where $D_i^{(j)}\in\Gamma_i$ such that $x^j\pi D_i^{(j)}\subset T$. Howevre for fixed $i$ the finite sum $J_i=\sum_{j=1}^{n} x_i^j D_i^{(j)}\in\Gamma_i$ by Lemma 1 and $\pi J_i=x^1\pi D_i^{(1)}+\cdots+x^n\pi D_i^{(n)}$. This shows that $\pi J_i\subset T\in\Gamma_1$.

Next we find an isomorphism from $\pi(R_i)_{\Gamma_i}$ to $(\pi R_i)_{\Gamma_1}$. Let $([f_i])\in(\pi R_i)_{\Gamma_i}$, where $f_i\in\mathrm{Hom}_{R_i}(D_i, R_i/(t_i(R_i))$ and $[f_i]$ is its equivalent class in $(R_i)_{\Gamma_i}$, and define a $\pi R_i$-homomorphism $f$ from $\pi D_i$ to $\pi R_i/t(\pi R_i)$ as $f_i((d_i))=(f_i(d_i))$. Since $t(\pi R_i)=\pi t(R_i)$, $\pi(R_i/t(R_i))\cong\pi R_i/t(\pi R_i)$ we have a well-defined map $\alpha$ from $\pi(R_i)_{\Gamma_i}$ to $(\pi R_i)_{\Gamma_1}$, as $\alpha([f_i])=[f]$, for if $f_i$ and $f_i'$ agree on $D_i$ for each $i$, then the corresponding $f$ and $f'$ agree on $\pi D_i$. It is routine to check that $\alpha$ is a one to one ring-homomorphism. Let $f: \pi D_i\to\pi R_i/t(\pi R_i)$ be a $\pi R_i$-homomorphism, $D\in\Gamma_1$ and define $f_i=\pi_i f e_i$, where $e_i$ is the ith-inclusion, $\pi_i$ is the ith-projection. Then $\alpha([f_i])=[f]$. Thus $\alpha$ is an isomorphism.

Note. (1) we agree that $n$ generators of right ideals are not necessary distinct.

(2) In proposition 1, $\Gamma_1$ also has a cofinal family of n-generated right ideals.

**Proposition** 2. *If* $\{(R_i, \Gamma_i), i\in I\}$ *is a family of pairs, then* $\Gamma_2=\{I\leqslant\pi_I R_i \mid I\supseteq\pi_I D_i, D_i\in\Gamma_i$ *and almost all* $D_i=R_i\}$ *is a Gabriel filter of* $\pi_I R_i$.

Proof. Similarly it only has to check $T_2$ for $\Gamma_2$. Let $I\leqslant\pi R_i$ and $D\in\Gamma_2$ such that $(/: d)\in\Gamma_2$for all $d\in D$. We can assume $D=\pi D_i, D_i\in\Gamma_i$ and except for $D_{i_k}, k=I, \cdots, n$, all other $D_i$ are equal to $R_i$. Let $e\in\pi D_i$be an element with $i_k$-th component$=0$, other component$=1$. It follows that there is a right ideal of the form $\pi J_i$ with $J_i\in\Gamma_i$ and almost all $Ji=Ri$ such that $I\supseteq e\pi J_i$. Also for each $d_{i_k}\in D_{i_k}$, there exists a right ideal $J_{i_k}^{(d)}\in\Gamma_{i_k}$ such that $I\supseteq e_{i_k}(d_{i_k}J_{i_k}^{(v)})$, where $e_{i_k}$ is the $i_k$ th inclusion. Now take $H_{i_k}=\sum d_{i_k}J_{i_k}^{(d)}$, the sum runs through all elements of $D_{i_k}$. We have $H_{i_k}\in\Gamma_{i_k}$and

$$(*)\ I\supseteq e\pi J_i+e_{i_1}(H_{i_1})+\cdots+e_{i_n}(H_{i_n})\,.$$

However the right side of (*) is of the form $\pi J_i$ with $J_i\in\Gamma_i$ and almost all $J_i=R_i$. Thus $I\in\Gamma_2$.

## 3. Products of perfect torsion theories

For a fixed ring $R$ with a perfect Gabriel filter $\Gamma$, we will investigate the notion of their products.

The following two theorems (Chapt. 13, [4]) motivate our definition.

**Theorem A.** *The following properties of a pair* $(R, \Gamma)$ *are equivalent*:

(1) $\mathrm{Ker}(M\to M\otimes_R R_\Gamma)=t(M)$ for all right $R$-module $M$.

(2) $\psi_R(I)R_\Gamma=R_\Gamma$for every $/\in\Gamma$.

**Theorem B.** *If* $\phi$: $A\to B$ *is a ring homomorphism. The following statements are equivalent:*

(1) $\phi$ *is an epimorphismand makes B into a flat left A-module.*

(2) *The family* $\Gamma$ *of right ideal I of A such that* $\phi(I)B=B$ *is a Gabriel filter, and there exists a ring isomorphism* $\sigma: B\to A_\Gamma$*such that* $\sigma\phi=\psi_A$.

(3) *The following two conditions are satisfied*;

(3a) *for every* $b\in B$, *there exists a finite subset* $T_n=\{(s_1,b_1), \cdots, (s_n, b_n)\}$ *of AxB such that* $b\phi(s_i)\in\phi(A)$ *aud* $\sum_i\phi(s)b_i=1$.

(3b) *if* $\phi(a)-0$, *then there exists a finite subset* $S_n=\{(s_1,b_1), \cdots, (s_n, b_n)\}$ *such that* $as_i-0$ *and* $\sum_i\phi(s_i)b_i=1$.

Note. A Gabriel filter $\Gamma$ of a ring $R$ is called perfect if it has properties listed in Theorem $A$. If $\Gamma$ is perfect, then

(1) $\Gamma$ has a cofinal family of finitely generated right ideals.

(2) $\Gamma=\{I\leqslant R\mid\psi_R(I)R_\Gamma=R_\Gamma\}$.

DEFINITION. If $\Gamma$ is a perfect Gabriel filter of $R$, for each $b\in R_\Gamma$define Ind $b=\inf|T_n|$, $T_n$ runs through all subsets of $R\times R_\Gamma$ that satisfy Theorem $B$, 3(a).

If $\psi_R(r)=0$, define Ind $r=$ Inf $|S_n|$, $S_n$ runs through all subsets of $R\times R_\Gamma$ that satisfy Theorem $B$, (3b). Then let

$$\text{Ind}\, R_\Gamma = \text{Max}\, \{ \sup_{b\in R_\Gamma} (\text{Ind}\, b),\ \sup_{\psi_R(r)=0} (\text{Ind}\, r) \} .$$

**Theorem** 3. *The following statements are* ***equivalent*** *for a perfect Gabriel filter* $\Gamma$ *of* $R$.

(1) $\Gamma$ *has a confinal family of n-generated right ideals.*

(2) $\Gamma_1=\{I\leqslant\pi R\,|\, I\supseteq\pi D_i,\ D_i\in\Gamma\}$ *is a Gabriel filter of* $\pi R$, *for any direct product of* $R$.

(3) *Ind* $R_\Gamma$ *is finite.*

Proof. (1)⇒(2). By Proposition 1.

(2)⇒(3). If $\Gamma_1$ is a Gabriel filter, then it is perfect. Suppose there is a sequence $\{b_1, b_2, \cdots, b_n, \cdots \,|\, b_n\in R_\Gamma\}$ such that Ind $b_n>$ Ind $b_{n-1}$. Consider the countable product $\pi R$ of $R$ and the element $x=(b_1, b_2, \cdots)$. Then we have $s_1, \cdots, s_t\in\pi R$, $x_1, \cdots$, tf,e(ττ,R)$_\Pi$^7rJ?$_\Gamma$ such that $x\psi(s_i)\in\psi\pi R$ and $\sum\psi(s_i)x_i=1$. Projecting to each component, Ind $b_n\leqslant t$ for each $n$. This is a contradiction. Similary we can prove that $\sup_{\psi_R(r)=0}$ {Ind $r$} is finite.

(3)⇒(1). If Ind $R_\Gamma$ is finite, then any direct product $\pi R_\Gamma$ of $R_\Gamma$ satisfies Theorem B, (3). So the product $\pi_{|\Gamma|} R_\Gamma$ is a ring of right quotient of $\pi_{|\Gamma|} R$ with respect to this perfect Gabriel filter $\Gamma=\{D\leqslant\pi_{|\Gamma|} R\,|\,\phi(D)\pi R_\Gamma=\pi R_\Gamma.\}$ Applying the well-ordering theorem to the family $\Gamma$, the right ideal $\pi_{D_i\in\Gamma} D_i$ is in $\Gamma$. So $\pi_{D_i\in\Gamma} D_i$ contains a $n$-generated right ideal $J\in\Gamma$. For each $i$ $J_i$, the $i$-th projection of $J$, is contained in $D_i$. Since $\psi_R(J_i)R_\Gamma=R_\Gamma$, $J_i\in\Gamma$. This shows that $\Gamma$ has a cofinal family of $n$-generated right ideals.

EXAMPLE. Let $Z$ be the ring of integers, $\Gamma=$ {all non-zero ideals of $Z$}, take a countable product $\pi Z$ of Z, then $\Gamma_0=\{I\leqslant\pi Z\,|\,I\supseteq\sum_\oplus D_i,\ D_i\in\Gamma\}$ is not a perfact Gabriel filter. However $\Gamma_1=\{I\leqslant\pi Z\,|\,I\supseteq\pi D_i,\ D_i\in\Gamma\}$ is perfact.

## 4. Applications to coalgebras

In this section we consider a subfunctor of the identity for the category of right C*-module $\mathcal{M}_{C^*}$ and study when this subfunctor defines a hereditary torsion theory. The main effect is to classify some types of cocommutative coalgebras. If $C$ is a coalgebra, for a right C*-module $M$ there is a unique maximal rational submodule $M^{\text{rat}}$ of $M$. Actually $M^{\text{rat}}=\{m\in M\,|\,\text{Ann}(m)$ is cofinite closed in $C^*\}$. There are some properties of $\mathcal{M}_{C^*}$.

(1) If (M, $\phi_M$) is a left $C$-comodule, $M$ can be considered as a right $C^*$-module by $mc^*=(c^*\otimes 1)\phi_M(m)$. Then $(M_{C^*})^{\text{rat}}=M$.

(2) Direct sum of rational $C^*$-modules is rational.

(3) $(C^{**})^{\mathrm{rat}}=C$.

(4) For a submodule $N$ of a C*-module $M$, $N^{\mathrm{rat}}=N\cap M^{\mathrm{rat}}$.

(5) Homomorphic image of a rational module is rational.

So we have a subfunctor rat of the identity on $\mathcal{M}_{C^*}$ just assigned each C*-module $M$ the maximal rational submodule $M^{\mathrm{rat}}$ and each homomorphism /: $M\to N$ the restriction map $\bar{f}$: $M^{\mathrm{rat}}\to N^{\mathrm{rat}}$.

DEFINITION. A coalgebra $C$ is said to have torsion rat functor if rat is a left exact radical of $\mathcal{M}_{C^*}$.

Note. If $C$ has the torsion rat functor, then

(1) the category of left $C$-comodules or equivalently of rational right $C^*$-modules is the torsion class.

(2) the corresponding Gabriel filter is

$$\Gamma = \{I \leqslant C^* \ \ I \text{ is cofinite closed in } C^*\} \ .$$

EXAMPLE. Let $V$ be an infinite dimensional vector space and $C=C(V)$ denote the connected coalgebra $K$© $V$ with

$$\Delta(v) = 1\otimes v+v\otimes 1 \qquad \forall v\in V$$
$$6(1) = 1$$
$$\varepsilon(v) = 0 \qquad \forall v\in V\,.$$

Take a basis $\{v_i|\, i\in I\}$ of $V$ and let $\{v_i^*\,|\, i\in \tilde{I}\}$ be its dual independent set in $V^*$. Extending this set to a basis $\{v_i^*\,|\, i\in I\}$ of $V^*$. We construct a linear map / from $C^*$ to $K$ as

$$\begin{cases} f(v_i^*) = 1 \text{ if } i\in I \\ f(1) = 1\,, \end{cases}$$

this element $f\notin C= C^{**\mathrm{rat}}$, however $fv^*= f(v^*)1\in C$ for any $v^*\in V^*$. So $(C^{**}/C^{**\mathrm{rat}})^{\mathrm{rat}}\neq 0$.

The following proposition is proved in [2, p. 521].

**Proposition.** *Suppose C is a coalgebra and* $0\to M'\to M\to M''\to 0$ *is an exact sequence of right* $C^*$*-modules with* $M'$ *and* $M''$ *rational. If the annihilator of each* $m''$EΞ M" *is a finitely generated right ideal, then M is rational.*

Note. From the proposition, we see that if C* is a right Noetherian, then $C$ has the torsion rat functor. In particular the universal cocommutative pointed irreducible coalgebra $B(V)$ over a finite dimensional vector space $V$ has the torsion rat functor.

**Proposition 4.** *If D is a subcoalgebra of C, then D has the torsion rat functor provided C has.*

Proof. There exists a ring epimorphism $\pi: C^* \twoheadrightarrow D^*$. Every $D^*$-module $M$ is a $C^*$-module by $mc^* = m\pi(c^*)$. Thus $(M_{D^*})^{\mathrm{rat}} = (M_{C^*})^{\mathrm{rat}}$ and $(M_{D^*}/M_{D^*}{}^{\mathrm{rat}})^{\mathrm{rat}} = (M_{C^*}/M_{C^*}{}^{\mathrm{rat}})^{\mathrm{rat}} = 0$.

**Corollary** 5. *For any pointed irreducible cocommutative coalgebra C, it has the torsion rat functor if and only if its space of primitive elements $P(C)$ is finite-dimensional.*

Proof. If *P(C)* is infinite-dimensional, the connected sub-coalgebra $D = K \oplus P(C)$ of C does not have the torsion rat functor. Conversely if *P(C)* is finite-dimensional there is an inclusion map from C to the universal cocommutative pointed irreducible coalgebra over *P(C)*. So by Proposition 4 *C* has the torsion rat functor.

**Theorem 6.** (*) *If $\{C_i \mid i \in I\}$ is a family of coalgebras and $C_i$ has the torsion rat functor for each $i \in I$. Then the direct sum $C = \sum_{\oplus} C_i$ has the torsion rat functor.*

Proof. Let $\Gamma_i = \{D_i \leqslant Cf \mid D_i$ is cofinite closed in $C_i^*\}$, and $\Gamma = \{D \leqslant C^* = \pi C_i^* D$ is cofinite closed in $C^*\}$. By proposition 2 $\Gamma_2 = \{I \leqslant \pi C_i^* \mid I \geqslant \pi D_i,\ D_i \in \Gamma_i$ and almost all $D_i = C_i^*\}$ is a Gabriel filter of $C^* = \pi C_i^*$. Hence it is sufficient to show that $\Gamma = \Gamma_2$. If $D \in \Gamma$, then $D = V^{\perp}$ for a finite dimensional subspace $V$ of $C = \sum_{\oplus} C_i$, and so $V \subseteq C_i \oplus \cdots \oplus C_{i_n}$ for some $n$.
For each $i$, let $V_i$ be the projection of $V$ to $C_i$. Then $V_i$ is a finite-dimensional subspace, almost all $V_i = 0$ and $V \subseteq \pi V_i$. Hence we have $\pi V_i^{\perp} \subseteqq V^{\perp} = D \in \Gamma_2$. Conversely suppose $I \in \Gamma_2$, since / contains a cofinite closed subspace $\pi D_i$, so / is also cofinite closed. Thus $\Gamma = \Gamma_2$.

**Corollary 7.** *Over an algebraically closed field, a cocommutative coalgebra has the torsion rat functor if and only if each space of primitives of its irreducible components is finite-dimensional.*

Proof. Over an algebrically closed field, a cocommutative coalgebra is a direct sum of its pointed irreducible components. So by Theorem 6 and Corollary 5, we have this result.

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(*) This theorem also appeared in [1], here we use the notion of products of torsion theories to give a different proof.

## References

[1] I-Peng Lin: *On homological properties of coalgebras,* Thesis, University of Illinois at Chicago Circle, 1973.
[2] D. Radford: *Coreflexive coalgebras,* J. Algebra 26 (1973), 512-535.
[3] W. Schelter: *Products of torsion theories,* Arch. Math. 22 (1971) 590-596.
[4] B. Stenstrom: Rings and Modules of Quotients, Lecture Notes in Math. 237, Berlin 1971.
[5] M. Sweedler: Hopf Algebra, Benjamin, W. A., New York, 1969.
[6] Y. Utumi: *On quotient rings,* Osaka J. Math. 8 (1956), 1-18.