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EXAMPLES OF FOLIATIONS WITH NON TRIVIAL EXOTIC CHARACTERISTIC CLASSES

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Introduction

An example of foliation of codimension one with non trivial Godbillon-Vey invariant ([3]) was constructed by R. Roussarie (see Bott [1]). Generalizing the Godbillon-Vey invariant, R. Bott [1] has defined exotic characteristic classes for foliations. In this paper, we shall construct examples of foliations with non trivial exotic characteristic classes.

Roussarie's example was constructed on a compact quotient space of $SL(2;\boldsymbol{R})$ by a discrete subgroup. This example may be regarded as an Anosov foliation arising from the geodesic flow on the unit tangent sphere bundle of a surface with constant negative curvature. This suggests us to consider such a foliation on the unit tangent sphere bundle of a closed $(q+1)$-manifold $(q\geqq 1)$ with constant negative curvature. In fact, our example is constructed as follows. Let G denote the identity component of the Lie group

$$O(q+1,\ 1)\ =\ \{X\in GL(q+2;\boldsymbol{R});\ {}^{t}XBX=B\}\ ,$$

where

$$B=\begin{pmatrix} I_{q+1} & 0 \\ 0 & -1 \end{pmatrix}.$$

Consider a compact subgroup

$$H=\left\{\begin{pmatrix} X & 0 \\ 0 & I_2 \end{pmatrix};\ X\in SO(q)\right\}$$

of G, and a closed subgroup K consisting of $X=(x_{ij})\in G$ such that

$$\det\begin{pmatrix} x_{q+1\,q+1} & x_{q+1\,q+2} \\ x_{q+2\,q+1} & x_{q+2\,q+2} \end{pmatrix}=1$$

and

$$x_{i\,q+1}+x_{i\,q+2}=0\ (i=1,\ \cdots,q)\,.$$

By a theorem of A. Borel [2], there exists a discrete subgroup D of G such that $D\backslash\bar{M}$ is a closed manifold, where $\bar{M}=G/H$. Foliate $\bar{M}$ into the fibres of the fibre bundle $\bar{M}=G/H\rightarrow G/K$, and consider the foliation on $M=D\backslash\bar{M}$ induced naturally from the G-invariant foliation on $\bar{M}$. Then it is proved that the foliation on M has non trivial exotic characteristic classes.

In §1, we review differential geometry which will be needed. In §2, we define exotic characteristic classes following R. Bott [1] and state our result precisely. §3 is devoted to construct examples of foliations with non trivial exotic characteristic classes. The proof of our result will be given in §4.

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1. Preliminaries

1.1. First, we shall fix some notations:
For a smooth manifold N, we put

$\mathfrak{X}(N)=$ {smooth vector fields on N},
$C^\infty(N)=$ {smooth real valued functions on N},
$A_C^*(N)=$ the space of complex smooth forms on N,
$A^*(N)=$ the space of (real) smooth forms on N,
$A^p(N)=\{\omega\in A^*(N);\omega$ is p-form$\}$,
$A_o^p(N)=\{\omega\in A^p(N);\omega$ has compact support$\}$,
$\Gamma(\xi)=$ the set of smooth cross-sections of a smooth vector bundle ξ over N.

For a C^∞-smooth codimension q foliation $\mathcal{F}$ on N, we denote by $\tau(\mathcal{F})$ (resp. $\nu(\mathcal{F})$) the subbundle of $\tau(N)$ tangent (resp. normal with respect to a Riemannian metric on N) to the foliation, where $\tau(N)$ denotes the tangent bundle of N.

1.2. Connections

Let N be a smooth manifold and ξ a smooth q-dimensional vector bundle over N.

(1.2.1) (1) A connection on ξ is an $\boldsymbol{R}$-bilinear map

$$\nabla:\ \mathfrak{X}(N)\times\Gamma(\xi)\rightarrow\Gamma(\xi)$$

such that

(i) $\nabla_X(fs)=X(f)s+f\nabla_X(s)$
(ii) $\nabla_{fX}(s)=f\nabla_X(s)$

for all $X\in\mathfrak{X}(N)$, $s\in\Gamma(\xi)$, $f\in C^\infty(N)$, where $\nabla_X(s)=\nabla(X,s)$.

(2) Let $S=\{s_1,\cdots,s_q\}$ be a smooth frame of ξ defined on some open set U in N. The connection form of ∇ relative to the frame S is a $q\times q$ matrix $\theta=(\theta_{ij})$ of 1-forms on U such that

$$\nabla_X(s_i)=\sum_{j=1}^{q}\theta_{ij}(X)s_j$$

for $i=1, \cdots, q$.

The curvature matrix of V associated to the frame S is a $qx\, q$ matrix $k=(k_{ij})$ of 2-forms on U such that

$$k_{ij} = d\theta_{ij} - \sum_{h=1}^{q} \theta_{ih} \wedge \theta_{hj}$$

for $i, j=1, \cdots, q$.

Let $(N, 3)$ be a C^∞-smooth codimension q foliation on N, and (,) be a Riemannian metric on $\nu(\mathcal{F})$ not necessarily induced from a Riemannian metric on N.

(1.2.2) (1) A metric connection on $\nu(\mathcal{F})$ is a connection V° on $\imath(F)$ such that

$$d(s_1, s_2)(X) = (\nabla^0_X(s_1), s_2) + (s_1, \nabla^0_X(s_2))$$

for $X \in \mathfrak{X}(N)$, $s_1, s_2 \in \Gamma(\nu(\mathcal{F}))$.

(2) A basic connection on $\nu(\mathcal{F})$ is a connection ∇^1 on $\nu(\mathcal{F})$ such that

$$\nabla^1_X(s) = \pi[X, \tilde{s}]$$

for $X \in \Gamma(\tau(\mathcal{F}))$, $s \in \Gamma(\nu(\mathcal{F}))$, where $\pi: \tau(N) \to \nu(\mathcal{F})$ is the natural projection, and $\tilde{s} \in \mathfrak{X}(N)$ is such that $\pi(\tilde{s}) = s$.

(3) Let $\nu(\mathcal{F}) \times \boldsymbol{R}$ denote the vector bundle over $N \times \boldsymbol{R}$ with the same fibre dimension as $\nu(\mathcal{F})$. Given a metric (resp. basic) connection V° (resp. ∇^1) on $\nu(\mathcal{F})$, a unique connection ∇^{01} on $\nu(\mathcal{F})$X R is defined by requiring

(i) On sections s which are constant in $\boldsymbol{R}$-direction, let $\nabla^{01}_{\partial/\partial t}(s)=0$;

(ii) If $X \in T_{(x,t)}(N \times \{t\})$, define

$$\nabla^{01}_X(s) = (1-t)\nabla^0_X(s) + t\nabla^1_X(s)$$

for $s \in \Gamma(\nu(\mathcal{F}) \times \boldsymbol{R})$.

Clearly, for a smooth frame $S=$ fo, $\cdots, s_q\}$ of $\imath(F)$ defined on some open set U in N, a smooth frame $S' = \{s'_1, \cdots, s'_q\}$ of $\nu(\mathcal{F}) \times \boldsymbol{R}$ on $U \times \boldsymbol{R}$ is defined by

$$s'_i(x, t) = (s_i(x), t), \quad i = 1, \cdots, q,$$

then connection form θ^{01} of ∇^{01} relative to the frame S' is represented as follows:

$$\theta^{01} = (1-t)\theta^0 + t\theta^1_,$$

where θ^0(resp. θ^1) is the connection form of ∇^0(resp. ∇^1) relative to the frame 5.

Let N be a Riemannian manifold, (,) be the Riemannian metric on N. The following is well known.

(1.2.3) *There exists a unique connection* V *on* $\tau(N)$ *satisfying the following conditions:*

(i) $\nabla_X(Y) - \nabla_Y(X) = [XY]$

for $X, Y \in \mathfrak{X}(N)$;

(ii) $d(Y_1, Y_2)(X) = (\nabla_X(Y_1), Y_2) + (Y_1, \nabla_X(Y_2))$

for $X, Y_1, Y_2 \in \mathfrak{X}(N)$.

This connection is called the Riemannian connection on N. Clearly, let ∇ be the Riemannian connection on N, and f be an isometry on N, then

$$f_*(\nabla_X(Y)) = \nabla_{f_*X}(f_* Y)$$

for $X, Y \in \mathfrak{X}(N)$.

1.3. Integration along the fibre

(1.3.1) (Bott [1]) *Let N be an oriented smooth manifold and π: $N \times [0, 1] \to N$ be the natural projection, then there exists a unique homomorphism of $C^\infty(N)$-modules*

$$\pi_*: A^p(N \times [0,1]) \to A^{p-1}(N), \qquad \text{for } p \geqq 1 ,$$

satisfying the equation

$$\int_{N \times [0,1]} \phi \wedge \pi^* \psi = \int_N \pi_* \phi \wedge \psi$$

for all $\phi \in A^p(N \times [0, 1])$, $\psi \in A_0^r(N)$, *where* $r = \dim N - p + 1$.

This homomorphism π_* is called integration along the fibre. Then it is easy to see the following.

(1.3.2) *Let $\bar{N}$, N be oriented smooth manifolds of dimension n, and $\bar{\pi}$: $\bar{N} \times [0, 1] \to \bar{N}$, π: $N \times [0, 1] \to N$ be the natural projections, then for any immersion f: $\bar{N} \to N$, the following diagram is commutative*:

$$\begin{array}{ccc} A^p(\bar{N} \times [0, 1]) & \xrightarrow{\bar{\pi}_*} & A^{p-1}(\bar{N}) \\ \uparrow (f \times id)^* & & \uparrow f^* \\ A^p(N \times [0,1]) & \xrightarrow{\pi_*} & A^{p-1}(N) \end{array} \qquad \text{for } p \geqq 1 .$$

2. Exotic characteristic classes and Theorem

In R. Bott [1], exotic characteristic classes for foliations have been defined as follows.

Let $q \geqq 1$ be an integer.

First, a cochain complex (WO_q, d) is defined. Let $R[c_1, \cdots, c_q]$ denote the graded polynomial aglebra over R generated by the elements c_i with degree $2i$. Set

$$R_q[c_1, \cdots, c_q] = R[c_1, \cdots, c_q] / \{\phi;\ \deg(\phi) > 2q\} .$$

Let $E(h_1, h_3, \cdots, h_r)$ denote the exterior algebra over R generated by the elements

h_i with degree $2i-1$, where r is the largest odd integer $\leqq q$. Then as a graded algebra over R

$$WO_q = \boldsymbol{R}_q[c_1, \cdots, c_q] \otimes E(h_1, h_3, \cdots, h_r),$$

and a unique antiderivation of degree 1

$$d: WO_q \to WO_q$$

is defined by requiring

$$\begin{aligned} d(c_i) &= 0, \qquad i = 1, \cdots, q \\ d(h_i) &= c_i, \qquad i = 1, 3, \cdots, r. \end{aligned}$$

Let £ be a smooth q-dimensional vector bundle over a manifold N and V a connection on ξ. For a curvature matrix k of V, local $2i$-forms $c_i(k)$ on N are defined by the following formula

$$\det(I_q + tk) = 1 + \sum_{i=1}^{q} t^i c_i(k).$$

Since $c_i(k)$ do not depend on the choice of the local frame of ξ, $c_i(k)$ define global $2i$-forms on N. Then a homomorphism of graded $\boldsymbol{R}$-algebras

$$\lambda(\nabla): \boldsymbol{R}[c_1, \cdots, c_q] \to A_C^*(N)$$

is defined by requiring

$$\lambda(\nabla)(c_i) = (\sqrt{-1}/2\pi)^i c_i(k), \qquad \text{for } i = 1, \cdots, q.$$

Let N be an oriented smooth manifold without boundary and $(N, \mathcal{F})$ a C^∞-smooth codimension q foliation on N. Let V° (resp. ∇^1) be a metric (resp. basic) connection on $\nu(\mathcal{F})$ and ∇^{01} be as in (1.2.2) (3). Then the followings hold.

(2.1) (1) $\lambda(\nabla^1)(\phi) \in A_C^*(N)$ *is a closed form for any* $\phi \in \boldsymbol{R}[c_1, \cdots, c_q]$, *and if* $\deg(\phi) > 2q$ *then* $\lambda(\nabla^1)(\phi) = 0$.

(2) $\lambda(\nabla^\circ)(\phi) = 0$ *for* $\phi \in \boldsymbol{R}[c_1, \cdots, c_q]$ *such that* $\deg(\phi)/2$ *is an odd integer.*

(3) *Let* $\pi: N X [0, 1] \to N$ *be the natural projection and* $i: N X [0, 1] \to N X \boldsymbol{R}$ *the inclusion mapping, then*

$$d(\pi_* i^* \lambda(\nabla^{01})(\phi)) = \lambda(\nabla^1)(\phi) - \lambda(\nabla^0)(\phi)$$

for $\phi \in \boldsymbol{R}[c_1, \cdots, c_q]$, *especially*

$$d(\pi_* i^* \lambda(\nabla^{01})(c_{2j-1})) = \lambda(\nabla^1)(c_{2j-1}),$$

where π_* *is the integration along the fibre.*

In view of (2.1), given a C^∞-smooth codimension q foliation $(N, \mathcal{F})$ on an oriented smooth manifold N without boundary, a homomorphism of cohain

complexes

$$\lambda_{(N,\mathcal{F})}: WO_q \to A_C^*(N)$$

is defined by requiring

$$\lambda_{(N,\mathcal{F})}(c_i) = \lambda(\nabla^1)(c_i),$$
$$\lambda_{(N,\mathcal{F})}(h_j) = \pi_* i^* \lambda(\nabla^{01})(c_j).$$

We used the notation $\lambda_{(N,}$30 in place of λ_E of Bott [1]. Here the homomorphism $\lambda_{(N,}$ 30 depends only on the choics of two connections V° and ∇^1 on $\nu(\mathcal{F})$. In cohomology, $\lambda_{(N,\mathcal{F})}$ induces a homomorphism of graded $\boldsymbol{R}$-algebras

$$\lambda^*_{(N,\mathcal{F})}: H^*(WO_q) \to H^*(N; C)$$

which depends only on the foliation *(N, F).*

The elements of $\lambda^*_{(N,\mathcal{F})}(H^*(WO_q))$ are called the exotic characteristic classes for the foliation $(N, \mathcal{F})$.

It is easy to see the foillowing lemma.

Lemma *2.2. Each canonical generator of $H^{2q+1}(WO_q)$ is represented by some $\phi\ h_j \in WO_q$, where $\phi \in \boldsymbol{R}_q[c_1, \cdots, c_q]$ is a monomial with degree $2(q-j+1)$.*

Then we have

Theorem. *For any integer $q \geqq 1$, there exists a C^∞-smooth codimensíon q foliation $(M, \mathcal{F})$ on a closed $(2q+1)$-manifolduch that all the exotic characteristic classes for the foliation which correspond to the canonical generators $[\phi\ h_j]$ of $H^{2q+1}(WO_q)$are non zero in $H^{2q+1}(M; C)$.*

REMARK. When $q=1$, the generator fo-AJ of $H^3(WO_1) \simeq \boldsymbol{R}$s the Godbillon-Vey invariant, and our foliation of codimension one is diffeomorphic to the foliation constructed by R. Roussarie (cf. [1]).

3. Construction of the foliation (M, $\mathcal{F}$)

Throughout this paper, integer $q \geqq 1$ is to be fixed, and all foliations are to be C^∞-smooth codimension q foliations. Let

$$O(q+1,\ 1) = \{X \in GL(q+2\boldsymbol{R});\ {}^tXBX = B\}, \text{where } B = \begin{pmatrix} ff+1 & \\ & \end{pmatrix}.$$

We can define subgroups $H \subset K \subset G$ of $O(q+1,\ 1)$ as follows:

(3.1) *Let G be the identity component of $O(q+1,\ 1)$. Then $H = \left\{\begin{pmatrix} X & 0 \\ 0 & I_2 \end{pmatrix}; X \in SO(q)\right\}$ is a compact subgroup of* G, *and G/H is an open $(2q+1)$-maifold.*

(3.2) *Let K be a subspace of G consisting of $X=(x_{ij}) \in G$such that*

$$\det\begin{pmatrix} x_{q+1\,q+1} & x_{q+1\,q+2} \\ x_{q+2\,q+1} & x_{q+2\,q+2} \end{pmatrix} = 1$$

and

$$x_{i\,q+1}+x_{i\,q+2}=0 \quad (i=1, \cdots, q)$$

then K is a subgroup of G, *and G/K is a q-maniford.*

Proof. The proof of (3.1) is trivial. We shall prove (3.2). Let $X=(x_{ij})\in GL(q+2; \boldsymbol{R})$ such that

$$x_{i\,q+1}+x_{i\,q+2}=0, \quad \text{for} \quad i=1, \cdots, q\,.$$

If $X\in G\subset O(q+1, 1)$, then the followings hold.

1) $\det\begin{pmatrix} x_{q+1\,q+1} & x_{q+1\,q+2} \\ x_{q+2\,q+1} & x_{q+2\,q+2} \end{pmatrix} = \pm 1\,,$

and $\det\begin{pmatrix} x_{q+1\,q+1} & x_{q+1\,q+2} \\ x_{q+2\,q+1} & x_{q+2\,q+2} \end{pmatrix} = 1$, if and only if

$$x_{q+1\,q+1}+x_{q+1\,q+2}=x_{q+2\,q+1}+x_{q+2\,q+2};$$

2) $x_{q+1\,i}-x_{q+2\,i}=0, \quad \text{for} \quad i=1, \cdots, q\,.$

If the above equality holds, (3.2) folows from 1) and 2) (q.e.d.)

Set $\bar{M}=G/H$, then $\bar{M}$ is an open $(2q+1)$-manifold and $\bar{M}$ is foliated into the fibres of the fibre bundle $\bar{M}=G/H\rightarrow G/K$. We denote this foliation by $(\bar{M}, \bar{\mathscr{F}})$. Clearly, the foliation $(\bar{M}, \bar{\mathscr{F}})$ is a G-invariant foliation of codimension q on $\bar{M}$.

By A. Borel [2], the connected semi-simple Lie subgroup G of $GL(q+2; C)$ has the discrete subgroup Γ of G which contains a normal torsion free subgroup D of finite index. Since H is compact subgroup of G, the subgroup D acts freely on $\bar{M}=G/H$ and $D\backslash\bar{M}$ is compact. Therefore we have.

(3.3) *There exists a discrete subgroup D of G such that $D\backslash\bar{M}$ is a closed $(2q+1)$-manifold.*

Set $M=D\backslash\bar{M}$. Since the foliation $(\bar{M}, \bar{\mathscr{F}})$ is G-invariant, the closed $(2q+1)$-manifold M has a codimension q foliation $(M, \mathscr{F})$ induced naturally from $(\bar{M}, \bar{\mathscr{F}})$. This foliation $(M, \mathscr{F})$ is the example of foliation with non trivial exotic characteristic classes.

4. Proof of Theorem

4.1. Naturality of the homomorphism $\lambda(N, \mathscr{F})$

Let N be an oriented manifold without boundary and (,) be a Riemannian metric on N. Let ∇ be the Riemannian connection on N. Given a foliation $(N, \mathscr{F})$ a metric connection ∇^{0} and a basic connection ∇^{1} on $\nu(\mathscr{F})$ are defined as follows:

$$\nabla^0_X(Y) = \pi\nabla_X(Y)$$
$$\nabla^1_X(Y) = \pi[X_{\tau(\mathcal{F})},\ Y] + \nabla^0_{X_{\nu(\mathcal{F})}}(Y)$$

for any $X \in \mathfrak{X}(N)$, $Y \in \Gamma(\nu(\mathcal{F}))$, where $\pi\colon \tau(N) \to \nu(\mathcal{F})$ is the natural projection, and $X_{\tau(\mathcal{F})} \in \Gamma(\tau(\mathcal{F}))X_{\nu(\mathcal{F})} \in \Gamma(\nu(\mathcal{F}))$ are such that $X = X_{\tau(\mathcal{F})} + X_{\nu(\mathcal{F})}$ Here, of course, we consider the Riemannian metric on $\nu(\mathcal{F})$ induced naturally from the Riemannian metric (,) on $\tau(N)$.

Then the homomorphism of cochin complexes

$$\lambda_{(N,\mathcal{F})}\colon \quad WO_q \to A^*_C(N)$$

is uniquely determined from the above connections ∇° and ∇^1, hence from the foliation $(N, \mathcal{F})$ and the Riemannian metric (,). Thus we denote this $\lambda(N, 30\,(\omega)$ by $\omega((N, \mathcal{F}), (\ ,\ ))$ for $\omega \in WO_q$.

Now, let $O(q+1, 1)$, G, H, K, $(\bar{M}, \bar{\mathcal{F}})$ and $(M$, ff) be as in Section 3. Let $\mathfrak{o}(q+1, 1)$ (resp. $\mathfrak{gl}_{q+2}$) denote the Lie algebra of $O(q+1, 1)$ (resp. $GL(q+2; \boldsymbol{R})$), then clearly

$$\mathfrak{o}(q+1, 1) = \{X \in \mathfrak{gl}_{q+2};\ {}^tXB + BX = 0\}\ ,$$

and a basis of $\mathfrak{o}(q+1, 1)$ is given by the following elements:

$$Z = \left(\begin{array}{c|c} O & O \\ \hline O & \begin{matrix} 0 & -2 \\ -2 & 0 \end{matrix} \end{array}\right)$$

$$\begin{array}{cl} H_{ij} = \left(\begin{matrix} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & -1 & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{matrix}\right) & \begin{matrix} \\ < i \\ \\ < j \\ \\ \end{matrix} \\ \qquad\qquad\ \ \hat{i} \qquad\ \ \hat{j} & \end{array}, \qquad 1 \leqq i < j \leqq q\,,$$

$$\begin{array}{cl} X_i = \left(\begin{array}{c|cc} O & 1 & -1 \\ \hline -1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right) & \begin{matrix} < i \\ \\ \\ \end{matrix} \\ \qquad\quad\ \hat{i} & \end{array}, \qquad i = 1, \cdots, q\,,$$

$$\begin{array}{cl} Y_i = \left(\begin{array}{c|cc} O & 1 & 1 \\ \hline -1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right) & \begin{matrix} < i \\ \\ \\ \end{matrix} \\ \qquad\quad\ \hat{i} & \end{array}, \qquad i = 1, \cdots, q\,.$$

It is known that $\{H_{ij}\}_{1\leqq i<j\leqq q}$ is a basis of the Lie algebra of H. Then $T_0(\bar{M})$ is identified naturally with the subspace of $T_e(G)$ spanned by $\{X_1, \cdots, X_q\ Y_1, \cdots, Y_q, Z\}$, where $o=H\in G/H=\bar{M}$.

In this time, we have

$$(Ad(g)(X_1), \cdots, Ad(g)(X_q), Ad(g)(Y_1), \ldots, Ad(g)(Y_q), Ad(g)(Z))$$
$$= (X_1, \cdots, X_q, Y_1, \cdots, Y_q, Z)\cdot\begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $g=\begin{pmatrix} A & 0 \\ 0 & I_2 \end{pmatrix}\in H$ $(A\in SO(q))$. Therefore, $\overline{\langle\ ,\ \rangle}_o=\sum_{i=1}^{q} X_i^*\otimes X_i^*+\sum_{i=1}^{q} Y_i^*\otimes Y_i^*+Z^*\otimes Z^*$ is an $Ad(H)$-invariant innerproduct on $T_o(\bar{M})$ where $\{X_1^*, \cdots, X_q^*, Y_1^*, \cdots, Y_q^*, Z^*\}$ denote the dual basis of $\{X_1, \cdots, X_q, Y_1, \cdots, Y_q, Z\}$. Hence, for any $u=g\,o\in\bar{M}(g\in G)$, an innerproduct $\overline{\langle\ ,\ \rangle}_u$ on $T_u(\bar{M})$ is defined by $\overline{\langle\ ,\ \rangle}_u=(g^{-1})^*\overline{\langle\ ,\ \rangle}_o$. Therefore we have

(4.1.1) $\overline{\langle\ ,\ \rangle}$ *is a G-invariant* ***Riemannian metric*** *on* $\bar{M}$ *and* $\bar{M}$ *is* ***orientable.***

Then we have the followings.

Lemma 4.1.2. *For any* $\omega\in WO_q, \omega((\bar{M}, \bar{\mathcal{F}}), \overline{\langle\ ,\ \rangle})$ *is a G-invariant differential form on* $\bar{M}$.

Lemma 4.1.3. *Let* $\langle\ ,\ \rangle$ *denots the* ***Riemannian metric*** *on M* ***induced*** *naturally from the* ***Riemannian*** *metric* $\overline{\langle\ ,\ \rangle}$ *on* $\bar{M}$*, then*

$$p^*\omega((M, \mathcal{F}), \langle\ ,\ \rangle) = \omega((\bar{M}, \bar{\mathcal{F}}), \overline{\langle\ ,\ \rangle})$$

for $\omega\in WO_q$*, where* p: $\bar{M}\to M$ *is the natural projection.*

Proof of (4.1.2). Let $\bar{\nabla}$ be the Riemannian connection on the Riemannian manifold $(\bar{M}, \overline{\langle\ ,\ \rangle})$. Since the Riemannian metric $\overline{\langle\ ,\ \rangle}$ on $\bar{M}$ is G-invariant,

$$g_*(\bar{\nabla}_X(Y)) = \bar{\nabla}_{g_*X}(g_*Y)$$

for any $g\in G$ and X, $Y\in\mathfrak{X}(\bar{M})$.

Since the foliation $(\bar{M}, \bar{\mathcal{F}})$ is G-invariant, g_* maps $\Gamma(\tau(\bar{\mathcal{F}}))$ (resp. $\Gamma(\nu(\bar{\mathcal{F}}))$) into $\Gamma(\tau(\bar{\mathcal{F}}))$ (resp. $\Gamma(\nu(\bar{\mathcal{F}}))$) and $g_*\pi=\pi g_*$ for $g\in G$, where π: $\tau(\bar{M})\to\nu(\bar{\mathcal{F}})$ is the natural projection.

Therefore we have

$$(*)\qquad \begin{aligned} g_*(\nabla^0_X(Y)) &= \nabla^0_{g_*X}(g_*Y), \\ g_*(\nabla^1_X(Y)) &= \nabla^1_{g_*X}(g_*Y), \qquad \text{for } g\in G. \end{aligned}$$

Let k^1(resp. k^0) be the curvature matrix of ∇^1(resp. ∇°) associated to some local frame $S=\{s_1, \cdots, s_q\}$ of $\nu(\bar{\mathcal{F}})$. Then by (*), $(g^{-1})^*k^1$(resp. $(g^{-1})^*k^0$) is the curvature matrix of ∇^1(resp. ∇°) associated to the local frame $g_*S=\{g_*s_1, \cdots, g_*s_q\}$. But $c_i(k^1)\in A^{2i}(\bar{M})$is independent of the choices of local frames, hence $c_i(k^1)$is G-invariant. Therefore

$$c_i((\bar{M}, \bar{\mathcal{F}}), \overline{\langle\ ,\ \rangle}) = \lambda(\nabla^1)(c_i) = (\sqrt{-1}/2\pi)^i c_i(k^1)$$

is also G-invariant.

Similarly, $\lambda(\nabla^{01})(c_i)\in A_C^{2i}(\bar{M}\times \boldsymbol{R})$is G-invariant. Hence it follows from (1.3.2) that $h_i((M, \mathcal{F}), \overline{\langle\ ,\ \rangle})$ is G-invariant. (q.e.d)

Proof of (4.1.3). Let ∇ be the Riemannian connection on $(M, \langle\ ,\ \rangle)$. Since the natural projection $p\colon (\bar{M}, \overline{\langle\ ,\ \rangle})\to(M, \langle\ ,\ \rangle)$ is a local isometry, locally we have

$$p_*(\bar{\nabla}_X(Y)) = \nabla_{p_*X}(p_*Y)\,.$$

Therefore, the proof is similar to that of (4.1.2). (q.e.d.)

4.2. Local frame of $\nu(\bar{\mathcal{F}})$

Let $o=H\in\bar{M}=G/H$. To calculate the connection forms, we define local vector fields around $o\in\bar{M}$ as follows:

Define a prametrization ϕ around $o\in\bar{M}$by

$$\begin{aligned}&\bar{\phi}(y_1, \cdots, y_q, x_1, \cdots, x_q, z)\\ &= \exp(\sum_{i=1}^q y_i Y_i)\exp(\sum_{i=1}^q x_i X)\exp(z\cdot Z)H\in\bar{M} = G/H\,.\end{aligned}$$

In the sequel, we use the vector notations such as $x=(x_1, \cdots, x_q), y=(y_1, \cdots, y_q)$

Set local vector fields $\bar{Z}, \bar{X}_1, \cdots, \bar{X}_q, \bar{Y}_1, \cdots, \bar{Y}_q$, around $o\in\bar{M}$, $\bar{Z}=\bar{\phi}_*(\partial/\partial z)$,

$$\bar{X}_i = \bar{\phi}_*(e^{-2z}\partial/\partial x_i), \qquad i = 1, \cdots, q\,,$$

$$\bar{Y}_j = \bar{\phi}_*(e^{2z}(\partial/\partial y_j+\sum_{k=1}^q x_k^2\partial/\partial x_j-2x_j\sum_{k=1}^q x_k\partial/\partial x_k+x_j\partial/\partial z)), \qquad j = 1, \cdots, q\,,$$

at $u=\bar{\phi}(y, x, z)\in\bar{M}$.

Then we have

Lemma 4.2.

(1) $\bar{X}_1, \cdots, \bar{X}_q, \bar{Z}$ *are tangent to the foliation* $(\bar{M}, \bar{\mathcal{F}})$.

(2) $\{\bar{X}_1, \cdots, X_q, Z, \bar{Y}_1, \cdots, \bar{Y}_q\}$ *is a local orthonormal frame of* $\tau(\bar{M})$ *with respect to the Riemannian metric* $\overline{\langle\ ,\ \rangle}$.

(3) *(bracket relations)*

$$[\bar{X}_i, \bar{Z}] = 2\bar{X}_i \qquad [\bar{X}_i, \bar{X}_j] = 0,$$
$$[\bar{Y}_j, \bar{Z}] = -2\bar{Y}_j,$$
$$[\bar{Y}_i, \bar{Y}_j]_u = 2e^{2z}(x_i(\bar{Y}_j)_u - x_j(\bar{Y}_i)_u)$$
$$[\bar{X}_i, \bar{Y}_j]_u = \begin{cases} (\bar{Z})_u - 2e^{2z} \sum\limits_{\substack{k=1 \\ (k\neq i)}}^{q} x_k(\bar{X}_k)_u, & i=j \\ 2e^{2z} x_i(\bar{X}_j)_u, & i \neq j, \end{cases}$$

where $u = \bar{\phi}(y, x, z) \in \bar{M}$, *and* $i, j = 1, \cdots, q$.

Proof. The bracket relations of (3) are calculated directly by the definitions of Z, X_i, $\bar{Y}_j$.

We shall prove (1) and (2).

First, we define a local parametrization ϕ around $e \in G$ and a local section σ around $o \in \bar{M}$ as follows:

Set

$$\phi(y, x, z, (h_{ij})_{1\leq i<j\leq q}) = \exp(\sum_{i=1}^{q} y_i Y_i)\exp(\sum_{i=1}^{q} x_i X_i)\exp(z\cdot Z)\exp(\sum_{1\leq i<j\leq q} h_{ij}H_{ij}) \in G.$$

The local section σ is defined by requiring

$$\sigma\bar{\phi}(y, x, z) = \phi(y, x, z, (0)).$$

Then the next (4.2.1) and (4.2.2) follows from tedious calculations, which will be left to the reader.

(4.2.1) (1) *For* $g_1 = \exp(\sum_{i=1}^{q} y_i^0 Y_i)$,

$$L_{g_1}(\phi(y, x, z, (h_{ij})) = \phi(y+y^0, x, z, (h_{ij}))$$

(2) *For* $g_2 = \exp(\sum_{i=1}^{q} x_i^0 X_i)$,

$$R_{g_2}(\phi(y, x, z, (h_{ij})) = \phi(y, \bar{x}, z, (h_{ij})).$$

where $\bar{x} = x + e^{-2z} x^0\ {}^t(\exp(\sum h_{ij}H_{ij}))$.

(3) $\qquad = \exp(z^0 Z + \sum_{1\leq i<j\leq q} h_{ij}^0 H_{ij})$

$$R_{g_3}(\phi(y, x, z, (h_{ij})) = \phi(y, x, z+z^0, (h_{ij}+h_{ij}^0)).$$

***Here**, of course, L_g(**resp.** R_g) denote the left (resp. right) translation by $g \in G$.*

(4.2.2) *Let* $g = \phi(y, x, z, (h_{ij})) \in G$, *and* g_1, g_2, g_3 *be as in* (4.2.1), *and let* $X_i' = \phi_*(\partial/\partial x_i)$, $Y_i' = \phi_*(\partial/\partial y_i)$, $Z' = \phi_*(\partial/\partial z)$. *Then,*

(1) $(L_{g_1})_*((X_i')_g) = (X_i')_{g_1 g}$

$(L_{g_1})_*((Y_j')_g) = (Y_i')_{g_1 g}$

$(L_{g_1})_*(Z_g') = Z_{g_1 g}'$

for $i=1, \cdots, q$;

(2) $(R_{g_2})_*((X_i')_g)=(X_i')_{gg_2}$

$(R_{g_2})_*((Y_i')_g)=(Y_i')_{gg_2}$

for $i=1, \cdots, q$;

$$(R_{g_2})_*(Z_g') = Z_{gg_2}'-2e^{-2z}\sum_{i,j=1}^{q} a_{ij}x_j^0(X_i')_{gg_2}$$

where a_{ij} *is the* (i, j) *compaonent of* $\exp(\sum h_{ij}H_{ij})$;

(3) $(R_{g_3})_*((X_j')_g)=(X_i')_{gg_3}$

$(R_{g_3})_*((Y_i')_g)=(Y_i')_{gg_3}$

$(R_{g_3})_*(Z_g')=Z_{gg_3}'$

for $i=1, \cdots, q$.

Then we have the following key lemma for the proof of Lemma 4.2.

(4.2.3) *The following* (a), (b), (c) *hold at* $g=\phi(y, x, z, (0))\in G$.

(a) $Z_g'=(L_g)_*Z$.

(b) $(X_j')_g=e^{2z}(L_g)_*X_i, j=1, \cdots, q$.

(c) $(Y_i')_g=(L_g)_*(e^{-2z}Y_j+2e^{2z}x_j\sum_{k=1}^{q}x_kX_k-e^{2z}\sum_{k=1}^{q}x_k^2X_j-x_jZ+\sum_{\substack{k=1\\(k\neq j)}}^{q}2x_kH_{kj})$,

for $j=1, \cdots, q$.

Here, of course, the elements $X_1, \cdots, X_q, Y_1, \cdots, Y_q, Z, H_{ij}$ *of* $\mathfrak{o}(q+1, 1)$ *are regarded as the elements of* $T_e(G)$, *and* $H_{ij}=-H_{ji}$ *foi* $>j$.

Proof. First, notice that

(*) $(X_i')_e=X_i$, for $i=1, \cdots, q$,

$(Y_i')_e=Y_i$, for $i=1, \cdots, q$,

$Z_e'=Z$.

For $g=(y, x, z, (0))$, set $g_1=\exp(\sum_{i=1}^{q}y_iY_i)$, $g_2=\exp(\sum_{i=1}^{q}x_iX_i)$, $g_3=\exp(z\cdot Z)$.

Then $g=g_1\cdot g_2\cdot g_3$.

We shall prove *(a)*. By (4.2.2),

$$\begin{aligned} Z_g' &= (L_{g_1})_*(Z_{g_2g_3}') \\ &= (L_{g_1})_*(R_{g_3})_*Z_{g_2}' \\ &= (L_{g_1})_*(R_{g_3})_*(R_{g_2})_*(Z_e'+2\sum_{k=1}^{q}x_k(X_k')_e)\,. \end{aligned}$$

Then, by (*),

$$\begin{aligned} Z_g' &= (L_{g_1})_*(R_{g_3})_*(R_{g_2})_*(Z+2\sum_{k=1}^{q}x_kX_k) \\ &= (L_g)_*(L_{(g_2g_3)^{-1}})_*(R_{g_2g_3})_*(Z+2\sum_{k=1}^{q}x_kX_k) \\ &= (L_g)_*Ad((g_2g_3)^{-1})\,(Z+2\sum_{k=1}^{q}x_kX_k) \\ &= (L_g)_*Ad(g_3^{-1})Ad(g_2^{-1})\,(Z+2\sum_{k=1}^{q}x_kX_k)\,. \end{aligned}$$

But, $Ad(g_2^{-1})(Z)=Z-2\sum_{k=1}^{q} x_k X_k$,

$Ad(g_2^{-1})(X_k)=X_k$, for $k=1, \cdots, q$,

and $Ad(g_3^{-1})(Z)=Z$.

Therfore, $Z'_g=(L_g)_*Z$.

(b) and (c) are proved similarly. (q.e.d.)

Now, let $\bar{P}$: $G\to\bar{M}=G/H$ be the natural projection, then clearly we have

$$\begin{aligned} \bar{P}_*(X_i) &= (\bar{X}_i)_o \qquad && i=1, \cdots, q\,, \\ \bar{P}_*(Y_j) &= (\bar{Y}_j)_o && j=1, \cdots, q\,, \\ \bar{P}_*(Z) &= (\bar{Z})_o && \\ \bar{P}_*(H_{ij}) &= 0, && 1\leqq i<j\leqq q\,, \end{aligned}$$

where $o=H\in\bar{M}=G/H$.

Hence, by the definition of the G-invariant Riemannian metric $\overline{\langle\ ,\ \rangle}$ on $\bar{M}$, the following lemma shows Lemma 4.2 (2).

(4.2.4) *Let σ be the local section defined as before, then*

(a) $(\bar{Z})_u=\sigma(u)_*(\bar{Z})_o$,

(b) $(\bar{X}_i)_u=\sigma(u)_*(\bar{X}_i)_o, i=1, \cdots, q$,

(c) $(\bar{Y}_j)_u=\sigma(u)_*(\bar{Y}_j)_o, j=1, \cdots, q$,

for any point u of some neighborhood of o in $\bar{M}$.

Proof. Clearly, $\bar{P}_*(L_{\sigma(u)})_*=\sigma(u)_*\bar{P}_*$, and $(\phi_*)_{\sigma(u)}=\sigma_*\circ(\bar{\phi}_*)_u$. Then, in view of the definitions of $\bar{X}_i$, $\bar{Y}_j$, $\bar{Z}$, we have (4.2.4) easily from (4.2.3).

(q.e.d.)

By the definition of Lie algebras of Lie groups, we have the following easily.

(4.2.5) *The following elements $\mathfrak{o}(q+1,\ 1)$ form a basis of the Lie algebra of K,*

$$Z,\ X_1, \cdots, X_q,\ H_{ij},\ 1\leqq i<j\leqq q\,,$$

where the subgroup K of $O(q+1,\ 1)$ is as in (3.2).

Since the foliation $(\bar{M}, \bar{\mathcal{F}})$ is G-invariant and each leaf of this foliation is a fibre of the fibre bundle $\bar{M}=G/H\to G/K$, we have Lemma 4.2 (1) from (4.2.4) and (4.2.5).

This completes the proof of Lemma 4.2.

REMARK. Consider Z, X_i, Y_j of $\mathfrak{o}(q+1,1)$ as left-invariant vector fields on G. It hold that $\bar{Z}=\bar{P}_*(Z)$ and $\bar{Z}$ is G-invariant, then we may define $\bar{Z}$ by $\bar{P}_*(Z)$. However it is impossible to define X_i (resp. $\bar{Y}_j$) by P_*X_i (resp. $\bar{P}_*Y_j$), for X_i, Y_j are not $Ad(H)$-invariant.

4.3. Calculation of $c_i((\bar{M}, \bar{\mathcal{F}}), \overline{\langle\ ,\ \rangle})$ and $h_j((\bar{M}, \bar{\mathcal{F}}), \overline{\langle\ ,\ \rangle})$.

Let Z^* (resp. $\bar{X}_i^*$, $\bar{Y}_j^*$) denote the dual one form of Z (resp. X_i, $\bar{Y}_j$) with respect to the G-invariant Riemannian metric $\overline{\langle\ ,\ \rangle}$ on $\bar{M}$.

Then we have

Lemma 4.3. *At* $o=H\in\bar{M}=G/H$,

(1) $c_i((M,\mathrm{ff}),\overline{\langle\ ,\ \rangle})=\alpha_i(\sqrt{-1}/2\pi)^i(d\bar{Z}^*)^i,\ \alpha_i>0,$
for $i=1,\ \cdots,\ q$;

(2) $h_j((M,\mathcal{F}),\overline{\langle\ ,\ \rangle})=\beta_j(\sqrt{-1}/2\pi)^j\bar{Z}^*\wedge(d\bar{Z}^*)^{j-1},\ \beta_j<0,$
for $j=1,3,\ \cdots,r$.

We shall prove Lemma 4.3. As usual, dx_i and rfy$_z$ are regarded as local 1-forms on $\bar{M}$ *by* the parametrization $\bar{\phi}$. It is easy to see the following.

(4.3.1) *Let* Z^*, $\bar{X}_i^*,\bar{Y}_j^*$ *be as above, then*

(1) $\bar{Y}_j^*=e^{-2z}dy_j$ *for* $j=1,\ \cdots,\ q$, *at* $u=\bar{\phi}(y_1,\ \cdots,\ y_q,\ x_1,\ \cdots,\ x_q,\ z)\in\bar{M}$;

(2) $d\bar{Z}=\sum_{i=1}^{q}dy_i\wedge dx_i=\sum_{i=1}^{q}\bar{Y}_i^*\wedge\bar{X}_i^*$.

Since the Riemannian metric $\overline{\langle\ ,\ \rangle}$ on $\bar{M}$ is given, connections V° and ∇^1 on $\nu(\mathcal{F})$ are uniquely defined as in Section 4.1. Then $\{\bar{Y}_1,\ \cdots,\bar{Y}_q\}$ is a local orthonormal frame of $\nu(\mathcal{F})$ by Lemma 4.2. Let $\theta^0=(\theta_{ij}^0)$ (resp. $\theta^1=(\theta_{ij}^1)$) be the connection form of ∇^0(resp. ∇^1) relative to the frame $\{\bar{Y}_1,\ \cdots,\ \bar{Y}_q\}$, then we have

(4.3.2) *At* $u=\phi(y_1,\ \cdots,y_q,x_1,\ \cdots,\ x_q,\ z)\in\bar{M}$

(1) $\theta_{ij}^0=\begin{cases}0 & i=j\\ 2e^{2z}(x_j\bar{Y}_i^*-x_i\bar{Y}_j^*), & i\neq j\end{cases}$

(2) $\theta_{ij}^1=\begin{cases}2\bar{Z}^*, & i=j\\ \theta_{ij}^0, & i\neq j.\end{cases}$

Proof. Let V be the Riemannian connection on the Riemannian manifold $(\bar{M},\overline{\langle\ ,\ \rangle})$ and $\theta=(\theta_{ij})$ be the connection form of V relative to the frame $\{\bar{Y}_1,\ \cdots,\ \bar{Y}_q,\ \bar{X}_1,\ \cdots,\ \bar{X}_q,\ \bar{Z}\}$ of $\tau(\bar{M})$.

Set $s_1=\bar{Y}_1,\ \cdots,\ s_q=\bar{Y}_q,\ s_{q+1}=\bar{X}_1,\cdots,\ s_{2q}=\bar{X}_q,\ s_{2q+1}=\bar{Z}$. Then by the definition of V,

$$d\overline{\langle s_i,\ s_j\rangle}(X)=\overline{\langle\nabla_X s_i,\ s_j\rangle}+\overline{\langle s_i,\ \nabla_X s_j\rangle}$$

for $X\in\mathfrak{X}(\bar{M})$. Hence

(i) $\theta_{ij}=-\theta_{ij}$, for $i,j=1,\ \cdots,\ 2q+1$.

Moreover $\nabla_{s_i}(s_j)-\nabla_{s_j}(s_i)=[s_i,\ s_j]$, then we have the followings (ii) (iii) from Lemma 4.2 (3).

For $i,j=1,\ \cdots,\ q$ and $i\neq j$.

(ii) $\theta_{ij}(s_j)=-2e^{2z}x_i,\ \theta_{ij}(s_i)=2e^{2z}x_j$.

(iii) $\theta_{jk}(s_i)=\theta_{ik}(s_j)$, for $k=1,\cdots,2q+1$ and $k\neq i,j$.

$\theta_{q+j\ k}(s_i)=\theta_{ik}(s_{q+j})$, for $k=1, \cdots, q$.

$\theta_{2q+1\ k}(s_i)=\theta_{ik}(s_{2q+1})$ for $k=1, \cdots, 2q+1$,

and $k \neq i$, $2q+l$.

Let $i, j=1, \cdots, q$. Now, let $k=1, \cdots, q$, and $k \neq i, j$, then by (i) and (iii),

$$\theta_{ij}(s_k)= -\theta_{ji}(s_k)= -\mathrm{MO} = \mathrm{MO} = \theta_{jk}(s_i)= -\theta_{kj}(s_i) = -\theta_{ij}(s_k)$$

Hence $\theta_{ij}(s_k)=0$. It is shown similarly by making use of (i) and (iii) that $\theta_{ij}(s_k)=0$ for $k=q+1, \cdots, 2q+1$. Therefore we have

$$\theta_{ij}(s_k) = \begin{cases} -2e^{2z}x_i\,, & k=j \\ 2e^{2z}x_j\,, & k=i \\ 0\,, & k \neq i, j \end{cases}$$

for $i,j=1,\quad, q$ and $i \Phi j$.

But by the definition of V°,

$$\theta^0_{ij} = \theta_{ij}, \quad \text{for} \quad i, j = 1, \cdots, q\,.$$

Hence we have (4.3.2) (1).

(4.3.2) (2) is shown easily by the following.

By the definition of ∇^1 and Lemma 4.2 (3),

$$\nabla^1_X(\bar{Y}_i) = \begin{cases} 0\,, & \text{if} \quad X=\bar{X}_1, \cdots, \bar{X}q \\ 2\bar{Y}_i\,, & \text{if} \quad X=\bar{Z} \\ \nabla^0_X(\bar{Y}_i), & \text{if} \quad X= \bar{Y}_1, \cdots \quad \bar{Y}_q \end{cases}$$

for $i=1, \cdots, q$. (q.e.d.)

Now, let $k^1=(k^1_{ij})$ be the curvature matrix of ∇^1 associated to the frame $\{\bar{Y}_1, \cdots, \bar{Y}_q\}$. Let ∇^{01} be the connection on $\nu(\bar{\mathcal{F}}) \times \boldsymbol{R}$ defined by V° and ∇^1 as in (1.2.2) (3), and $\theta^{01}=(\theta^{01}{}_{ij})$ be the connection form of ∇^{01}, that is,

$$\theta^{01} = (1-t)\theta^0+t\theta^1\,,$$

and $k^{01}=(k^{01}{}_{ij})$ be the curvature matrix of ∇^{01}. Then we have

(4.3.3) (1) *At* $o=H\in\bar{M}=G/H$,

$$k^0_{ij} = \begin{cases} 2d\bar{Z}^* & i=j \\ 2(dx_j \wedge dy_i - dx_i \wedge dy_j), & i \neq j\,. \end{cases}$$

(2) *At* $(o, t) \in o \times \boldsymbol{R} \subset \bar{M} \times \boldsymbol{R}$,

$$k^{01}_{ij} = \begin{cases} 2dt \wedge \bar{Z}^* + 2td\bar{Z}^*, & i=j \\ 2\pi^*(dx_j \wedge dy_i - dx_i \wedge dy_j), & i \neq \end{cases}$$

where $\pi: \bar{M}\times \boldsymbol{R} \to \bar{M}$ *in the natnral projection.*

Proof. By (4.3.2) (1), proof of (1) is trivial. By the definition of θ^{01},

$$\theta^{01}_{ij} = \begin{cases} 2t\bar{Z}_*, & i=j \\ 2e^{2z}(x_j\bar{Y}^*_i - x_i\bar{Y}^*_j), & i \neq j, \end{cases}$$

then by (4.3.1), we have (2). (q.e.d.)

By the definition of the determinant of matrices, we have the following.
(4.3.4) *Let* $K=(K_{ij})$ *be a qxq matrix of 2-forms. Assume that*

$$K_{ij} = \begin{cases} \omega, & i=j \\ \gamma_j\wedge\eta_i - \gamma_i\wedge\eta_j, & i \neq j, \end{cases}$$

where $\gamma_1, \cdots, \gamma_q, \eta_1, \cdots, \eta_q$ *are* 1*-forms. Then*

$$c_i(K) = \sum_{\substack{0\leq n\leq i \\ n \text{ are even}}} a_{in}\cdot\omega^{i-n}\wedge\Big(\sum_{k=1}^{q}\eta_k\wedge\gamma_k\Big)^n,$$

for $i=1, \cdots, q$ *and each* a_{in} *is a positive number.*

Now, $c_i((M,\mathcal{F}), \langle\ ,\ \rangle)$ and $h_j((M, \text{ff}), \langle\ ,\ \rangle)$ are calculated as follows. In view of (4.3.1), we have Lemma 4.3 (1) from (4.3.3) (1) and (4.3.4). Similarly, by (4.3.3) (2),

$$c_j(k^{01}) = \sum_{\substack{0\leq n\leq j \\ n \text{ are even}}} a_{jn}\cdot 2^n(j-n)t^{j-n-1}dt\wedge\bar{Z}^*\wedge(d\bar{Z}^*)^{j-1}$$
$$+ \text{ (terms which do not contain } dt),$$

for $j=1, \cdots, q$, at $(o, t)\in\bar{M}\times\boldsymbol{R}$.
Let $i: \bar{M}\times[0, 1]\hookrightarrow\bar{M}\times\boldsymbol{R}$ be the inclusion mapping, and $\pi: \bar{M}\times[0, 1]\to\bar{M}$ be the natural projection, then then by the definition of Integration along the fibre π_*,

$$h_j((\bar{M}, \bar{\mathcal{F}}), \overline{\langle\ ,\ \rangle}) = (\sqrt{-1}/2\pi)^j\pi_* i^* c_j(k^{01})$$
$$= \beta_j(\sqrt{-1}/2\pi)^j\cdot\bar{Z}^*\wedge(d\bar{Z}^*)^{j-1},\ \beta_j>0.$$

This completes the proof of Lemma 4.3.

4.4. Proof of Theorem

Let ω be an element of WO_q with degree $2q+1$ such that $\omega=\phi\ h_j$ for some monomial $\phi\in\boldsymbol{R}_q[c_1, \cdots, c_q]$ as in Lemma 2.2. By Lemma 4.2 (1), $\{\bar{X}_1, \cdots, X_q, \bar{Y}_1, \ \ , \bar{Y}_q,$ is a local frame of $\tau(M)$. Then by (4.3.1) (2), $(2q+1)$-form $\bar{Z}^*\wedge(d\bar{Z}^*)^q = q!\cdot\bar{Z}^*\wedge(\bar{Y}^*_1\wedge\bar{X}^*_1)\wedge\cdots\wedge(\bar{Y}^*_q\wedge\bar{X}^*_q)$ is non zero around $o\in\bar{M}$. Hence $\omega((\bar{M}, \bar{\mathcal{F}}), \overline{\langle\ ,\ \rangle})$ is non zero at $o\in\bar{M}$ by Lemma 4.3. On the other hand, $\omega((\bar{M}, \bar{\mathcal{F}}), \overline{\langle\ ,\ \rangle})$ is a G-invariant form on $\bar{\text{M}}$ by Lemma 4.1.2.

Therefore $\omega((\bar{M}, \bar{\mathcal{F}}), \overline{\langle\ ,\ \rangle})$ is nowhere zero on $\bar{M}$. In view of Lemma 4.1.3, $\omega((M, \mathcal{F}), \langle\ ,\ \rangle)$ is a nowhere zero $(2q+1)$-form on the closed orientable $(2q+1)$-manifold M. Therefore $\omega((M, \mathcal{F}), \langle\ ,\ \rangle)$ represents a non zero cohomology class of $H^{2q+1}(M\ C)$.

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