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ON STABLE JAMES NUMBERS OF QUATERNIONIC PROJECTIVE SPACES

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In [4], we have defined the stable James numbers $k_s(X, A)$ for some finite CW-pairs (X, A) and computed $d_C(n)=k_s(CP^n, CP^1)$. In this note we estimate $d_H(n)=k_s(HP^n, HP^1)$, where HP^n denotes the quaternionic projective space of topological dimension $4n$. We obtain

Theorem. *For $n\geqq 2$*

(0) *$d_H(n)$ is a factor of $(2n)!\,(2n-2)!\cdots 4!$, in particuler none of the prime factors of $d_H(n)$ is greater than $2n$,*

$$\text{(i)}\quad 2j+1\leqq\begin{cases} \nu_2(d_H(n))\leqq 3j+3 & \text{for}\quad n=2^j\,, \\ \nu_2(d_H(n))\leqq 3j+6 & \text{for}\quad 2^j+1\leqq n<2^{j+1}\,, \end{cases}$$

$$\text{(ii)}\quad 2j\leqq\nu_3(d_H(n))\leqq 2j+2 \qquad \text{for}\quad 3^j\leqq n<2\cdot 3^j\,,$$

$$2j+1\leqq\begin{cases} \nu_3(d_H(n))\leqq 2j+2 & \text{for}\quad 2\cdot 3^j\leqq n<\dfrac{3^{j+2}+1}{4}\,, \\ \nu_3(d_H(n))\leqq 2j+4 & \text{for}\quad \dfrac{3^{j+2}+1}{4}\leqq n<3^{j+1}\,, \end{cases}$$

(iii) *for a prime $p\geqq 5$*

$$\nu_p(d_H(n))=2j \qquad \text{for}\quad p^j\leqq n<\frac{p^{j+1}+1}{4}\,,$$

$$2j\leqq\nu_p(d_H(n))\leqq 2j+2 \quad \text{for}\quad \frac{p^{j+1}+1}{4}\leqq n<\frac{p+1}{2}p^j\,,$$

$$2j+1\leqq\nu_p(d_H(n))\leqq 2j+2 \quad \text{for}\quad \frac{p+1}{2}p^j\leqq n<p^{j+1}\,,$$

where $\nu_p(m)$ denotes the exponent of p in the prime factorization of m.

Recall that $d_H(n)=$the index of the image of $i^*\colon \{HP^n, S^4\}\to\{S^4, S^4\}$, where $\{X, Y\}$ denotes the set of stable homotopy classes of stable maps $X\to Y$ and $i\colon S^4=HP^1\to HP^n$ the natural inclusion. Then obviously $d_H(1)=1$.

1. Lower bound of $d_H(n)$

In this section we use K-theories. We introduce the following notations: ξ_n=the canonical quaternionic line bundle over HP^n; $g_H=\xi_1-1\in\widetilde{KSp}(S^4)$; $g_R=g_H\wedge g_H\in\widetilde{KO}^{-4}(S^4)$; $\xi_n=g_H\wedge(\xi_n-1)\in\widetilde{KO}^{-4}(HP^n)$; η=the canonical complex line bundle over $S^2=CP^1$; $g_C=\eta-1\in\tilde{K}(S^2)$; ε: $KO^*(\ )\to K^*(\ )$, the complexification; c: $KSp(\ )\to K(\ )$, the scaler restriction; ch: $K^*(\ )\to H^*(\ ;Q)$, the Chern character; $y_{2k}=g_R^{-k}\in KO^{8k}$; $y_{2k+1}\in KO^{8k+4}$, the generator such that $\varepsilon(y_{2k+1})=2g_C^{-4k-2}$; $z_n=c(\xi_n-1)\in\tilde{K}(HP^n)$; $t\in H^4(HP^n;Z)$, the first symplectic Pontrjagin class of ξ_n. Then we have

$$\begin{aligned}\varepsilon(y_{2k}\xi_n^{2k+1}) &= g_C^2z_n^{2k+1}\,,\\ \varepsilon(y_{2k-1}\xi_n^{2k}) &= 2g_C^2z_n^{2k}\,,\\ ch(z_n) &= \exp(\sqrt{t})+\exp(-\sqrt{t})-2\,,\end{aligned}$$

and $\widetilde{KO}^{-4}(HP^n)$ is the free group with basis $\xi_n,\ y_1\xi_n^2,\cdots,y_{n-1}\xi_n^n$, and $K(HP^n)$ is the truncated polynomial ring over Z with generator z_n and the relation $z_n^{n+1}=0$.

Choose $f\in\{HP^n,S^4\}$ such that the composition $S^4\xrightarrow{i}HP^n\xrightarrow{f}S^4$ is of degree $d_H(n)$. Put

$$f^*(g_R)=\sum_{j=1}^n a_jy_{j-1}\xi_n^j\,,\quad a_j\in Z\,.$$

And put $2a_{2j}=b_{2j}$ and $a_{2j+1}=b_{2j+1}$. Then, by the above equations, we have

$$d_H(n)t=f^*\cdot ch\cdot\varepsilon(g_R)=ch\cdot\varepsilon\cdot f^*(g_R)=\sum_{j=1}^n b_j(\exp(\sqrt{t})+\exp(-\sqrt{t})-2)^j$$

in $H^*(HP^n;Q)$ and hence $b_1=d_H(n)$. Put $t=x^2$, then we have

$$b_1x^2\equiv\sum_{j=1}^n b_j(\exp(x)+\exp(-x)-2)^j \bmod x^{2n+2}\,.$$

Differentiating this equation twice, we have

$$2b_1\equiv 2b_1+\sum_{j=1}^{n-1}(j^2b_j+2(2j+1)(j+1)b_{j+1})(\exp(x)+\exp(-x)-2)^j \bmod x^{2n}\,.$$

Hence

$$j^2b_j+2(2j+1)(j+1)b_{j+1}=0\qquad\text{for}\ \ j\leqq n-1\,,$$

and therefore

$$j!\,b_1=(-1)^j2^j\cdot3\cdot5\cdots(2j+1)(j+1)b_{j+1}\qquad\text{for}\ \ j\leqq n-1\,.$$

That is, we have

$$(2j)!\,d_H(n)=2^{2j}\cdot3\cdot5\cdots(4j+1)(2j+1)a_{2j+1}\qquad\text{for}\ \ j\leqq\left[\frac{n-1}{2}\right],$$

$$(2j-1)!\,d_H(n) = -2^{2j-1}\cdot 3\cdot 5\cdots(4j-1)(2j)2a_{2j} \qquad \text{for } j\leqq\left[\frac{n}{2}\right].$$

Put

$$\tau_p(n) = \max_{\substack{j\leqq\left[\frac{n-1}{2}\right]\\ k\leqq\left[\frac{n}{2}\right]}} \{\nu_p(2^{2j}\cdot 3\cdot 5\cdots(4j+1)(2j+1))-\nu_p((2j)!)\,,$$
$$\nu_p(2^{2k-1}\cdot 3\cdots(4k-1)(2k)2)-\nu_p((2k-1)!)\}\,.$$

Then obviously $\tau_p(n)\leqq\nu_p(d_H(n))$. Elementary calculation shows that

$$\tau_2(n) = 2j+1 \quad \text{for} \quad 2^j\leqq n<2^{j+1} \quad \text{and} \quad j\geqq 1\,,$$

and for an odd prime p

$$\tau_p(n) = \begin{cases} 2j & \text{for} \quad p^j\leqq n<\dfrac{p+1}{2}p^j \\ 2j+1 & \text{for} \quad \dfrac{p+1}{2}p^j\leqq n<p^{j+1}\,. \end{cases}$$

Thus we obtain the lower estimates of Theorem.

2. Upper bound of $d_H(n)$

The canonical fibration $S^{4n-1}\xrightarrow{\tilde{p}_{n-1}}HP^{n-1}$ factorizes as the composition of the canonical fibrations $S^{4n-1}\xrightarrow{p_{2n-1}}CP^{2n-1}\longrightarrow HP^{n-1}$. The order of p_{2n-1} as a stable map is $(2n)!$ [2], [5]. Hence the stable order of $\tilde{p}_{n-1}$ is a factor of $(2n)!$. Therefore $d_H(n)$ is a factor of $(2n)!\,d_H(n-1)$. This implies Theorem (0).

Lemma 1. *Let X be a simply connected finite CW- complex with a base point. Then the natural inclusion $SP^m(X)\xrightarrow{\iota_m}SP^\infty(X)$ induces isomorphisms $\pi_k(SP^m(X))\to\pi_k(SP^\infty(X))$ for $k\leqq m$, where $SP^m(X)$ and $SP^\infty(X)$ denote the m-fold symmetric product of X and the infinite symmetric product of X respectively* [1].

Proof. There is a commutative diagram [3] for $j\geqq 1$

$$\begin{array}{ccc} H_j(SP^m(X)) & \xrightarrow{\quad\iota_{m*}\quad} & H_j(SP^\infty(X)) \\ \downarrow\cong & & \downarrow\cong \\ \sum_{k=1}^{m} H_j(SP^k(X), SP^{k-1}(X)) & \to & \sum_{k=1}^{\infty} H_j(SP^k(X), SP^{k-1}(X)) \end{array}$$

where the bottom map is the natural inclusion. Then, it follows from $H_j(SP^k(X), SP^{k-1}(X))=0$ for $1\leqq j<k$ that ι_{m*}: $H_j(SP^m(X))\to H_j(SP^\infty(X))$ are isomorphic for $j\leqq m$. Since $SP^m(X)$ is simply connected, the result follows

from the theorem of J.H.C. Whitehead.

The obstructions to extending the natural inclusion $S^4 \to SP^{4n-1}(S^4)$ over HP^n lie in $H^{4j}(HP^n, S^4) \otimes \pi_{4j-1}(SP^{4n-1}(S^4))$ for $2 \leqq j \leqq n$. Lemma 1 shows that $SP^{4n-1}(S^4)$ and $SP^\infty(S^4) = K(Z, 4)$, the Eilenberg-MacLane complex, have the same $4n$-type. Hence, in particuler, we have $\pi_{4j-1}(SP^{4n-1}(S^4)) = 0$ for $j \leqq n$. Therefore we have a map f: $HP^n \to SP^{4n-1}(S^4)$ which factorizes $S^4 \xrightarrow{i} SP^{4n-1}(S^4)$ as $S^4 \subset HP^n \xrightarrow{f} SP^{4n-1}(S^4)$. This implies that $d_H(n)$ is a factor of $k_s^{4n-1,4} = k_s(SP^{4n-1}(S^4), S^4)$. $k_s^{4n-1,4}$ and $k_s^{4n-1,5}$ are factors of $k_s^{4n-1,5}$ and $k^{4n-1,5}$ respectively [4]. Hence $d_H(n)$ is a factor of $k^{4n-1,5}$.

We require the following theorem of Ucci [6]:

$$\nu_2(k^{m,2t+1}) \leqq \phi(2t)\beta_2(m)\,,$$
$$\nu_p(k^{m,2t+1}) = t\beta_p(m) \quad \text{for an odd prime } p$$

where $\beta_p(m)$ is defined by $p^{\beta_p(m)} \leqq m < p^{\beta_p(m)+1}$ and $\phi(s)$ is the number of integers u such that $0 < u \leqq s$ and $u \equiv 0$, 1, 2, or 4 mod 8.

By this theorem, we have

$$\nu_2(d_H(n)) \leqq 3\beta_2(4n-1)\,,$$
$$\nu_p(d_H(n)) \leqq 2\beta_p(4n-1) \quad \text{for an odd prime } p.$$

Then the following lemma completes the proof of Theorem.

Lemma 2.

$$\text{(i)} \quad \beta_2(4n-1) = \begin{cases} j+1 & \text{for} \quad n = 2^j\,, \\ j+2 & \text{for} \quad 2^j+1 \leqq n < 2^{j+1}\,, \end{cases}$$

$$\text{(ii)} \quad \beta_3(4n-1) = \begin{cases} j+1 & \text{for} \quad 3^j \leqq n < \dfrac{3^{j+2}+1}{4}\,, \\ j+2 & \text{for} \quad \dfrac{3^{j+2}+1}{4} \leqq n < 3^{j+1}\,, \end{cases}$$

(iii) *for a prime* $p \geqq 5$

$$\beta_p(4n-1) = \begin{cases} j & \text{for} \quad p^j \leqq n < \dfrac{p^{j+1}+1}{4}\,, \\ j+1 & \text{for} \quad \dfrac{p^{j+1}+1}{4} \leqq n < p^{j+1}\,. \end{cases}$$

The proof of this lemma is easy, and we omit it.

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