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REAL n-PLANE BUNDLES OVER AN (n+1)-COMPLEX

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1. Introduction

Let X be a finite dimensional complex. We consider the general problem of classifying real n-plane bundles over X, which are in a natural one-to-one correspondence with $[X; BO(n)]$, the set of homotopy classes of maps from X to $BO(n)$. Alternatively, if ξ is a real stable bundle over X, we consider the problem of classifying n-plane bundles stably equivalent to ξ, which are in one-to-one correspondence with $[X; BO(n); f] = (i_\sharp)^{-1}[f] \subset [X; BO(n)]$, where $f: X \to BO$ is any map which classifies ξ.

Line bundles over X are classified by $w_1 \in H^1(X; Z_2)$, while 2-plane bundles are classified by w_1 and $W_2 \in H^2(X; Z[w_1])$, a (twisted if $w_1 \neq 0$) integer class which reduces to w_2. Oriented 3-plane bundles over a 4-complex were classified by Dold and Whitney [3], while James and Thomas enumerated n-plane bundles over an n-complex for n odd, n-plane bundles over an $(n+1)$-complex for $n = 3(4)$, and oriented n-plane bundles over an n-complex for n even. [4] (Note that "oriented" and "orientable" are equivalent concepts for bundles of odd, but not even, dimensions.) In [9] this result was extended to the case of n-plane bundles over an n-complex for all n while in [6] the James and Thomas result was restated for a few low-dimensional cases in a somewhat more explicit form.

In [7], a spectral sequence approach was used, which (in theory) completely enumerates $[X\ Y]$ in all cases where X is a finite complex and $\pi_1(Y)$ is Abelian. In fact, all real and complex bundles over P_k, $k \leqslant 5$, are tabulated.

Nomura [16] has classified n-plane bundles over P_{n+1} for $n \equiv 1(4)$ in most cases, and n-plane bundles over P_{n+2} for $n = 3(4)$ is some cases.

In the present paper, we use the approach of affine actions, developed in [9]. For the sake of space, it shall be assumed here that the reader is familiar with the constructions and notations of that paper. A general enumeration result is given for n-plane bundles over X of dimension, m, provided $m \leqslant 2n-2$ (the metastable assumption), and provided $m \leqslant n+2$. We give specific results for $X = P_m$, real projective m-space, if $m = n+1$, $n \geqslant 3$, or if $m \leqslant n+2$, $n \equiv 3(4)$, $n \geqslant 7$.

After Nomura, let $N_n(\xi\ X)$ be the number of equivalence classes of n-plane bundles over X stably equivalent to ξ. Let η be the Hopf bundle over P_m, if $m\geqslant 1$. Clearly $N_n(k\eta; P_m)=0$ if $m>n$ and $\binom{k}{n+1}$ is odd, since $w_{n+1}(k\eta)\neq 0$. In the cases covered by Theorems 1.1 and 1.2, that is the only obstruction to n-dimensionality, in effect.

Theorem 1.1. *Let* $n\geqslant 3$, $\binom{k}{n+1}$ *even.*

Case I [James and Thomas]: *if* $n=3(4)$

$$N_n(k\eta;\,P_{n+1}) = \begin{cases} 2 \text{ if } \binom{k-1}{n-1} \text{ is even} \\ 1 \text{ if } \binom{k-1}{n-1} \text{ is odd} \end{cases}$$

Case II [Nomura, except for the last case]: *if* $n\equiv 1(4)$

$$N_n(k\eta;\,P_{n+1}) = \begin{cases} 1 \text{ if } \binom{k}{2} \text{ odd}, \binom{k-1}{n-1} \text{ odd} \\ 2 \text{ if } \binom{k}{2} \text{ odd}, \binom{k-1}{n-1} \text{ even} \\ 2 \text{ if } \binom{k}{2} \text{ even}, \binom{k-1}{n-1} \text{ odd} \\ 3 \text{ if } \binom{k}{2} \text{ even}, \binom{k-1}{n-1} \text{ even} \end{cases}$$

Case III: *if* $n\equiv 0(4)$

$$N_n(k\eta;\,P_{n+1}) = \begin{cases} 2 \text{ if } k \text{ odd} \\ 3 \text{ if } k\equiv 0(4), \binom{k-1}{n-1} \text{ odd} \\ 2 \text{ if } k\equiv 0(4), \binom{k-1}{n-1} \text{ even}, \binom{k}{n} \text{ odd} \\ 5 \text{ if } k\equiv 0(4), \binom{k-1}{n-1} \text{ even}, \binom{k}{n} \text{ even} \\ 2 \text{ if } k=2(4), \binom{k}{n} \text{ odd} \\ 5 \text{ if } k=2(4), \binom{k}{n} \text{ even} \end{cases}$$

Case IV: *if* $n\equiv 2(4)$

$$N_n(k\eta;\, P_{n+1}) = \begin{cases} 1 \ \textit{if } k \ \textit{odd}, \ n=6 \\ 2 \ \textit{if } k \ \textit{even}, \ n=6, \ \binom{k-1}{n-1} \ \textit{even} \\ 2 \ \textit{if } k = 3(4), \ n \geqslant 10 \\ 4 \ \textit{if } k = 1(4), \ n \geqslant 10 \\ 6 \ \textit{if } k \ \textit{even}, \ n \geqslant 10, \ \binom{k-1}{n-1} \ \textit{even} \\ 4 \ \textit{if } k \ \textit{even}, \ n \geqslant 10, \ \binom{k-1}{n-1} \ \textit{odd} \\ 1 \ \textit{if } k \ \textit{even}, \ n=6, \ \binom{k-1}{n-1} \ \textit{odd} \end{cases}$$

Theorem 1.2. *Let* $n = 3(4)$, $n \geqslant 7$, *and* $\binom{k}{n+1}$ *even. Then*

$$N_n(k\eta;\, P_{n+2}) = \begin{cases} 2 \ \textit{if } \binom{k-1}{n-1} \ \textit{odd} \ \text{[Nomura]} \\ 3 \ \textit{if } \binom{k-1}{n-1} \ \text{even} \end{cases}$$

Theorems 1.1 and 1.2 are condensed versions of 3.8 and 3.10, below.

2. The Main theory

We shall utilize the tecniques of Becker, McClendon, and the author with regards to constructions over, and over-and-under a fixed space. [2, 8, 12]

If $\pi: E \to B$ is a fibration and $f: X \to B$ is a map, let $[X; E]_f$ be the set of fiber-homotopy classes of liftings of f to E, and let $[X; E; f] = (\pi_\sharp)^{-1}[f] \subset [X; E]$ be the set of homotopy classes of liftings of f to E. Recall that $[X; E; f]$ is the set of orbits of a left action [9, 15]:

$$\mu\colon \pi_1(B^X, f) \times [X\ ;\ E]_f \to [X;\ E]_f$$

Furthermore, if $\dim X \leqslant 2n$, where each fiber of π is n-connected (the meta-stable assumption), $[X;\ E]_f$ is an Abelian affine group and μ is an affine action, i.e., $\mu(\alpha,\ )$ is an affine automorphism for each $\alpha \in \pi_1(B^X, f)$. In that case, we also have a left action γ of $\pi_1(B^X, f)$ on $[X\ E]_f{}^0$, the difference group of $[X;\ E]_f$. Writing ax for $\gamma(\alpha, x)$ for all $x \in [X; E]_f{}^0$, we have $\alpha(a+x) = a + \alpha x$ for all $a \in [X; E]_f$, $x \in [X; E]_f{}^0$. More generally, if h^* is any B-twisted cohomology theory, a left action

$$\gamma\colon\ \pi_1(B^X, f) \times h^*(X, A, f) \to h^*(X,\ A, f)$$

can always be defined, for a subcomplex A; and γ is functorial in all the obvious ways.

Let $G=O$, U, or Sp, and let $r=1$, 2, or 4, respectively. Let π: $BG(n)\to BG$ be a fibration replacing the usual inclusion, and let $\mathcal{S}_n=\mathcal{S}_nG$ be the Ω_{BG}-spectrum associated with π (see [8]). Let $\Gamma^{n+1}\in H^1(BG;\mathcal{S}_n)$ be the single obstruction to section of π. If ξ is a stable vector bundle over a complex X of dimension m, classified by f: $X\to BG$, we can define a characteristic class $\Gamma^{n+1}\xi\in H^1(X; f^{-1}\mathcal{S}_n)$, the single (metastable) obstruction to n-dimensionality of ξ. We have [2, 8]:

REMARK 2.1. If *g.d.* $\xi\leqslant n$ (*g.d.*=geometric dimension), $\Gamma^{n+1}\xi=0$. This condition is sufficient if $m\leqslant 2rn+2r-3$.

Proof. The fiber of $BG(n)\to BG$ is $r(n+1)-2$ connected.

Now let $A_n(\xi; X)=[X;BG(n)]_f$, the set of n-plane bundles *stabilized to* ξ, an Abelian affine group in the metastable range, i.e., $m\leqslant 2rn+2r-4$.

REMARK 2.2. (I) If a, $b\in A_n(\xi;X)$, a unique difference class $\Delta(a, b)\in H^0(X; f^{-1}\mathcal{S}_n)$ is defined, such that $\Delta(a, c)=\Delta(a, b)+\Delta(b, c)$ for all a, b, $c\in A_n(\xi; X)$ (II) If $m\leqslant 2rn+2r-4$, $\Delta(a, b)=0$ if and only if $a=b$; while for any $a\in A_n(\xi;X)$ and $x\in H^0(X; f^{-1}\mathcal{S}_n)$, there exists $b=a+x\in A_n(\xi; X)$ such that $\Delta(a, b)=x$. It follows that (III) If $m\leqslant 2rn+2r-4$, $H^0(X;f^{-1}\mathcal{S}_n)$ corresponds in a natural way to $A_n^0(\xi; X)$, the difference group of $A_n(\xi; X)$; provided the latter is non-empty.

Recall that Λ_ξ: $KG^{-1}(X)\simeq\pi_1(BG^X,f)$ in a natural way [4]. We thus have a left action:

$$\mu:\ KG^{-1}(X)\times A_n(\xi;X)\ \to\ A_n(\xi;X)$$

and $V_n(\xi; X)$, the set of equivalence classes of n-plane bundles stably equivalent to ξ, corresponds in a natural way to the set of orbits of μ. Write αa for $\mu(\alpha, a)$, for any $\alpha\in KG^{-1}(X)$, $a\in A_n(\xi;X)$. Thus $N_n(\xi;X)$ is simply the cardinality of $V_n(\xi;X)$.

We summarize our general results for classification of real bundles in low codimension cases. Throughout, let ξ be a real stable bundle over a complex X of dimension m, classified by f: $X\to BO$; let $Y=K(Z_2,1)\times K(Z_2,2)$, and let β: $BO\to Y$ be a map such that $\beta^*(\iota_1\otimes 1)=w_1$ and $\beta^*(1\otimes\iota_2)=w_2$. Without loss of generality, X is connected, thus we may identify $H^0(X; Z_2)$ with Z_2.

Theorem 2.3. *Let* $m\leqslant 2n-2$, $m\leqslant n+2$. *Then* (I) *There is an* Ω_Y*-spectrum* $\mathcal{T}_n$ *such that* $H^i(X; (\beta f)^{-1}\mathcal{T}_n)\cong H^i(X; f^{-1}\mathcal{S}_n)$ *for* $i\geqslant 0$ *(canonical isomorphism)* (II) *There is a universal characteristic class* $\phi^{n+1}\in H^1(BO;\beta^{-1}\mathcal{T}_n)$ *such that* $\phi\xi=f^*\phi^{n+1}=\Gamma^{n+1}\xi$. (III) *The constructions of* $\mathcal{T}_n$, ϕ^{n+1} *are independent of* X *and* ξ.

Proof. Let $\mathcal{S}_n^{(2)}$ be the second stage of the Postnikov tower for $\mathcal{S}_nO$. $\mathcal{S}_n^{(2)}$ has homotopy width three or less. By McClendon [14], $\mathcal{S}_n^{(2)}=\beta^{-1}\mathcal{T}_n$ for some

Ω_Y-spectrum $\mathcal{I}_n$, since β is a 3-equivalence. Let $\phi^n=P_\sharp\Gamma^n$, where $P\colon \mathcal{S}_n\to\mathcal{S}_n^{(2)}$ is the projection. The remaining details are trivial, and we are done.

Let $h_i=\sigma w_{i+1}\in H^i(OZ_2)$ for all $i\geqslant 0$, where σ is the looping suspension. We have a short exact sequence:

$$\text{(2-1)}\qquad 0\to[X;Spin]\to KO^{-1}(X)\xrightarrow{h_0,\,h_1}H^0(X;Z_2)+H^1(X;Z_2)\cong\pi_1(Y^X,\beta f)\to 0$$

From 2.3 and [9, diagram (3–1)], we have

REMARK 2.4. Let $m\leqslant 2n-2$, $m\leqslant n+2$. Then there is a homomorphism $\nu_\xi\colon [X;\text{Spin}]\to A_n^0(\xi;X)$ such that $\mu(\alpha,a)=a+\nu_\xi\alpha$ for all $\alpha\in[X;Spin]$, $a\in A_n(\xi;X)$.

We may express ν_ξ in a very specific way. For any $\alpha\in KO^{-1}(X)\beta$ $F_{\xi,\alpha}=\beta\circ f\circ p_X$, where $F_{\xi,\alpha}\colon X\times S\to BO$ is a homotopy of f representing $\Lambda_\xi\alpha$, and $p_X\colon X X S\to X$ is the projection. Thus

$$H^i(X\times S;(\beta\circ F_{\xi,\alpha})^{-1}\mathcal{I}_n) = H^i(X;(\beta\circ f)^{-1}\mathcal{I}_n)+H^i(X\times S,X(\beta\circ f\circ p_X)^{-1}\mathcal{I}_n)$$

and $\nu_\xi\alpha$ can be uniquely defined by the equation

$$F_{\xi,\alpha}{}^*\phi^n=\phi^n\otimes 1+\nu_\xi\alpha\otimes\sigma$$

where $\sigma\in\pi_S^1(S)$ is the fundamental class of S in stable cohomotopy.

We now consider the action

$$\gamma\colon KO^{-1}(X)XA_n^0(\xi\ \ X)\to A_n^0(\xi\ \ X)$$

where we write ax for $\gamma(\alpha,x)$. 2.5 can be restated as follows:

REMARK 2.5. For $m\leqslant 2n-2$, $m\leqslant n+2$, and fixed $x\in A_n^0(\xi\ \ X)$, αx depends only on $h_0\alpha$ and $h_1\alpha$. Equivalently, If $h_0\alpha=h_1\alpha=0$, $\alpha x=x$ for all $x\in A_n^0(\xi;X)$.

We now give very specific expressions for γ in the case $m=n+1$, and the case $m=n+2$ for $m\equiv 3(4)$. We shall assume that the reader is familiar with the procedure for construction of the first two stages of the relative modified Postnikov tower for $BO(n)\to BO$ in all cases.

Generally, for any $u\in H^1(X;Z_2)$, let $Z[u]$ be the sheaf of integers over X twisted by u; thus $Z[0]=Z$, and $Z[u+v]=Z[u]\otimes Z[v]$. Let $\rho\colon H^i(X;Z[u])\to H^i(X;Z_2)$ be reduction modulo 2, and let $\delta[u]\colon H^i(X;Z_2)\to H^{i+1}(X;Z[u])$ be the Bokstein homomorphism associated with the exact sequence of sheaves $Z[u]\to Z[u]\to Z_2$. Let $w_i=w_i\xi$, the i^{th} Stiefel Whitney class of ξ, for all $i\geqslant 0$.

Theorem 2.6. *Let* $n\equiv 1(4)$, $n\geqslant 5$, $\dim X=m\leqslant n+1$, *and assume* $w_{n+1}=0$. *Let* $\theta=Sq^2+w_2\cup$. *Then* (I) *We have an exact sequence:*

$$H^{n-1}(X;Z_2)\xrightarrow{\theta}H^{n+1}(X;Z_2)\xrightarrow{\lambda}A_n^0(\xi\ \ X)\xrightarrow{p}H^n(X\ \ Z_2)\to 0$$

(II) *If* $x \in H^n(X; Z_2)$ *and* $z \in p^{-1}x$, *then* $2z = \lambda Sq^1 x$.
(III) *If* $x \in H^n(XZ_2)$ *and* $z \in p^{-1}x$, *and if* $\alpha \in KO^{-1}(X)$,

$$\alpha z = z + \lambda((h_1\alpha + w_1 \cup h_0\alpha) \cup x)$$

Theorem 2.7. *Let* rc^3(4), $n \geqslant 7$, $m \leqslant n+2$; *and assume* $w_{n+1} = 0$. *Let* $\theta = (Sq^2 + w_2 \cup + w_1 \cup) Sq^1$. (I) *We have an exact sequence:*

$$H^{n-1}(X; Z_2) \to \hat{\mathrm{tf}}^{n+2}(X; Z_2) \to A_n^0(\xi; X) \xrightarrow{p} H^n(X; Z_2) \to 0$$

(II) *If* $x \in H^n(XZ_2)$ *and* $z \in p^{-1}x$, *then* $2z = \lambda(Sq^2 + w_2 \cup + w_1^2 \cup)x$
(III) *If* $x \in H^n(XZ_2)$ *and* $z \in p^{-1}x$, *and* $\alpha \in KO^{-1}(X)$, *then*

$$\alpha z = z + \lambda((h_1\alpha + w_1 \cup h_0\alpha) \cup Sq^1 x)$$

Theorem 2.8. *Let* $n = 0(4)$, $n \geqslant 4$, $m \leqslant n+1$; *and assume* $\delta[w_1]w_{:} = 0$. *Let* $\theta(x, y) = (Sq^2 + w_2 \cup)\rho x + w_1 \cup y$. (I) *We have an exact sequence:*

$$H^{n-1}(X; Z[w_1]) + H^n(X; Z_2) \xrightarrow{\theta} H^{n+1}(X; Z_2) \xrightarrow{\lambda} A_n^0(\xi; X) \xrightarrow{p}$$
$$H^n(X; Z[w_1]) + H^{n+1}(X; Z_2) \to 0$$

(II) // $x \in H^n(X; Z[w_1])$, $y \in H^{n+1}(X; Z_2)$, $2x = 0$; *and* $z \in p^{-1}(x, y)$, *pick* $w \in H^{n-1}(X; Z_2)$ *such that* $\delta_1[w_1]w = x$. *Then* $2z = \lambda(Sq^1\rho x + (Sq^2 + w_2 \cup)w)$.
(III) *If* $x \in H^n(X; Z[w_1])$, $y \in H^{n+1}(X; Z_2)$ *and* $z \in p^{-1}(x, y)$, *and if* $\alpha \in KO^{-1}(X)$, *then* $\alpha z = (-1)^{h_0\alpha} z + \lambda((h_1\alpha + w_1 \cup h_0\alpha) \cup x + h_0\alpha \cup y)$ (Recall $H^0(X; Z_2) = Z_2$; let $(-1)^0 = 1$, $(-1)^1 = -1$.)

Theorem 2.9. *Let* $n = 2(4)$, $n \geqslant 6$, $m \leqslant n+1$; *and assume* $\delta[w_1]w_n = 0$. *Let* $\theta(x, y) = (Sq^2 + w_2 \cup)\rho x + Sq^1 y$. (I) *We hai e an exact sequence:*

$$H^{n-1}(X; Z[w_1]) + H^n(X; Z_2) \xrightarrow{\theta} H^{n+1}(X; Z_2) \xrightarrow{\lambda} A_n^0(\xi \;\; X) \xrightarrow{p}$$
$$H^n(X; Z[w_1]) + H^{n+1}(X; Z_2) \to 0$$

(II) // $x \in H^n(X; Z[w_1])$, $y \in H^{n+1}(X; Z_2)$, $2x = 0$, *and* $z \in p^{-1}(x, y)$, *pick* $w \in H^{n-1}(X; Z_2)$ rocA *that* $\delta[w_1]w = x$. *Then*

$$2z = \lambda(Sq^1\rho x + (Sq^2 + w_2 \cup)w + y).$$

(III) // $x \in H^n(X; Z[w_1])$, $y \in H^{n+1}(; Z_2)$, *and* $z \in p^{-1}(x, y)$, *then for any* $\alpha \in KO^{-1}(X)$, $\alpha z = (-1)^{h_0\alpha} z + \lambda((h_1\alpha + w_1 \cup h_0\alpha) \cup x + h_0\alpha \cup y)$

Proof of 2.6-2.9. (I) is obtained from the McClendon spectral sequence [12], while (II) is computed using the extension results of [11]. (III) follows from 4.2, 4.6, and 4.7. (Note that ${}^{\alpha}w_1 = h_0\alpha$, ${}^{\alpha}w_2 = h_1\alpha + w_1 \cup h_0\alpha$, in the notation of §4.)

In the range $m \leqslant 2n-2$, $m \leqslant n+2$, knowledge of the three things suffices to enumerate $V_n(\xi;X)$: namely, ν_ξ, 7, and a function s_a: $H^0(X;Z_2)+H^1(X Z_2) \rightarrow A_n^0(\xi;X)$, for any fixed $a \in A_n(\xi X)$. [9, Thm 3.1] We shall see that s_a is determined by its values on generators, although it is not a homomorphism.

Without loss of generality, X is connected. Let $\rho \in KO^{-1}(X)$be classified by a map which takes $X$to a single point of O which does not lie in the identity component. For any $u \in H^1(X;Z_2)$, let $\psi[u] \in KO^{-1}(X)$be classified by the composition $X \xrightarrow{u} P_\infty \xrightarrow{W} SO$where Wisthe Whitehead map: recall that $W^*h_i = u^i$ for all $\imath$. We define (for fixed $a \in A_n(\xi;X)$), for $(x, y) \in H^0(X;Z_2)+H^1(X;Z_2)$;

$$s_a(x, y) = \alpha_{x,y}a - a \quad \text{where } \alpha = \begin{cases} \psi[y] & \text{if } x=0 \\ \rho+\psi[y] & \text{if } x=\bar{1} \end{cases}$$

For any $(x, y) \in H^0(X;Z_2)+H^1(X;Z_2)$, and any $z \in A_n^0(\xi; X)$, let $\gamma'(x, y, z) = (x, y)z = \rho_{x,y}z \in A_n^0(\xi;X)$. It is clear that γ' is an action if $m \leqslant 2n-2$ and $m \leqslant n+2$.

From 2.4 and Theorem 3.1 of [9], we immediately have:

REMARK 2.10. Let X be a connected complex of dimension m, where $m \leqslant 2n-2$, $m \leqslant n+2$; and let ξ be a real stable vector bundle over Xwhich has an n-dimensional stabilization a. Let $\Gamma \subset H^1(X;Z_2)$ be any generating set. Then, in order to enumerate $V_n(\xi;X)$, it is sufficient to compute

(i) the Abelian group $A_n^0(\xi\ X)$

(ii) the homomorphism ν_ξ: $[X; Spin] \rightarrow A_n^0(\xi;X)$

(iii) the action γ': $H^0(X; Z_2)+H^1(X; Z_2) \times A_n^0(\xi; X) \rightarrow A_n^0(\xi;X)$

(iv) $s_a(1, 0) = s_a\rho \in A_n^0(\xi;X)$, and $s_a(0, u) = s_a\psi[u] \in A_n^0(\xi;X)$ for all $u \in \Gamma$.

The following lemma, which follows from James and Thomas [4, 1.4] will be a useful aid in computing s_a. Let $\rho^\wedge$: $A_n^0(\xi; X) \rightarrow H^n(X;Z_2)$ be the reduction defined in the obvious way, to wit, if $x = a-b$, where a and b are stabilized vector bundles, classified by liftings g_a, g_b: $X \rightarrow BO(n)$ of f: $X \rightarrow BO$, $\rho^\wedge x$ is the difference class of g_a and g_b defined by w_{n+1} in $\dot{B}O$.

Lemma 2.11. *Let ξ be a real stable bundle over a connected complex X of dimension m, $m \leqslant 2n-2$, and let $a \in A_n(\xi;X)$.* (I) *If $\alpha \in KO^{-1}(X)$, $\rho^\wedge(\alpha a - a) = \sum_{i=0}^m h_i\alpha \cup w_{n-i}\xi$.* (II) *If $m \leqslant n+2$ $\rho^\wedge s_a\rho = w_n\xi$, and $\rho^\wedge s_a\phi[u] = \sum_{i=1}^m u^i \cup w_{n-i}\xi$ for all $u \in H^1(X; Z_2)$.*

Proof. (I) follows immediately from the James and Thomas formula, and (II) is an immediate corollary of (I).

3. Applications to projective spaces

Tensor products. Let ξ be a stable real, or complex vector bundle over a complex X, and let L be a line bundle over X. Let $\xi^{(n)}$ be the virtual ra-plane bundle representing ξ; we define $L\otimes_n\xi$ to be the stable bundle represented by the virtual n-plane bundle $L\otimes\xi^{(n)}$. Let t and t_n, respectively, classify $\otimes$ and $\otimes_n$ such that we have a commutative diagram:

$$\begin{array}{ccc} BG(1)\times BG(n) & \xrightarrow{t} & BG(n) \\ \downarrow 1\times\pi & & \downarrow \pi \\ BG(1)\times BG & \xrightarrow{t_n} & BG \end{array}$$

If $a\in A_n(\xi;X)$, we can thus define $L\otimes a\in A_n(L\otimes_n\xi;X)$ as follows: if g: $X\to BG(n)$ classifies a, $t(l,g)$ classifies $L\otimes a$; where l: $X\to BG(1)$ classifies L.

REMARK 3.1. (I) $L\otimes: A_n(\xi;X)\to A_n(L\otimes_n\xi;X)$ is one-to-one and onto: in fact its inverse is $L'\otimes$, where L' is the line bundle conjugate to L (note that $L'=L$ in the real case). (II) In the metastable range, i.e., $\dim X\leqslant 2rn+2r-4$, L® is an affine isomorphism.

Proof. (I) is obvious; (II) requires some manipulation of the base spaces; we leave the details to the reader.

Let $l\otimes_n: BG^X\to BG^X$ be given by $l\otimes_n f'=t_n(l,f')$ for any $f': X\to BG$. If f: $X\to BG$ classifies ξ, and Λ_ξ: $\pi_1(BG^X,f)\cong KG^{-1}(X)$ is the James-Thomas isomorphism, let $\delta_{\xi,L,n}: KG^{-1}(X)\to KG^{-1}(X)$ be the composition

$$KG^{-1}(X)\overset{(\Lambda_\xi)^{-1}}{\cong}\pi_1(BG^X,f)\xrightarrow[\cong]{(l\otimes_n)_\xi}\pi_1(BG^X,\ l\otimes_n f)\overset{\Lambda_{L\otimes_n\xi}}{\cong}KG^{-1}(X)$$

In the metastable range, define $L\otimes x\in A_n^0(L\otimes_n\xi;X)$ for all $x\in A_n^0(\xi\ X)$ by $L\otimes(a-b)=L\otimes a-L\otimes b$, for all a, $b\in A_n(\xi;X)$. By simply chasing the definitions, we can easily check that:

REMARK 3.2. (I) For all $a\in A_n(\xi;X)$ and all $\alpha\in KG^{-1}(X)$, $L\otimes\alpha a=\delta_{\xi,L,n}\alpha(L\otimes a)$. (II) If $m\leqslant 2nr+2r-4$, $L\otimes\alpha x=\delta_{\xi,L,n}\alpha(L\otimes x)$ for all $x\in A_n^0(\xi;X)$ and $\alpha\in KG^{-1}(X)$.

And from the splitting theorem, we can compute

Lemma 3.3. *In the real case,* i.e., $G=O$, $t_n{}^*w_k=\sum_{i=0}^k\binom{n-k+1}{i}w_1{}^i\otimes w_{k-i}$. *(Where it is understood that if* $a<0$, $b\geqslant 0$, $\binom{a}{b}$ *is defined (modulo 2) to be* $\binom{2^N+a}{b}$ *far sufficiently large* N.)

The null-element. Let ξ be real, henceforth. We define $\nu_{\xi,n}\in KO^{-1}(X)$ to be the element corresponding to $N_{\xi,n}\in\pi_1(BO^X,f)$ where $N_{\xi,n}$ is the composition

$$X\times S^1 \xrightarrow{1\times f} BO(1)\times BO\xrightarrow{t_n}BO$$

where ε_1 represents the generator of $\pi_1 BO(1)$, and T reverses coordinates. We call the $\nu_{\xi,n}$ *null-element* of ξ in dimension n.

Lemma 3.4. *Let ξ be real. Then* (I) $\nu_{\xi,n}$ *is functorial in X, i.e., if $g: X'\to X$ is a map and $\xi'=g^{-1}\xi$, then $\nu_{\xi',n}=g^!\nu_{\xi,n}$.* (II) *For all $a\in A_n(\xi;X)$, $\nu_{\xi,n}a=a$.* (III) *For all $k\geqslant 0$, $h_k\nu_{\xi,n}=\binom{n-k}{1}w_k\xi$.*

Proof. (I) is obvious. (II) holds since, if $g: X\to BO(n)$ is a lifting of f which represents a, $N_{\xi,n}$ lifts to $t\circ(\varepsilon_1\times g)\circ T: X\times S\to BO(n)$, a self-homotopy of g. (III) follows immediately from 3.3.

Projective spaces. Henceforth, let P_r be real projective r-space for any integer $r\geqslant 0$, and let $u\in H^1(P_r;Z_2)$ be the generator, if $r\geqslant 1$. Let $\psi=\psi[u]\in KO^{-1}(P_r)$. It is well-known and easily computable that

Lemma 3.5. *Let $r\geqslant 1$. Then* (I) *if $r\not\equiv 3(4)$, $KO^{-1}(P_r)\cong Z_2+Z_2$, with generators p and ψ.* (II) *If $r=3(4)$, $KO^{-1}(P_r)\cong Z_2+Z_2+Z$ with generators p, ψ, and τ, where $\tau\in[P_r;\ Spin]$ is classified by $P_r\to P_r/P_{r-1}=S^r\xrightarrow{\varepsilon_r}Spin$ where ε_r represents the generator of $\pi_r(Spin)\cong Z$.* (III) *$KO^0(P_r)=[P_r;\ BO]\cong Z_{2^s}$, generated by η, the canonical line bundle over P_r (i.e., $w_1\eta=u$), where s is the number of positive integers less than or equal to r which are equivalent to* 0, 1, 2, *or* 4 *modulo* 8. (IV) *For all $i\geqslant 1$, $h_i\rho=0$, and $h_i\psi=u^i$; while $h_0\rho=1$ and $h_0\psi=0$. If $r=2(4)$, $h_r\tau=u^r$ for $r=2$ or 6; while $h_i\tau=0$ if $i\neq r$ or $r>6$.* (V) *For any $0\leqslant k<2^s$, $w_i(k\eta)=\binom{k}{i}u^i$ for all $i\geqslant 0$.*

Lemma 3.6. *For $r\geqslant 1$ and $0\leqslant k<2^s$ (where s is computed as in* 3.5 (III)), (I) $\delta_{k\eta,\eta,n}\psi=\psi$. (II) $\delta_{k\eta,\eta,n}\rho=\rho+\psi$. (III) *If $r=2(4)$, $\delta_{k\eta,\eta,n}\tau=\tau$.*

Proof. Clearly $\delta_{k\eta,\eta,n}\psi$ and $\delta_{k\eta,\eta,n}\rho$ must be 2-torsion elements; the stated results for (I) and (II) are the only answers which agree with 3.3. To prove (III), consider the covering map $c: S^r\to P_r$. Now $c^{-1}\eta=1$, the trivial line bundle, and Ker $c^!$ is generated by p and ψ. Thus $\delta_{k\eta,\eta,n}\tau=\tau+\lambda_1\rho+\lambda_2\psi$ for λ_1, $\lambda_2\in Z_2$. Again by 3.3, $\lambda_1=\lambda_2=0$.

Lemma 3.7. *Let $r\geqslant 1$, $0\leqslant k<2^s$. Then*

$$\nu_{k\eta,n}=\begin{cases}0 & \text{if } n \text{ is even}\\ P & \text{if } n \text{ odd, } k \text{ even}\\ \rho+\psi & \text{if } n \text{ odd, } k \text{ odd}\end{cases}$$

Proof. For $r\neq 2(4)$, the stated result is the only possible which agrees with 3.3. If $r=2(4)$, the result still holds, by 3.4(I), since $k\eta$ lives in P_{r+1}.

Notation. We give standard names, x_{01}, x_{00}, and x_{10}, for generators of the Abelian group $A_n{}^0(k\eta;\, P_r)$ (which can be computed by one of the theorems 2.6-2.9), for $r=n+1$, $n\geqslant 3$. If $n\not\equiv 3(4)$, let $x_{01}=\lambda u^{n+1}$, while if $n=3(4)$, let $x_{01}=0$. Let x_{00} be defined by the equation $px_{00}=u^n$ if n is odd, $px_{00}=(\delta u^{n-1},\, 0)$ if n and k are both even. If n is even and k is odd, x_{00} shall not be defined. If n is even, x_{10} shall be defined by the equation $px_{10}=(0,\, u^{n+1})$. For odd n, x_{10} is not defined.

Note that x_{01} is always uniquely defined, but may be zero, while x_{00} and x_{10} are not always defined, have intederminacy x_{01}, and are never zero. Intuitively, if e^n and e^{n+1} are the bottom two cells of the Stiefel manifold V_n, and ι, η are the generators of the stable 0- and 1-stems in the homotopy of spheres, we may write $x_{00}=e^n\otimes\iota\otimes u^n$, $x_{01}=e^n\otimes\eta\otimes u^{n+1}$, and $x_{10}=e^{n+1}\otimes\eta\otimes u^{n+1}$.

Define endomorphisms χ_0 and χ_1 on $A_n{}^0(k\eta;\, P_r)$ for $r=n+l$, $n\geqslant 3$, as follows: $\chi_0 x_{01}=\chi_1 x_{01}=0$ in all cases; $\chi_0 x_{00}=x_{01}$ and $\chi_1 x_{00}=0$ if x_{00} exists; and $\chi_0 x_{10}=0$ and $\chi_1 x_{10}=x_{01}$ if x_{10} exists.

Theorem 3.8. *Let $n\geqslant 3$, and let k be any integer such that* $\binom{k}{\text{[illegible]}}$ *is even.* (I) *If $n\neq 6$, the homomorphism $\nu_{k\eta}$: $[P_{n+1};\, Spin]\to A_n{}^0(k\eta;P_{n+1})$ is zero, while if $n=6$, $\nu_{k\eta}\tau=\pm x_{10}$.* (II) *The group $A_n{}^0(k\eta;P_{n+1})$, the automorphisms $\gamma(\rho,\ )$ and $\gamma(\psi,\ )$ on $A_n{}^0(k\eta;\, P_{n+1})$, the elements $s_a\rho$, $s_a\psi\in A_n{}^0(k\eta;P_{n+1})$ (for some choice of $a\in A_n(k\eta;\, P_{n+1})$), and the resulting value of $N_n(k\eta;\, P_{n+1})$ are as in Table A.*

Proof. The groups $A_n{}^0(k\eta;\, P_{n+1})$, with their generators and relations, come from theorems 2.5-2.8, (I) and (II). The actions γ(p,) and $\gamma(\psi,\ )$ are then from 2.5-2.8 (III). If $n\not\equiv 2(4)$, $\nu_{k\eta}=0$ since $[P_{n+1};\, Spin]=0$. If $n=2(4)$, $[P_{n+1};\, Spin]\cong Z$generated by r (cf. 3.5) and $\nu_{k\eta}\tau=\pm x_{10}$ by 2.11 if $n=6$. If $n\geqslant 14$, $\nu_{k\eta}=0$ since $P_{n+1}\to S^{n+1}\to Spin$ can be lifted to $Spin(n)$(see Barratt and Mahowald [1]). If $n=10$ and $k=3(4)$, $\nu_{k\eta}=0$ since $\rho^\wedge$: $A_{10}{}^0(k\eta;\, P_{11})\to H^{10}(P_{11};\, Z_2)$ is mono. Let $n=10$, ft=4ί+Λ $0\leqslant j\leqslant 2$. Since $\pi_{12}(BSp)\to\pi_{12}(BSpin)$ is onto, ε_{12}: $S^{12}\to BSpin$ classifies a quaternionic bundle. Let $(\Lambda_{k\eta})^{-1}$: $KO^{-1}(P_{11})\cong\pi_1(BO^{P_{11}}$, fbe the James-Thomas isomorphism. Then $(\Lambda_{k\eta})^{-1}\tau$ is represented by $F_{\tau,k\eta}:P_{11}\text{ X } S\to BO$ which classifies $Q\otimes j\eta$, where Q is a bundle with a quaternionic structure. One may easily verify that $e_3 Q=0$, hence $g.d.Q\leqslant 8$. Thus $F_{\tau,k\eta}$ lifts to $BO(10)$, whence $\nu_{k\eta}=0$.

The values of $s_a\rho$ and $s_a\psi$ can now be computed up to the natural indeterminacy caused by the choice of a, using 2.11, 3.1, 3.2, 3.3, 3.6, and 3.7; and in the case $n=2(4)$, by lemma 3.9 below. This completes the proof of 3.8.

Lemma 3.9. (I) *Let $n=2(4)$, $\binom{k}{2}$even, $\binom{\text{[illegible]}}{\text{[illegible]}}$ even. Then $\rho a=a$ for some $a\in A_n(k\eta\ \ P_{n+1})$.* (II) *Let $n=2(4)$, $k\equiv 1(4)$, $\binom{k}{n+1}$ even. Then $(\rho+\psi)a=a$for some $a\in A_n(k\eta,\, P_{n+1})$.*

Table A: n-plane bundles over $P_{n+1}$of stable type $k\eta$,for $n\geq 3$, $\binom{k}{n+1}$ even.

	$A_n^0(k\eta\, P_{n+1})$	generators & relations	$\gamma(\rho,\ )$	$\gamma(\psi,\ )$	$s_a\rho$	$s_a\psi$	$N_n(k\eta;P_{n+1})$
$n\equiv 3(4)$	Z_2	#00, #01 $x_{01}=2x_{00}=0$	1	1	$\binom{k}{n}x_{00}$	$\binom{k-1}{n-1}x_{00}$	2 if $\binom{k-1}{n-1}$ even 1 if $\binom{k-1}{n-1}$ odd
$n=1(4)$ $k\equiv 0(4)$	Z_4	x_{00}, #01 $2x_{00}=x_{01}$ $2x_{01}=0$	1	$1+\chi_0$	0	$\binom{k-1}{n-1}x_{00}$	3 if $\binom{k-1}{n-1}$ even 2 if $\binom{k-1}{n-1}$ odd
$n=1<4)$ $k=1(4)$	Z_4	#00, #01 $2x\ =x_{01}$ $2x\ =0$	$1+\ \prime_0$	$1+\chi_0$	$\binom{k}{n}x_{00}$	$\binom{k}{n}x_{00}$	3 if $\binom{k-1}{n-1}$ evcn 2 if $\binom{k-1}{n-1}$ odd
$n=1(4)$ $k\equiv 2(4)$	Z_2	#00, #01 $2x_{00}=x_{01}=0$	1	1	0	$\binom{k-1}{n-1}x_{00}$	2 if $\binom{k-1}{n-1}$ even 1 if $\binom{k-1}{n-1}$ odd
$n\equiv 1(4)$ $A=3(4)$	Z_2	#00, #01 $2x_{00}=x_{01}=0$	1	1	$\binom{k}{n}x_{00}$	$\binom{k-1}{n-1}x_{00}$	2 if $\binom{k-1}{n-1}$ even 1 if $\binom{k-1}{n-1}$ odd
$n\equiv 0(4)$ k odd	Z_2	#10, #01 $2x_{10}=x_{01}=0$	1	1	0	0	2
$n\equiv 0(4)$ $k\equiv 0(4)$	Z_4+Z_2	x_{00}, #01, #10 $2x_{00}=x_{01}$ $2x_{10}=2x_{01}$ $=0$	$1+\chi_0+\chi_1$	$1+\chi_0$	$\binom{k}{n}x_{00}$	$\binom{k-1}{n-1}x_{00}$ $+\binom{k-1}{n}x_{10}$	3 if $\binom{k-1}{n-1}$ odd 2 if $\binom{k}{n}$odd & $\binom{k-1}{n-1}$ even 5 if both even
$n\equiv 0(4)$ $k=2(4)$	$Z_2+Z_2+Z_2$	#00, #10, #01 $2x_{00}=2x_{10}$ $=2x_{01}=0$	$1+\chi_1$	$1+\chi_0$	$\binom{k}{n}x_{00}$	$\binom{\ '\ \ \ '}{n}x_{10}$	5 if 0 even 2 if $\binom{k}{n}$ odd
$n=2(4)$ $k=3(4)$	Z_2	#10, #01 $2x_{10}=x_{01}=0$	1	1	0	0	2 if $n\geq 10$ 1 if $n=6$
$n\equiv 2(4)$ $k\equiv 1(4)$	Z_4	#10, #01 $2x_{10}=x_{01}$ $2x_{01}=0$	1	1	0	0	4 if $n\geq 10$ 1 if $n=6$
$n\equiv 2(4)$ $k\equiv 0(4)$	Z_4+Z_2	#00, #01, #10 $2x_{10}=x_{01}$ $2x_{00}=2x_{01}$ $=0$	1	$1+\chi_0$	0	$\binom{k-1}{n-1}x_{00}$ $+\binom{k-1}{n}x_{10}$	6 if $n\geq 10$, $\binom{k-1}{n-1}$ even 2 if $n=6$, $\binom{k-1}{n-1}$ even
$n\equiv 2(4)$ $k\equiv 2(4)$	Z_4+Z_2	#00, x_{01}, #10 $2x_{00}=2x_{10}$ $=x_{01}$ $2x_{01}=0$	$1+\chi_0$	$1+Z.$	$\binom{k}{n}x_{00}$	$\binom{k-1}{n-1}x_{00}$	4 if $n\geq 10$, $\binom{k-1}{n-1}$ odd 1 if $n=6$, $\binom{k-1}{n-1}$ odd

Proof. (I) Let $\sum$ be the line bundle classified by $P_{n+1}\times S \xrightarrow{p_S} S \xrightarrow{\sim 1} BO(1)$. If $k=4i+j$ $j=0$ or 1, $4i\eta$ has a quaternionic structure, and $e_{n+2}(4i\eta)=0$, thus $g.d.k\eta \leqslant n-1$. $F_{\rho,k\eta}: P_{n+1}\times S\rightarrow BO$ classifies $k\eta\oplus\sum$, hence can be lifted to $BO(n)$. Thus $\rho a=a$ for some a. (II) If $k\equiv 1(4)$, $(k-1)(\eta\otimes\sum)$ has a quaternionic structure, hence $F_{\rho+\psi,k\eta}$, which classifies $k(\eta\otimes\sum)$, lifts to $BO(n)$: thus $\rho a=\psi a$ for some a.

Theorem 3.10[1]**.** *Let* $n=3(4)$, $n\geqq 7$, *and let* k *be any integer such that* $\binom{k}{n+1}$ *is even. Let* x_{00} *be an element of* $A_0^n(k\eta; P_{n+2})$ *with* $p_{00}x=u^n$ *and let* $x_{11}=\lambda u^{n+2}$. *Then*

(i) $A_n^0(k\eta; P_{n+2})\cong Z_4, 2x_{00}=x_{11}$ *for* $k=0$, $3(4)$,
$A_n^0(k\eta; P_{n+2})\cong Z_2+Z_2, 2x_{00}=2x_{11}=0$ *for* $k=1$, $2(4)$;

furthermore $A_n^0(k\eta; P_{n+2})\rightarrow A_n^0(k\eta; P_{n+1})$ *is onto.*

(ii) $\gamma(\psi, x_{00})=x_{00}+x_{11}$, *and* $\gamma(\rho, x_{00})=x_{00}+\binom{k}{1}x_{11}$.

(iii) *For any* $a\in A_n(k\eta; P_{n+2})$, $s_a\psi=\binom{k-1}{n-1}x_{00}$, *with indeterminacy* x_{11} *(based on the choice of* a*). Furthermore,* $s_a\rho=0$ *for* k *even,* $s_a\rho=s_a\psi$ *for* k *odd; both with no indeterminacy.*

Proof. Similar to 3.8.

For the sake of uniformity, the above information is displayed in table B.

Table B[2]: n-plane bundles over P_{n+2} of stable type k for $n=3(4)$, $n\geq 7$, $\binom{k}{n+1}$ even.

	$A_n^0(k\eta\ P_{n+2})$	generators & relations	$\gamma(\rho,\ )$	$\gamma(\psi,\ )$	$s_a\rho$	$s_a\psi$	$N_n(k\eta; P_{n+2})$
$k\equiv 0(4)$	Z_4	x_{00}, x_{11} $2x_{00}=x_{11}$ $2x_{11}=0$	1	$1+x$	0	$\binom{k-1}{n-1}x_{00}$	3 if $\binom{k-1}{n-1}$ even 2 if $\binom{k-1}{n-1}$ odd
$k\equiv 1(4)$	Z_2+Z_2	#00, #11 $2x_{00}=2x_{11}$ $=0$	1+Z	$1+z$	$\binom{k}{n}x_{00}$		
$k\equiv 2(4)$	Z_2+Z_2	#00, #11 $2x_{00}=2x_{11}$ $=0$	1	$1+\chi$	0		
$k\equiv 3(4)$	Z_4	#00, #11 $2x_{00}=x_{11}$ $2x_{11}=0$	$1+X$	$1+\chi$	$\binom{k}{n}x_{00}$		

1), 2) The corrected statement of Theorem 3.10, and a corresponding correction of Table B, were supplied by the referee.

4. Actions

Let A* be any Y-twisted cohomology theory satisfying the axioms given in [2]. Fix, for the moment, a C.W.-pair (X,A) and a map /: $X\to Y$. We have a natural left action

$$\gamma:\ \mathcal{H}\times h^*(X,\ A,f)\to h^*(X,\ A,f)$$

where $\mathcal{H}=\pi_1(Y^{X}, f)$, defined as follows: if $\alpha\in\mathcal{H}$ is represented by F_α: $X\times S\to Y$, where $F_\alpha(x,*)=f(x)$ for all $x\in X$, let F_α also denote the composition $X\times I\to X\times S\xrightarrow{F_\alpha} Y$. We have isomorphisms, where i_0 and i_1: $X\to X\times I$ are the inclusions along 0 and 1, respectively:

$$h^*(X\times I,\ A\times I,\ F_\alpha)\ \overset{i_0^*}{\underset{i_1^*}{\rightrightarrows}}\ h^*(X,\ A,f)$$

Let $\gamma(\alpha,\ x)=\alpha x=i_0^*(i_1^*)^{-1}x$ for all $x\in h^*(X,\ A,f)$.

Let $\varepsilon\in Z_2$: we shall write $(-1)^\varepsilon=\pm1\in Z$. For $\alpha\in\mathcal{H}$, we shall define a long exact *action sequence* for the pair $(\varepsilon,\ a)$ and the theory h^*:

$$\text{(4–1)}\qquad \cdots\xrightarrow{\nu} h^{k-1}(X,\ A,f)\xrightarrow{\phi_\alpha^\varepsilon} h^{k-1}(X,\ A,f)$$
$$\xrightarrow{\chi} h^{k+1}(X\times T_\varepsilon,\ X\times jS\cup A\times T_\varepsilon,\ F_\alpha)\xrightarrow{\nu} h^k(X,A,f)$$
$$\xrightarrow{\phi_\alpha^\varepsilon} h^k(X,\ A,f)\xrightarrow{\chi}\cdots$$

where T_ε is the torus if $\varepsilon=0$, the Klein bottle if $\varepsilon=1$, $S\to T_\varepsilon\xrightarrow{\pi} S$ is the fibration, and j: $S\to T$ is the section. By a slight abuse of notation, write F_α: $X\times T\xrightarrow{1\times\pi} X\times S\xrightarrow{F_\alpha} Y$. For $*\in h^*(X,\ A,\ f)$, $\phi_\alpha^\varepsilon x$ is given to be $\alpha x-(-1)^\varepsilon x$. We let ν be the composition

$$\begin{array}{ccc} h^{k+1}(X\times T_\varepsilon,\ X\times jS\cup A\times T_\varepsilon,F_\alpha) & \xrightarrow{(1\times i)^*} & h^{k+1}(X\times iS,A\times iS\cup X,f\circ p_X)\\ & & \cong\big\downarrow s\\ & & h^k(X,\ A,f)\end{array}$$

where p_X is projection to X and s is suspension. Let X be defined by commutativity of the diagram:

$$\begin{array}{ccc} h^{k-1}(X,A,\ /) & \xrightarrow{\chi} & h^{k+1}(X\times T_\varepsilon,\ X\times jS\cup A\times T_\varepsilon,\ F_\alpha)\\ \cong\big\downarrow & & \big\uparrow j^*\\ h^k(X\times iS,\ X\cup A\times iS,f\circ p_X) & & h^{k+1}(X\times T_\varepsilon,\ X\times(iS\vee S)\cup A\times T_\varepsilon,\ F_\alpha)\\ \cong\big\updownarrow & & \cong\big\downarrow a^*\\ h^{k+1}(X\times iS\times I,X\times L\cup A\times iS\times I,\ F_0) & \underset{\cong}{\xrightarrow{\{F_t\}^*}} & h^{k+1}(X\times iS\times I,X\times L\cup A\times iS\times I,\ F_1)\end{array}$$

where $q: X\times iS\times I\to X\times T_\varepsilon$ is the obvious quotient map, $L=iS\times\{0, 1\}\cup\{*\}\times I\subset iS\times I$, and $F_t(x, y, u)=(F_\alpha\circ q)(xy, 1-t(1-u))$ for all $t, u\in I$, $x\in X$, $y\in iS$.

It is fairly straightforward to prove that the action sequence is exact and natural with respect to stable Y-twisted cohomology operations, i.e., $\phi_\alpha{}^\varepsilon\psi$ — $ty\varphi u>$ $\nu\psi=\psi\nu$, and $\chi\psi=\psi\chi$ for any such operation ψ. We leave the details to the reader.

Now suppose that $'h^*$, "A* are Y-twisted cohomology theories classified by Ω_Y-spectra $'\mathcal{E}$, $''\mathcal{E}$, respectively. Let ψ: $'h^*\to''h^*$ be a stable Y-twisted cohomology operation of degree, say, rf, classified by an Ω_Y-spectrum map ψ: $'\mathcal{E}\to''\mathcal{E}$. If we let A* be the Y-twisted cohomology theory classified by $\mathcal{E}$, the fiber of ψ, we obtain the long exact sequence associated to ψ:

$$\text{(4–2)}\quad \cdots\ 'h^{k-1}(X, A, /)\xrightarrow{\psi}{}''h^{k+d-1}(X, A, /)\xrightarrow{\lambda} h^k(X, A, f)$$
$$\xrightarrow{p}{}'h^k(X, A, f)\xrightarrow{\psi}{}''h^{k+d}(X, A, f)\to\cdots$$

where λ, p are now stable Y-twisted cohomology operations of degrees $d-1$ and 0, respectively,

We now examine the following question. Suppose we know the action of $\mathcal{H}$ on $'h^*(X,A, f)$ and $''h^*(X, A, f)$. How can we determine the action of $\mathcal{H}$ on $h^*(X, A, f)$? We give a partial answer, which suffices for our applications to vector bundles.

Let tfeKer-v/r in the sequence above, and suppose $\alpha x=(-1)^\varepsilon x$ for some $\alpha\in\mathcal{H}$, $\varepsilon\in Z_2$. If $z\in h^k(X,A, /)$ and $pz=x$, it is clear that $(-1)^\varepsilon z$ differs from αz by λw for some $w\in{}''h^{k+d-1}(X,A, f)$. This element $w=\Phi_\alpha{}^\varepsilon x$ clearly has indeterminancy. In fact (as is trivial to check), we have a homomorphism

$$\Phi_\alpha{}^\varepsilon=\lambda^{-1}\phi_\alpha{}^\varepsilon p^{-1}\colon \operatorname{Ker}\psi\cap\operatorname{Ker}\phi_\alpha{}^\varepsilon\to{}''h^*(X, A, f)/\operatorname{Im}\psi+\operatorname{Im}\phi_\alpha{}^\varepsilon$$

Analogous to diagram (3-2) of [11] and Fig. 2 of [10], we have a commutative diagram with exact rows and columns:

$$\begin{array}{ccc}
 & & 'h^{k-1}(X, A, f)\\
 & & \downarrow\psi\\
''h^{k+d-1}(X, A, f) & \xrightarrow{\phi_\alpha{}^\varepsilon} & ''h^{+d-1}(X, A, f)\\
\downarrow\lambda & & \downarrow\lambda\\
h^k(X, A, f) & \xrightarrow{\phi_\alpha{}^\varepsilon} & h^k(X, A, f)
\end{array}$$

$$\begin{array}{ccccccccc}
 & & 'h^{k-1}(X, A, f) & \xrightarrow{\chi} & 'h^{k+1}(X\times T_\varepsilon,\ X\times jS\cup A\times T_\varepsilon F_\alpha) & \xrightarrow{\nu} & 'h^k(X, A, f) & \xrightarrow{\phi_\alpha{}^\varepsilon} & 'h^k(X A, f)\\
 & & \downarrow\psi & & \downarrow\psi=\psi_\alpha{}^\varepsilon & & \downarrow\psi & & \\
''h^{k+d-1}(X,A,f) & \xrightarrow{\psi_\alpha} & 'h^{k+d-1}(XA, f) & \xrightarrow{\chi} & ''h^{k+d+1}(X\times T_\varepsilon,\ X\times jS\cup A\times T_\varepsilon, F_\alpha) & \longrightarrow & ''h^{k+d}(X, A, f) & &
\end{array}$$

We define a homomorphism:

$$*\Phi_\alpha^\varepsilon = \chi^{-1}\psi\nu^{-1}\colon \operatorname{Ker}\psi \cap \operatorname{Ker}\phi_\alpha^\varepsilon \to {}''h^*(X, A, f)/\operatorname{Im}\psi + \operatorname{Im}\phi_\alpha^\varepsilon$$

and, analogous to Theorem 2.5 of [10] and Theorem 3.2 of [11], we have

Theorem 4.1. $\Phi_\alpha^\varepsilon = -*\Phi_\alpha^\varepsilon$

Proof. Analogous to that of Theorem 2.5, [10].

The usefulness of Φ_α^ε as a computational tool is illustrated by the following remark:

REMARK 4.2. Let $\varepsilon\colon \mathcal{A} \to Z_2$ be a homomorphism such that (in sequence (4–2)) $\alpha x = (-1)^{\varepsilon\alpha} x$ and $\alpha y = (-1)^{\varepsilon\alpha} y$ for all $\alpha \in \mathcal{A}$ and all $x \in {}'h^k(X, A, f)$, $y \in {}''h^{k+d-1}(X, A, /)$. If $\Gamma \subset \mathcal{A}$ is a generating set, knowledge of $\Phi_\alpha^{\varepsilon\alpha} x$ for all $x \in \operatorname{Ker}\psi \subset {}'h^k(X, A, f)$ and all $\alpha \in \Gamma$ suffices to determine the action of $\mathcal{A}$ on $h^k(X, A, f)$.

Proof. We leave to the reader.

We proceed to compute $*\Phi_\alpha$ in certain cases which are applicable to vector bundles. Henceforth, assume that $'h^*$ is an ordinary twisted cohomology theory of type Z or Z_2, i.e., with coefficients in a sheaf G over Y, where each stalk of G is isomorphic to either Z or Z_2. Thus $G = Z_2$ or $G = Z[u]$ for some $u \in H^1(Y; Z_2)$. The same assumption shall be made concerning $''h^*$, namely that it is also an ordinary twisted theory of type Z or Z_2, and is determined by a sheaf H over Y.

We shall make specific computations of the homomorphisms

$$\nu\colon {}'h^*(X \times T_\varepsilon, X \times jS \cup A\, T_\varepsilon, F_\alpha) \to {}'h^*(X, A, f)$$
$$\psi\colon {}'h^*(X \times T_\varepsilon, X \times jS \cup A \times T_\varepsilon, F_\alpha) \to {}''h^*(X \times T_\varepsilon, X \times iS \cup A \times T_\varepsilon, F_\alpha)$$
$$\chi\colon {}'h^*(X, A, f) \to {}''h^*(X \times T_\varepsilon, X \times jS \cup A \times T_\varepsilon, F_\alpha)$$
$$\phi_\alpha^\varepsilon\colon {}'h^*(X, A, f) \to {}'h(X, A, f)$$
$$\phi_\alpha^\varepsilon\colon {}''h^*(X, A, f) \to {}''h(X, A, f)$$

in each possible case. If $'h^*$, $''h^*$ are in fact direct sums of ordinary theories, no additional complication ensues, since ψ will be a matrix of ordinary operations, and ν, χ, and ϕ_α shall each be a vector consisting of the corresponding homomorphisms in the ordinary component cases.

Recall that $F_\alpha; X \times S \to Y$ classifies $\alpha \in \mathcal{A}$; as before, we let $F_\alpha = F_\alpha \circ (1 X \pi)$: $X X T_\varepsilon \to Y$. If $y \in {}^kH(Y; Z_2)$, let ${}^\alpha y \in H^{k-1}(X Z_2)$ be defined by the equation $F_\alpha^* y = f^* y \otimes 1 + {}^\alpha y \otimes \sigma$ where $\sigma \in H^1(S; Z)$ is the fundamental class of S. We shall assume X is connected, thus ${}^\alpha u \in H^0(X; Z_2) = Z_2$ if $u \in H^1(Y; Z_2)$. If ${}^\alpha u = \varepsilon$, and $y \in H^k(Y; Z[u])$, let ${}^\alpha y \in H^{k-1}(X; Z[f^* u])$ be defined by $F_\alpha^* y = f^* y \otimes 1 + {}^\alpha y \otimes \sigma$.

The cohomology of T_ε is well-known and easily computed. Let $a=\rho\pi^*\sigma \in H^1(T_\varepsilon; Z_2)$, and let $B\in H^1(T_\varepsilon; Z[\varepsilon a])$ be uniquely defined by the equation $j^*B=0$, $i^*B=\sigma$; and let $b=\rho B$. We have:

REMARK 4.3. $H^*(T_\varepsilon; \boldsymbol{Z}_2)$ is generated by a and b, subject only to the relations $a^2=0$ and $b^2=\varepsilon ab$.

Lemma 4.4. *We have isomorphisms (for any $u\in H^1(Y; Z_2)$):*

(I) ι: $H^{k-1}(X, A; Z_2)+H^{k-2}(X, A; Z_2)\cong H^k(X T_\varepsilon, X\times jS\cup A\times T_\varepsilon;\ Z_2)$, *where* $\iota\ (x, x')=x\otimes b+x'\otimes ab$.

(II) *L:* $H^{k-1}(X,A;\ Z[f^*u])+H^{k-2}(XA;\ Z[f^*u])\cong H^k(X\ T_\varepsilon,\ X\times jS\cup A\times T_\varepsilon;\ Z[f^*u])$, *where* $\iota(x, x')=x\otimes B+x'\otimes\pi^*\sigma\cup B$; *if* ${}^\alpha u=\varepsilon$.

(III) ι: $H^{k-2}(X,A;\ Z_2)\cong H^k(X\times T X\times jS\cup A\times T_\varepsilon;\ Z\ [f^*u])$, *where* $\iota x=\delta[F_\alpha{}^*u](x\otimes b)$; *if* ${}^\alpha u\neq\varepsilon$.

Lemma 4.5. *Consider the action sequence for* $h^*(X,A,f)=H^*(X,A\ ;f^{-1}G)$, *where* $G=Z_2$ *or* $Z[u]$, $u\in H^1(Y; Z_2)$. *Then*

(I) *//* $G=Z_2$, $\phi_\alpha{}^\varepsilon x=0$; $\chi x=\iota(0,x)$ *and* $\nu\iota(x,x')=x$ *forany* $x, x'\in H^*(X, A; Z_2)$.

(II) *If* $G=Z[u]$, ${}^\alpha u=\varepsilon$, *then* $\phi_\alpha{}^\varepsilon x=0$, $\chi x=\iota(0,x)$, *and* $\nu\iota(x, x')=x$ *for any* $x, x'\in H^*(X, A; Z[f^*u])$.

(III) *If* $G=Z[u]$; ${}^\alpha u\neq\varepsilon$, *then* $\phi_\alpha{}^\varepsilon x=\pm 2x$, $\chi x=\iota\rho x$, *and* $\nu\iota z=\delta[f^*u]z$ *for any* $x\in H^*(X, A;\ Z[f^*u])$, $z\in H^*(X, A;\ Z_2)$.

Proof of 4.4 and 4.5. Straightforward; by 4.3 and the definitions of $\phi_\alpha{}^\varepsilon$, χ, and ν.

We give a useful way of expressing ψ: $H^*(X\times T_\varepsilon, X\times jS\cup A\ T_\varepsilon;\ F_\alpha{}^{-1}G)\to H^*(X\times T_\varepsilon, X\times jS\cup A\ T_\varepsilon;\ F_\alpha{}^{-1}H)$, for G and H both Z_2 or $Z[u]$.

Let $\mathfrak{A}$ be the mod 2 Steenrod algebra. Let $\kappa: \mathfrak{A}\to\mathfrak{A}$ be the derivation of degree -1 defined by Kristensen [5]; we write $\theta'=\kappa\theta$. The following property also defines θ': if $x\in H^*(W; Z_2)$, $y\in H^1(W; Z_2)$, where W is any space, $\theta(\chi\cup y)=\theta x\cup y+\theta' x\cup Sq^1 y$.

Lemma 4.6. (I) *Let* $G=H=Z_2$, $\psi=y\cup\theta$ *for some* $y\in H^*(Y; Z_2)$, $\theta\in\mathfrak{A}$. *Then for* $x\in H^{k-1}(X, A\ \ Z_2)$, $x'\in H^{k-2}(X,A\ \ Z_2)$, $\psi\iota(x, x')=\iota(f^*_y\ \cup\theta x', f^*y\cup\theta x'+{}^\alpha y\cup\theta x+\varepsilon f^*y\cup\theta' x)$. (II) *Let* $G=Z[u], H=Z_2$, $\psi=\rho$, *reduction modulo* 2. *//* $u=\varepsilon$, $\psi\iota(x, x')=\iota(\rho x, \rho x')$ *for* $x\in H^{k-1}(X, A;\ Z[f^*u])$, $x'\in H^{k-2}(X, A;\ Z[f^*u])$. *If* ${}^\alpha u\neq\varepsilon$, $\psi\iota(x, x')=\iota(x+(Sq^1+f^*u\cup)x')$. (III) *Let* $G=Z_2$, $H=Z[v]$, $\psi=\delta[v]$. *If* ${}^\alpha v\neq\varepsilon$, $\psi\iota(x, x')=\iota(x+Sq^1x'+f^*v\cup x')$ *far* $x\in H^{k-1}(X, A;\ Z_2)$, $x'\in H^{k-2}(X, A; Z_2)$. *If* $v^\alpha=\varepsilon$, $\psi\iota(x, x')=\iota(\delta[f^*v]x,\ \delta[f^*v]x')$ *for* $x\in H^{k-1}(X, A;\ Z_2)$, $x'\in H^{k-2}(X,A;\ Z_2)$. (IV) *Let* $G=Z[u], H=Z[v]$, $\psi=D\cup$, *where* $D\in H^*(Y; Z[u+v])$. *If* ${}^\alpha u=\varepsilon$, $x\in H^{k-1}(X,A;\ Z[f^*u])$, *and* $x'\in H^{k-2}(X, A;\ Z[f^*u])$ *then* $\psi\iota(x, x')=\iota(f^*D\cup x, f^*D\cup x'+{}^\alpha D\cup x)$ *if* ${}^\alpha v=\varepsilon$, *while* $\psi\iota(x, x')=\iota({}^\alpha(\rho D)x\cup x$

$+f^*(\rho D)\cup x')$ *if* $v^{\alpha}\neq\varepsilon$. *On the other hand, if* ${}^{\alpha}u\neq\varepsilon$, *and* $x\in H^{k-2}(X, A;\ Z_2)$, $\psi\iota x=\iota(\delta[f^*v](\rho f^*D\cup x\delta[f^*v]({}^{\alpha}(\rho D)\cup x))$ *far* ${}^{\alpha}v=\varepsilon$, *while if* ${}^{\alpha}v\neq\varepsilon$, $\psi\iota x=\iota(\rho f^*D\cup x+{}^{\alpha}D\cup f^*u\cup x+{}^{*}D\cup Sq^1x)$.

Proof. Case (I) is elementary. For cases (II) and (III), the formulas given are the only ones consistent with case (I) in the respective universal examples. Case (IV) is proved somewhat similarly (it is necessary to observe that if ${}^{\alpha}u\neq{}^{\alpha}v$, $(Sq^1+f^*u\cup+f^*v\cup)^{\alpha}(\rho D)=f^*(\rho D)$) we leave the details to the reader.

If $'h^*$, $''h^*$ can both be expressed as direct sums of ordinary twisted cohomologies of types Z_2 and Z, $\Phi_{\alpha}^{\varepsilon}$ can be easily computed using 4.1, 4.4, 4.5, and 4.6. We write the specific results in all cases where each has only one summand.

Corollary 4.7. *Let* $f\colon X\to Y$, (X, A) *a C.W.-pair*, X *connected*, $\alpha\in\mathcal{H}=\pi_1(Y^X, f)$, *and* $\varepsilon\in Z_2$. *Let* $'h^*$, $''h^*$ *be twisted cohomology theories determined by sheaves of local coefficients G and H, respectively, over* Y, *and let* $\psi\colon {}'h^*\to{}''h^*$ *be a stable* Y *-twisted cohomology operation. Let* $x\in H^*(X, A\ f^{-1}G)$, *such that* $\psi x=0$, $\phi_{\varepsilon}{}^{\alpha}\alpha=0$. *Then (where in each case,* $y_\lambda\in H^*(Y;Z_2)$, $\theta_\lambda\in\mathfrak{A}$ *for each* $\lambda\in\Lambda$, *some finite indexing set; and* u, $v\in H^1(Y;Z_2)$): (I) *If* $G=H=Z_2$ *and* $\psi=\sum_{\lambda\in\Lambda}y_\lambda\cup\theta_\lambda$. *Then* $\Phi_{\alpha}^{\varepsilon}x=\sum_{\lambda\in\Lambda}{}^{\alpha}y_\lambda\cup\theta_\lambda x+\varepsilon f^*y\cup\theta_\lambda'x$. (II) // $G=Z[u]$, $H=Z_2$, *and* $\psi=\sum_{\lambda\in\Lambda}y_\lambda\cup\theta_\lambda\rho$, *then*

$$\Phi_{\alpha}^{\varepsilon}x=\begin{cases}\sum_{\lambda\in\Lambda}{}^{\alpha}y_\lambda\cup\theta_\lambda\rho x & \text{if } \varepsilon={}^{\alpha}u\\ \sum_{\lambda\in\Lambda}{}^{\alpha}y_\lambda\cup\theta_\lambda\rho x+f^*y\cup\theta_\lambda'\rho x+f^*y_\lambda\cup\theta_\lambda z & \text{if } \varepsilon\neq{}^{\alpha}\varepsilon\end{cases}$$

uwhere, if $\varepsilon\neq{}^{\alpha}u$, $z\in H^*(X,A;\ \mathrm{Z})$ *is chosen such that* $\delta[f^*u]z=x$. (III) $G=Z_2$, $H=Z[v]$, $\psi=\delta[v]\sum_{\lambda\in\Lambda}y_\lambda\cup\theta_\lambda$. *Then*

$$\Phi_{\alpha}^{\varepsilon}X=\begin{cases}\delta[f^*v]\sum_{\lambda\in\Lambda}{}^{\alpha}y_\lambda\cup\theta_\lambda x+\varepsilon f^*y_\lambda\cup\theta_\lambda'x & \text{if } \varepsilon={}^{\alpha}v\\ w+\delta[f^*v]\sum_{\lambda\in\Lambda}{}^{\alpha}y_\lambda\cup\theta_\lambda x+\varepsilon f^*y_\lambda\cup\theta_\lambda'x & \text{if } \varepsilon\neq{}^{\alpha}v\end{cases}$$

where, if $\varepsilon\neq{}^{\alpha}v$, $w\in H^*(X, A; Z[f^*v])$ *is chosen such that* $\rho w=\sum_{\lambda\in\Lambda}f^*y_\lambda\cup\theta_\lambda x$. (IV) *If* $G=Z[u]$, $H=Z[v]$, *and* $\psi=D\cup+\delta[v]\sum_{\lambda\in\Lambda}y_\lambda\cup\theta_\lambda$ *for some* $D\in H^*(Y;\ Z[u+v])$, *then*

$$\Phi_{\alpha}^{\varepsilon}x=\begin{cases}\pm{}^{\alpha}D\cup x+\delta[f^*v]\sum_{\lambda\in\Lambda}{}^{\alpha}y_\lambda\cup\theta_\lambda\rho x+\varepsilon f^*y_\lambda\cup\theta_\lambda'\rho x & \text{if } \varepsilon={}^{\alpha}u={}^{\alpha}v\\ w+\delta[f^*v]\sum_{\lambda\in\Lambda}{}^{\alpha}y_\lambda\cup\theta_\lambda\varepsilon\rho x+\varepsilon f^*y_\lambda\theta_\lambda'\rho x & \text{if } \pounds={}^{\alpha}u\,\Phi\,{}^{\alpha}v\\ \delta[f^*v]({}^{\alpha}(\rho D)\cup z+\sum_{\lambda\in\Lambda}f^*y_\lambda\cup\theta_\lambda z+{}^{\alpha}y_\lambda\cup\theta_\lambda z+\varepsilon f^*y_\lambda\cup\theta_\lambda'\rho x & \text{if } \varepsilon\neq{}^{\alpha}u\end{cases}$$

where (when necessary), $w\in H^*(X, A;\ Z[f^*v])$ *and* $z\in H^*(X, A;\ Z_2)$ *are chosen by the equations* $\rho w={}^{\alpha}(\rho D)\cup\rho x+\sum_{\lambda\in\Lambda}f^*y_\lambda\cup\theta_\lambda\rho x$, *and* $\delta[f^*u]z=x$.

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