Daley, R.P.
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A NOTE ON A RESULT OF KAMAE*

ROBERT P. DALEY

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In [3] Kamae proved that for each constant c, there exists a finite string y such that

$$K(y)-K(y|x)>c$$

for all but finitely many finite strings x, where $K(\ )$ and $K(\ \backslash\ \cdot)$ are the unconditional and conditional minimal-program complexity measures respectively of Kolmogorov [4]. By considering infinite sequences we are able to obtain a slightly stronger statement of this result.

Let X^∞ denote the set of all infinite binary strings. For $x\in X^\infty$ let x^n denote the initial segment of x of length n, i.e., $x^n=x(1)\ \ x(n)$. To simplify matters we will associate with each finite binary string y the integer n whose binary representation is 1 y. By this means we will consider complexity expressions of the form $K(x^n)$, $K(x^n|m)$ and $K(m)$ for $x\in X^\infty$ and integers n and m. Then $K(x^n|n)$ is Kolmogorov's restricted conditional complexity. By a recursively enumerable sequence we mean the characteristic sequence of a recursively enumerable set. By "$\overset{\infty}{\exists}n$" and "$\overset{\infty}{\forall}n$" we mean respectively "There exist infinitely many integers n" and "For all but finitely many integers n."

Theorem. *There exists a recursively enumerable sequence x, such that*

$$\forall c\overset{\infty}{\exists}n\overset{\infty}{\forall}m.\quad K(x^n|n)-K(x^n|m)>c\,.$$

Proof. We need the following two lemmas.

Lemma. *For every recursively enumerable sequence x,*

$$\exists c_1\forall n\overset{\infty}{\forall}m.\quad K(x^n|m)\leq K(n)+c_1\,.$$

Proof. Let h be a total recursive function which enumerates the 1's of x, i.e., $x(i)=1\Leftrightarrow\exists j.\,h(j)=i$. Define the total recursive function $f(n,m)$ as follows:
Step 1: Enumerate via h the first m 1's of the sequence x, i.e. compute $h(1),\cdots,h(m)$.
Step 2: Output the following finite string y of length n— $y(i)=1\Leftrightarrow i$ appears on

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the list generated in Step 1, i.e. $\exists j \leq m. h(j)=i$.

Using f and the asymptotically optimal algorithm for $K(\ \backslash\ \cdot)$ the result follows.

Lemma. *There exists a recursively enumerable sequence x such that* $\forall n.$ $K(x^{2^n} \mid 2^n) > n-1$.

Proof. Let A by the asymptotically optimal algorithm for $K(\ \mid\ )$ and define the sequence x as follows: For each n and each m such that $1 \leq m \leq 2^{n-1}$, $x(2^{n-1}+m)=1 \Leftrightarrow m^{th}$ digit of $A(m, 2^n)$ is 0. Clearly x is recursively enumerable and $x^{2^n} \neq A(y,2^n)$ for all programs y of length $\leq n-1$.

Combining these two lemmas we have that there exists a recursively enumerable sequence x such that

$$\exists c_1 \forall n \overset{\infty}{\forall} m. \quad K(x^{2^n} \mid 2^n) - K(x^{2^n}\ m) > n-1-c_1-K(2^n).$$

While it is true that $K(2^n)$ for most integers of the form 2^n is about n, there are infinitely many such integers which have descriptions of arbitrarily (in an effective sense) short length relative to n, i.e. $\forall c \overset{\infty}{\exists} n. n-K(2^n) > c$. (see [1]) Combining these two inequalities yields the result. Q.E.D.

Recently, Chaitin (see [2]) has shown that a sequence x is recursive $\Leftrightarrow \exists c \forall n. K(x^n) \leq K(n)+c$. Combining this result with the first lemma above we have immediately

Corollary. *For every non-recursive recursively enumerable sequence x,*

$$\forall c \overset{\infty}{\exists} n \overset{\infty}{\forall} m. \quad K(x^n) - K(x^n \mid m) > c.$$

However, the above theorem is the best result obtainable for any recursively enumerable sequence in view of the following result which is proved in [1].

Theorem. *For every recursively enumerable sequence x*

$$\exists c \overset{\infty}{\exists} n. \quad K(x^n \mid n) \leq c\ .$$

THE UNIVERSITY OF CHICAGO

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