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# IMMERSIONS AND EMBEDDINGS OF ORBIT MANIFOLDS $D_p(l, m)$ OF $S^{2l+1}\times S^m$ BY DIHEDRAL GROUP $D_p$

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1. Let a dihedral group $D_p, p$ an odd prime, act on $S^{2l+1}\times S^m$ by $g^i t^j(z,x) =(\rho^i c^j(z), (-1)^j x)$ where $g$ and $t$ are generators of order $p$ and 2 respectively, $c(z)$ is the conjugate point of $z$ and p=exp $2\sqrt{-1}/p$. We denote by $D_p(l, m)$ the orbit space [6].

Let $\phi(m)$=the number of integers $s$ with $0<s\leqq m$ and $s\equiv 0$, 1, 2 or 4 mod 8;

$\bar{\sigma}(l, m)$=the largest integer $s$ with $2^{s-1}\binom{l+m+s+1}{s}\not\equiv 0 \bmod 2^{\phi(m)}$;

$L(l, p)$=the largest integer $s$ with $s\leqq\left[\frac{[illegible]}{2}\right]$ and $\binom{l+s}{s}\not\equiv 0$ mod $p^{1+[l-2s/p-1]}$;

$$\sigma^*(l, m)=\begin{cases}\max(\bar{\sigma}(l, m),\ 2L(l, p)) & \text{if } m>0\,, \\ 2L(l, p) & \text{if } m=0\,.\end{cases}$$

In this paper we obtain

**Theorem 1.1.**
(i) *$D_p(l, m)$ cannot be immersed in $R^{2l+m+\sigma^*(l,m)}$*,
(ii) *$D_p(l, m)$ cannot be embedded in $R^{2l+m+\sigma^*(l,m)+1}$*.

In §2, we discuss about $\widetilde{KO}(D_p(l,m))$, $l\not\equiv 0$ mod 4. In §3, we study the tangent bundle of $D_p(l, m)$. In §4, the Grothendieck operators $\gamma^i$ in $KO(D_p(l,m))$ are computed and Theorem 1.1 is proved.

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2. Let G be a finite group. The symbol $\Lambda$ will denote either $O$ or $U$. Let $K\Lambda_G(X)$ be equivariant $K\Lambda$-group of a G-space $X$. It is well-known that if the action of $G$ is free then $K\Lambda_G(X)\cong K\Lambda(X/G)$. There is canonical homomorphism from the representation ring $RA(G)$ to $K\Lambda_G(X)$ which maps a representation space $M$ over real field or complex field to an equivariant $G$-bundle $X\times M$. If $X$ is a free G-space then there is a homomorphism

$$\pi_\Lambda\colon\ R\Lambda(G) \to K\Lambda_G(X) \cong K\Lambda(X/G)\,.$$

Let $a\colon H \to G$ be a homomorphism and $/\colon Y \to X$be an equivariant map from an $H$-space $Y$ to a $G$-space $X, f(hy)=\alpha(h)f(y)$. The equivariant map / induces the homomorphism

$$f_\Lambda{}^!\colon\ K\Lambda_G(X) \to K\Lambda_H(Y) \qquad [8]\,.$$

We take a $Z_2$-action on $S^m$, a $Z_p$-action on $S^{2l+1}$ and a $D_p$-action on $S^{2l+1} \times S^m$ defined by $t\ x=-x$, $g\ z=\rho z$ and $g\ t(x,z)=(\rho c(z),-x)$ respectively [4]. These orbit space are an $m$-dimensional real projective space $RP^m$, a $(2l+1)$-dimensional lens space $L^l(p)$ and $D_p(l,m)$. There exist equivariant maps

$$\begin{array}{llll} i\colon & S^{2l+1} \to S^{2l+1}\times S^m, & i(z) = (z, (1, 0, \cdots, 0))\,, \\ j\colon & S^m \to S^{2l+1}\times S^m, & j(x) = ((1, 0, \cdots, 0), x) \end{array}$$

and

$$p\colon\ S^{2l+1}\times S^m \to S^m, \qquad p(z, x) = x$$

compatible with injections $\tilde{i}\colon Z_p \to D_p$, $\tilde{j}\colon Z_2 \to D_p$ and a projection $p\colon D_p \to Z_2$ respectively. It follows immediately that $j_\Lambda{}^! p_\Lambda{}^! = 1$.

Let $H$ be a normal subgroup of a finite group $G$ and $A$ be a representation of $H$. Denote by $A^G$ the induced representation (§2 in [4]). Throughout this section, we suppose that $l \not\equiv 0 \bmod 4$. Consider the following commutative diagram

$$\begin{array}{ccc} \widetilde{RU}(Z_p) & \xrightarrow{\pi_U} \quad \widetilde{\ } \qquad & \widetilde{\ }(L^l(p)) \\ \Big\downarrow r_1 & & \Big\downarrow r \\ \widetilde{RO}(Z_p) & \xrightarrow{\pi_0} \widetilde{KO}_{Z_p}(S^{2l+1}) \cong & \widetilde{KO}(L^l(p)) \end{array}$$

where $r_1$, $r$ are real restrictions. It follows from T. Kambe [5] and N. Mahammed [7], that $r$ and $\pi_U$ are surjective. Therefore we have that $\pi_0$ is surjective. We define the homomorphism

$$i_*^\Lambda\colon\ \widetilde{K\Lambda}_{Z_p}(S^{2l+1}) \to \widetilde{K\Lambda}_{D_p}(S^{2l+1}\times S^m)$$

by $i_*^\Lambda(S^{2l+1}\times M)=S^{2l+1}\times S^m \times M^{D_p}$ where $M$ is a representation space of $Z_p$ and $M^{D_p}$ is the induced representation space.

**Lemma 2.1.** *The following diagram is commutative.*

$$\begin{array}{ccc} \widetilde{KU}_{Z_p}(S^{2l+1}) & \underset{i_U^!}{\overset{i_*^U}{\rightleftarrows}} & \widetilde{KU}_{D_p}(S^{2l+1}\times S^m) \\ \Big\uparrow c & & \Big\uparrow c \\ \widetilde{KO}_{Z_p}(S^{2l+1}) & \underset{i_O^!}{\overset{i_*^O}{\rightleftarrows}} & \widetilde{KO}_{D_p}(S^{2l+1}\times S^m) \end{array}$$

*where c is the complexification.*

Proof. This lemma is obtained by the naturality of complexification $c$ and $(cM)^{D_p}=c(M^{D_p})$, where $M$ is a representation space of $Z_p$ over real field. q.e.d.

**Proposition** 2.2. *For* $\eta\in\widetilde{KO}_{Z_p}(S^{2l+1})$,

$$i_O^! i_*^O(\eta)=2\eta\,.$$

Proof. Since $r$ is surjective, there is $\beta$ in $\widetilde{KU}_{Z_p}(S^{2l+1})$ with $r(\beta)=\eta$. $cr(\beta)=\beta+\bar{\beta}$ is an element of $KU_{Z_p}(S^{2l+1})^{Z_2}$ which is a subgroup of $KU_{Z_p}(S^{2l+1})$ consisting of elements fixed under the conjugation automorphism. We obtain $i_U^! i_*^U(cr(\beta))=2cr(\beta)$ (cf [4] Proposition 2.1). By Lemma 2.1, $ci_O^! i_*^O(\eta)=c(2\eta)$. Since $c$ is injective, we have $i_O^! i_*^O(\eta)=2\eta$. q.e.d.

By the same way as Theorem 2.2 in [4], we have the following.

**Theorem** 2.3. *The homomorphism*

$$\theta:\ \widetilde{KO}_{Z_p}(S^{2l+1})\oplus\widetilde{KO}_{Z_2}(S^m)\to\widetilde{KO}_{D_p}(S^{2l+1}\times S^m)$$

*giveh by* $\theta(\eta,\nu)=i_*^O(\eta)+p_O^!(\nu)$ *is injective.*

3. Consider maps $i: L^l(p)\to D_p(l,m)$, $j: RP^m\to D_p(l,m)$ and $p: D_p(l,m)\to RP^m$ which are induced by $i, j$ and $p$ in §2. Let $\pi: L^l(p)\to CP^l$ be a canonical projection. Denote by $\eta$ and $\xi$ the canonical line bundles over the complex projective space $CP^l$ and the real projective space $RP^m$ respectively.

**Proposition 3.1.** (cf. [3], [9]). *There is a real 2-plane bundle* $\eta_1$ *over* $D_p(l,m)$ *satisfying the following conditions:*

(i) $i^!\eta_1$ *is equivalent to* $r\pi^!\eta$,

(ii) $\eta_1$ *for* $l=0$ *is the 2-plane bundle* $1\oplus p^!\xi$,

(iii) $\eta_1\otimes p^!\xi$ *is equivalent to* $\eta_1$,

(iv) $j^!\eta_1$ *is equivalent to* $1\oplus\xi$,

*where r is the real restriction.*

Proof. Each point of $D_p(l,m)$ can be represented by $[z,x]$ under the identification $(z,x)=(\rho^k c(z),-x)$ for $z\in S^{2l+1}\subset C^{l+1}$, $x\in S^m\subset R^{m+1}$. Then the total space $E(\eta_1)$ of $\eta_1$ is defined as set of all triples $[(z,x),y]$ under the identification $((z,x),y)=((\rho^k c(z),-x),\rho^k\bar{y})$, where $y\in C$ and $z$, $x$ are as above. Let $U_{\alpha\beta}$ be the set of points $[z,x]$ of $D_p(l,m)$ such that $z_\alpha$ and $X\beta$ are non-zero. $\{U_{\alpha\beta}:\alpha=0,1, \ ,l;\ \beta=0,1,\cdots,m\}$ is an open covering of $D_p(l,m)$. The projection of the bundle $\eta_1$ is given by $p_1([(z,x),y])=[z,x]$

Define $\phi_{\alpha\beta}:\ U_{\alpha\beta}\times R^2\to p_1^{-1}(U_{\alpha\beta})$ by

$$\phi_{\alpha\beta}([z, x], y) = \begin{cases} [(z, x), z_\alpha y] & \text{if} \quad x_\beta > 0 , \\ [(z, x), z_\alpha \bar{y}] & \text{if} \quad x_\beta < 0 . \end{cases}$$

Then $\phi_{\alpha\beta}$ is a chart of $\eta_1$ over $U_{\alpha\beta}$ and the transition functions are given as follows:

$$g_{(\gamma,\delta)(\alpha,\beta)}[z, x] = \begin{cases} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} & x_\beta, x_\delta > 0 , \\ \begin{pmatrix} a & -b \\ -b & -a \end{pmatrix} & x_\beta > 0, x_\delta < 0 , \\ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} & x_\beta < 0, x_\delta > 0 , \\ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} & x_\beta, x_\delta < 0 , \end{cases}$$

where $z_\alpha/z_\gamma = a + b\sqrt{-1}$, $a, b \in R$.

Let $U_\alpha = \{[z_0, \cdots, z_l] : z_\alpha \neq 0\} \subset L^l(p)$, then the transition function of $\eta$, $g_{\alpha\beta}$: $U_\alpha \cap U_\beta \to GL(1,C)$, is given by $g_{\alpha\beta}[z] = z_\beta/z_\alpha$ Take $V_\alpha = \{[x_0, \cdots, x_m] : x_\alpha \neq 0\} \subset RP^m$, then the transition function of $\xi$, $g'_{\alpha\beta}$: $V_\alpha \cap V_\beta \to O(1)$, is given by the following

$$g'_{\alpha\beta}[x] = \begin{cases} 1 & x_\alpha x_\beta > 0 , \\ -1 & x_\alpha x_\beta < 0 . \end{cases}$$

Therefore, we can complete the proof. q.e.d.

Let $\tau(M)$ denote the tangent bundle of manifold $M$.

**Proposition 3.2.**

$$\tau(D_p(l, m)) \oplus 1 \cong p^! \tau(RP^m) \oplus (l+1)\eta_1 .$$

Proof. Let $\langle\,|\,\rangle$ and $(\,|\,)$ denote the real and complex inner products of $R^{m+1}$ and $C^{l+1}$ respectively. The total space of the real tangent vector bundle of $D_p(l, m)$ can be represented as the set of all pairs $[(z, x), (u, v)]$ with $z \in S^{2l+1}$, $x \in S^m$, $u \in C^{l+1}$, $v \in R^{m+1}$, $(z|u) = 0$ or $u = r\sqrt{-1} \cdot z$ for some $r \in R$, and $\langle x|v \rangle = 0$, under the identification $((z, x), (u, v)) = ((\rho^k \bar{z}, -x), (\rho^k \bar{u}, -v))$. We have the following decomposition

$$\tau(D_p(l, m)) = p^! \ \tau(RP^m) \oplus \zeta ,$$

where the total space $E(\zeta)$ of $\xi$ is the set of all triple $[(z, x), u]$ with $(z|u) = 0$ or $u = r\sqrt{-1}\ z$ for some $r \in R$, under the identification $((z, x), u) = ((\rho^k \bar{z}, -x), \rho^k \bar{u})$ in $S^{2l+1} \times S^m \times C^{l+1}$. The total space $E((l+1)\eta_1)$ of the $(l+1)$-fold bundle sum $(l+1)\eta_1$ can be represented as the set of all triple $[(z, x), u]$ with the identification $((z, x), u) = ((\rho^k \bar{z}, -x), \rho^k \bar{u})$ in $S^{2l+1} \times S^m \times C^{l+1}$. Then we have $E((l+1)\eta_1)$

$\supset E(\zeta)$. Consider the trivial line bundle $\theta$ over $D_p(l, m)$ whose total space is represented as $[(z, x), t\ z]$ modulo the identification $((z, x), t\ z)=((\rho^k\bar{z}, -x), t\ \rho^k\bar{z})$, where $z\in S^{2l+1}$, $x\in S^m$ and $t\in R$. Then we have

$$\tau(D_p(l, m))\oplus\theta\cong p^!\tau(RP^m)\oplus(l+1)\eta_1. \qquad \text{q.e.d.}$$

4. We use $\lambda^i$- and $\gamma^i$-operations in $KO$-theory [2] to study the immersion and embedding of the manifold $D_p(l, m)$. Put $\tau_0(D_p(l,m))=\tau(D_p(l,m))-(2l+m+1)$ and $x=p^!\xi-1$, $z=\eta_1-2$ and $y=z-x(=\eta_1-1-p^!\xi)$ in $\widetilde{KO}(D_p(l,m))$. It follows from Proposition 3.2, that

$$-\tau_0(D_p(l,m)) = -(m+1)x-(l+1)z = -(l+m+2)x-(l+1)y$$

and

$$(4.1) \qquad \gamma_t(-\tau_0(D_p(l, m)) = \gamma_t(x)^{-(l+m+2)}\gamma_t(y)^{-(l+1)}.$$

Then, we have the following.

**Lemma 4.1.** (i) $\gamma_t(z)=1+zt-yt^2$,
(ii) $x\cdot y=0$, $\gamma_t(x)=1+xt$,
(iii) $\gamma_t(y)=1+yt-yt^2$.

Proof. (i) Comparing the transition functions $\lambda^2(\eta_1)$ with one of $p^!\xi$, we have $\lambda^2(\eta_1)\cong p^!\xi$. Therefore we obtain $\lambda_t(\eta_1)=1+\eta_1t+p^!\xi t^2$ and $\lambda_t(z)=\lambda_t(\eta_1-2)=\lambda_t(\eta_1)\lambda_t(1)^{-2}=(1+\eta_1t+p^!\xi t^2)/(1+t)^2$. Hence, $\gamma_t(z)=\lambda_{t/(1-t)}(z)=1+zt-yt^2$.

(ii) We note that $p^!\xi\otimes p^!\xi\cong 1$. And recall $p^!\xi\otimes\eta_1\cong\eta_1$, from Proposition 3.1, (iii). We have

$$x\cdot y = (p^!\xi-1)\cdot(\eta_1-1-p^!\xi) = 0$$

in $\widetilde{KO}(D_p(l,m))$. Since $\gamma_t(\xi-1)=1+(\xi-1)t$, we have

$$\gamma_t(x) = \gamma_t(p^!\xi-1) = 1+xt.$$

(iii) Making use of the relation $x\ y=0$, we have

$$\gamma_t(y)= \gamma_t(z-x) = \gamma_t(z)\ \gamma_t(x)^{-1} = 1+yt-yt^2. \qquad \text{q.e.d.}$$

Noting that $x^2=-2x$, we obtain the following proposition from (4.1) and Lemma 4.1.

**Proposition 4.2.**

$$\gamma_t(-\tau_0(D_p(l, m))) = \left(-\sum_{s=0}^{\infty}\binom{m+l+s+1}{s}2^{s-1}xt^s\right).$$

$$\left(\sum_{k=0}^{\infty}(-1)^k\binom{l+k}{k}y^k(t-t^2)^k\right).$$

Proof of Theorem 1.1.

Since $pj=1$, $j^!x=\xi-1$ is the generator of $\widetilde{KO}(RP^m)$ and $j^!x$ is of order $2^{\phi(m)}$ by J.F. Adams [1]. On the other hand, $i^!y=r(\pi^!\eta-1_C)$ by $i^!p^!\xi-1$ and Proposition 3.1. (i). By T. Kambe [5], $i^!y, i^!y^2, \cdots, i^!y^{p-1/2}$ are additive generators of $p$-components of $\widetilde{KO}(L^l(p))$ and $i^!y^k$ is of order $p^{1+[(l-2k)/(p-1)]}$. We investigate the power of $t$ having non-zero coefficient $\gamma^k(-\tau_0(D_p(l,m))$ in the expansion of $\gamma_t(-\tau_0(D_p(l,m))$ and apply the theorem of Atiyah [2] to the non-immersion and non-embedding of $D_p(l, m)$ in $R^{2l+m+k+1}$. Then, we obtain the theorem.

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