Nakaoka, M.
Osaka J. Math.
12 (1975), 197-208

CONTINUOUS MAPS OF MANIFOLDS WITH INVOLUTION III

MINORU NAKAOKA

(Received February 28, 1974)

Introduction

Let N and M be m-dimensional closed manifolds on each of which an involution T is given, and let $f:N\to M$ be a continuous map. In the preceding paper [7], on the assumption that the involutions T of M and N are both free the author introduced a mod 2 integer $\hat{\chi}(f)$ called the equivariant Lefschetz number of f, and proved that if $\hat{\chi}(f)\neq 0$ then f has an equivariant point. In this paper the result will be generalized to the case when the involution T of N is not necessarily free.

The former result was proved through the use of the equivariant point index $\hat{I}(f)$, which is constructed from the class $\Delta_\infty\in H^m(S^\infty\underset{T}{\times}M^2)$ requiring that the involution T of N is free (see [7]). Taking in place of $S^\infty\underset{T}{\times}M^2$ the pair of the symmetric product of M and its diagonal, we define a new equivariant point index $\hat{I}(f)$ provided that the involution of M is free and the involution of N is non-trivial. The new result will be proved by making use of the new index.

Recently, to show that certain homotopy classes in closed manifolds cannot be realized by embedded sphere, R. Fenn [5] has proved a theorem of the Borsuk-Ulam type. In this paper, $\hat{I}(f)$ will be also used to generalize the Fenn theorem.

Throughout this paper, the homology and cohomology with coefficients in $\boldsymbol{Z}_2$ are to be understood.

1. Preliminaries

Let N be a compact polyhedron with PL(=piecewise linear) involution T. We denote by F the fixed point set of T, and by N_T the quotient of N with respect to T. Let $\pi: N\to N_T$ be the projection, and put $F_T=\pi(F)$. The following facts are well known (see [1], [2], [6], [9], [10]).

There are the transfer homomorphism

$$\phi^*: H^q(N)\to H^q(N_T, F_T)$$

and the Smith homomorphism

$$\mu\colon H^q(N_T, F_T) \to H^{q+1}(N_T, F_T)\,.$$

These are natural with respect to equivariant maps, and the following two sequences are exact.

$$\begin{aligned}(1.1)\quad &\cdots \to H^q(N, F) \xrightarrow{\phi^*\circ j^*} H^q(N_T, F_T) \xrightarrow{\mu} H^{q+1}(N_T, F_T)\\ &\xrightarrow{\pi^*} H^{q+1}(N, F) \to \cdots\,,\\ (1.2)\quad &\cdots \to H^q(N) \xrightarrow{\phi^*} H^q(N_T, F_T) \xrightarrow{j^*\circ\mu} H^{q+1}(N_T)\\ &\xrightarrow{\pi^*} H^{q+1}(N) \to \cdots\,,\end{aligned}$$

where $j^*\colon H^*(N, F)\to H^*(N)$ and $j^*\colon H^*(N_T, F_T)\to H^*(N_T)$ are induced by the inclusions.

We have

(1.3) **Lemma.** *Let N be an n-dimensional closed PL manifold with a non-trivial PL involution T. Then it holds*

$$\phi^*\colon H^n(N)\cong H^n(N_T, F_T)\,.$$

Proof. Since $H^n(N)\cong \boldsymbol{Z}_2$ and $H^{n+1}(N_T)=0$ in (1.2), it suffices to prove that $\phi^*\colon H^n(N)\to H^n(N_T, F_T)$ is not trivial. Suppose this is trivial. Then we have $H^n(N_T, F_T)=0$, and so $H^n(N, F)=0$ by (1.1). By duality we have $H_0(N-F)=0$ which implies $N=F$. Since this contradicts to the non-triviality of T, we have proved (1.3).

For each $t=1, 2, \cdots$, we define a homomorphism

$$E_t\colon H^q(F) \to H^{q+t}(N_T, F_T)$$

to be the composite

$$\begin{aligned}H^q(F)\overset{\pi^*}{\cong} H^q(F_T) &\xrightarrow{\delta^*} H^{q+1}(N_T, F_T) \xrightarrow{\mu} H^{q+2}(N_T, F_T)\\ &\xrightarrow{\mu} \cdots \xrightarrow{\mu} H^{q+t}(N_T, F_T)\,,\end{aligned}$$

where δ^* is the coboundary homomorphism.

For a compact polyhedron M, we consider the product $M^2=M\times M$ and define an involution T on M^2 by $T(x_1, x_2)=(x_2, x_1)$. The quotient $(M^2)_T$ is the *symmetric product*. If $D\colon M\to M^2$ denote the diagonal map, DM is the fixed point set of $T\colon M^2\to M^2$. We have the homomorphisms

$$\begin{aligned}&\phi^*\colon H^q(M^2) \to H^q((M^2)_T, (DM)_T)\\ &\mu\ \colon H^q((M^2)_T, (DM)_T) \to H^{q+1}(M^2)_T, (DM)_T)\,,\end{aligned}$$

and also the homomorphisms

$$E_t\colon H^q(M) \to H^{q+t}((M^2)_T, (DM)_T) \qquad (t\geqq 1)$$

by identifying M with DM.

R. Thom [9] and R. Bott [2] studied the cohomology of symmetric products in connection with the Steenrod operations (see also [1], [5], [9]). In particular they proved

$$(1.4) \qquad \phi^*(\alpha\times\alpha) = \sum_{t=1}^{q} E_t Sq^{r-t}(\alpha) \qquad (\alpha\in H^q(M)).$$

Based on the results of R. Thom, the author determined in [6] the cohomology structure of $((M^2)_T, (DM)_T)$ as follows.

(1.5) **Proposition.** *Let $\{\alpha_1, \alpha_2, \cdots, \alpha_s\}$ be a basis of $H^*(M)$. Then the totality of elements in the following* i), ii) *is a basis of $H^*((M^2)_T, (DM)_T)$:*

i) $E_t(\alpha_i)$ $\quad (1\leqq t\leqq deg\ \alpha_i,\ \ 1\leqq i\leqq s)$,

ii) $\phi^*(\alpha_i\times\alpha_j)$ $\quad (1\leqq i<j\leqq s)$.

Furthermore, the totality of elements in i) *is a basis of the kernel of the homomorphism $\pi^*\colon H^*((M^2)_T, (DM)_T)\to H^*(M^2, DM)$.*

2. The class ϑ

Let Y be an n-dimensional (topological) manifold without boundary, and let $U\in H^n(Y^2, Y^2-DY)$ denote the orientation class of Y over $\boldsymbol{Z}_2$, that is, an element whose restriction to $H^n(y\times(Y, Y-y))$ is a generator for any $y\in Y$. Let X be a closed manifold contained in Y, and let $U'\in H^n(X\times(Y, Y-X))$ denote the restriction of U. Define a homomorphism

$$(2.1) \qquad \gamma_U\colon H_q(X) \to H^{n-q}(Y, Y-X)$$

by sending a to the slant product $a\backslash U'$ (see Chap 6, §10 of [8]).

(2.2) **Lemma.** *If Y is a closed manifold, we have*

$$j^*\gamma_U(a)\frown[Y] = i_*(a)$$

for $a\in H_*(X)$, where $[Y]$ is the mod 2 fundamental class of Y, and $j^*\colon H^*(Y, Y-X)\to H^*(Y)$ and $i_*\colon H_*(X)\to H_*(Y)$ are induced by the inclusions.

Proof. Let $U_1\in H^n(X\times Y)$ denote the restriction of U, and $\alpha\in H^*(Y)$ be any element. It follows that

$$\begin{aligned}&\langle\alpha, j^*\gamma_U(a)\frown[Y]\rangle = \langle\alpha, (a\backslash U_1)\frown[Y]\rangle\\ &= \langle\alpha\smile(a\backslash U_1), [Y]\rangle = \langle a\backslash(U_1\smile 1\times\alpha), [Y]\rangle\\ &= \langle U_1\smile 1\times\alpha, a\times[Y]\rangle.\end{aligned}$$

Take a basis $\{\alpha_1, \alpha_2, \cdots, \alpha_s\}$ of $H^*(Y)$, and put

$$d_{ij} = \langle \alpha_i \smile \alpha_j, [Y] \rangle \in \boldsymbol{Z}_2\,,$$
$$U_0 = \sum_{i,j} c_{ij}\alpha_i \times \alpha_j \qquad (\iota_{ij} \in \boldsymbol{Z}_2)\,,$$

where $U_0 \in H^n(Y^2)$ is the restriction of U. It is known that

$$\sum_k c_{ik} d_{kj} = \delta_{ij}$$

(see p. 347 of [8]). Therefore we see that

$$\begin{aligned}
&\langle \alpha_k, j^*\gamma_U(a) \frown [Y] \rangle \\
&= \langle \sum_{i,j} c_{ij}(i^*\alpha_i) \times \alpha_j \smile 1 \times \alpha_k,\ a \times [Y] \rangle \\
&= \sum_{i,j} c_{ij} \langle (i^*\alpha_i) \times (\alpha_j \smile \alpha_k),\ a \times [Y] \rangle \\
&= \sum_{i,j} c_{ij} \langle i^*\alpha_i, a \rangle \langle \alpha_j \smile \alpha_k, [Y] \rangle \\
&= \sum_{i,j} c_{ij} d_{jk} \langle i^*\alpha_i, a \rangle \\
&= \sum_i \delta_{ik} \langle \alpha_i, i_* a \rangle \\
&= \langle \alpha_k, i_* a \rangle\,.
\end{aligned}$$

This completes the proof.

REMARK. It is known that the homomorphism γ_U of (2.1) is an isomorphism (see p. 351 of [8]).

Assume now that there is given on Y a free involution T such that $T(X) = X$. Then the quotient Y_T is also a manifold without boundary, and X_T is a closed manifold contained in it. Let $\pi\colon Y \to Y_T$ denote the projection, and $V \in H^n((Y_T)^2, (Y_T)^2 - DY_T)$ the orientation class of Y_T over $\boldsymbol{Z}_2$. Then we have

(2.3) **Lemma.** *The diagram*

$$\begin{array}{ccc}
H_q(X) & \xrightarrow{\gamma_U} & H^{n-q}(Y, Y-X) \\
\Big\uparrow{\scriptstyle \phi_*} & & \Big\uparrow{\scriptstyle \pi^*} \\
H_q(X_T) & \xrightarrow{\gamma_V} & H^{n-q}(Y_T, Y_T - X_T)
\end{array}$$

is commutative, where ϕ_* is the transfer homomorphism. In particular, it holds

$$\pi^*\gamma_V[X_T] = \gamma_U[X]$$

for the mod 2 fundamental classes of X and X_T.

Proof. Consider a chain map $\phi\colon C_*(Y_T) \to C_*(Y)$ and a cochain map $\rho\colon C^*(Y^2) \to C^*((Y_T)^2)$ defined as follows:

$$\phi(\pi_\sharp c) = c + T_\sharp c\,,$$
$$\langle \rho u, (\pi\times\pi)_\sharp\sigma\rangle = \langle u, \sigma\rangle + \langle u, (T\times 1)_\sharp\sigma\rangle\,,$$

where $c\in C_*(Y)$, $u\in C^*(Y^2)$, $\sigma\in C_*(Y^2)$. Then it is easily checked that

$$\pi^\sharp(c'\backslash\rho u) = \phi(c')\backslash u$$

for $c'\in C_*(Y_T)$. The homomorphism $\phi_*\colon H_q(X_T)\to H_q(X)$ is induced by ϕ, and it follows from the definition of orientation class that

$$\rho^*(U) = V$$

for the homomorphism $\rho^*\colon H^*(Y^2, Y^2-DY)\to H^*((Y_T)^2, (Y_T)^2-DY_T)$ induced by ρ. Consequently it is easily seen that the diagram in the lemma is commutative. Since $\phi_*[Y_T]=[Y]$, the proof completes.

Let M be an m-dimensional closed manifold with a free involution T. Regard M^2 as a manifold with involution by the switching map T. Then an equivariant map

$$\Delta\colon M\to M^2$$

is given by $\Delta(x)=(x, Tx)$. The image ΔM is an invariant submanifold of M^2, and the map $\Delta_T\colon M_T\to(M^2)_T$ is a homeomorphism onto $(\Delta M)_T$. Since T is free, $\Delta M\cap DM$ and $(\Delta M)_T\cap(DM)_T$ are empty.

Obviously $(M^2)_T-(DM)_T$ is a $2n$-dimensional manifold without boundary, and $(\Delta M)_T$ is a closed manifold contained in it. Consider the homomorphisms

$$H_m((\Delta M)_T)\xrightarrow{\gamma_V} H^m((M^2)_T-(DM)_T, (M^2)_T-(DM)_T-(\Delta M)_T)$$
$$\xleftarrow[\cong]{i^*} H^m((M^2)_T, (M^2)_T-(\Delta M)_T)\xrightarrow{k^*} H^m((M^2)_T, (DM)_T)\,,$$

where V is the orientation class of the manifold $(M^2)_T-(DM)_T$, and i and k are the inclusions. We define $\vartheta=\vartheta(M)\in H^m((M^2)_T, (DM)_T)$ by

$$\vartheta = k^*i^{*-1}\gamma_V[(\Delta M)_T]\,. \tag{2.4}$$

(2.5) **Lemma.** *For the homomorphisms $\pi^*\colon H^m((M^2)_T, (DM)_T)\to H^m(M^2)$ and $\Delta_*\colon H_m(M)\to H_m(M^2)$, we have*

$$\pi^*\vartheta(M)\frown[M^2] = \Delta_*[M]\,.$$

Proof. Let W be the orientation class of the manifold M^2. Then it follows from (2.2) that

$$j^*\gamma_W[\Delta M]\frown[M^2] = \Delta_*[M]$$

for the homomorphism $j^*\colon H^*(M^2, M^2-\Delta M)\to H^*(M^2)$ induced by the inclusion. Therefore it suffices to prove

$$\pi^*\vartheta = j^*\gamma_W[\Delta M].$$

Let U denote the orientation class of M^2-DM. Then by (2.3) we have

$$\pi^*\gamma_V[(\Delta M)_T] = \gamma_U[\Delta M]$$

for $\pi^*: H^*((M^2)_T-(DM)_T, (M^2)_T-(DM)_T-(\Delta M)_T)\to H^*(M^2-DM, M^2-DM-\Delta M)$. Since U is the restriction of W, it follows that

$$\begin{aligned}\pi^*\vartheta &= j^*i^{*-1}\pi^*\gamma_V[(\Delta M)_T] \\ &= j^*i^{*-1}\gamma_V[\Delta M] = j^*\gamma_W[\Delta M],\end{aligned}$$

where $i^*: H^*(M^2, M^2-\Delta M)\cong H^*(M^2-DM, M^2-DM-\Delta M)$ is the excision isomorphism. This completes the proof.

We have the following (compare Prop. 3.2 of [7]).

(2.6) **Proposition.** *Let M be a closed PL manifold with a free involution (not necessarily PL). Take a basis $\{\alpha_1, \alpha_2, \cdots, \alpha_s\}$ of $H^*(M)$, and put*

$$\Delta_*[M] = \sum_{i,j}\varepsilon_{ij}a_i\times a_j,$$

where $a_i=\alpha_i\frown[M]$. Then we have

$$\varepsilon_{ij} = \varepsilon_{ji}, \quad \varepsilon_{ii} = 0,$$

and also

$$\vartheta(M)\equiv \sum_{i<j}\varepsilon_{ij}\phi^*(\alpha_i\times\alpha_j) \quad \textit{mod Ker } \pi^*$$

for the homomorphism $\pi^*: H^*((M^2)_T, (DM)_T)\to H^*(M^2, D(M))$ and the transfer $\phi^*: H^*(M^2)\to H^*((M^2)_T, (DM)_T)$.

Proof. By (1.5) we can put

$$\vartheta = \sum_{i<j}\varepsilon_{ij}'\phi^*(\alpha_i\times\alpha_j) \quad \text{mod Ker } \pi^*.$$

Then it follows from (2.5) that

$$\begin{aligned}\Delta_*[M] &= \sum_{i<j}\varepsilon_{ij}'j^*\pi^*\phi^*(\alpha_i\times\alpha_j)\frown[M^2] \\ &= \sum_{i<j}\varepsilon_{ij}'(\alpha_i\times\alpha_j+\alpha_j\times\alpha_i)\frown([M]\times[M]) \\ &= \sum_{i<j}\varepsilon_{ij}'(a_i\times a_j+a_j\times a_i),\end{aligned}$$

where $j^*: H^*(M^2, DM)\to H^*(M^2)$ is induced by the inclusion. Therefore we have $\varepsilon_{ij}=\varepsilon_{ji}=\varepsilon_{ij}'$, $\varepsilon_{ii}=0$, and the proof completes.

In virtue of (2.6) we have the following theorem (see Theorem 3.4 of [7]).

(2.7) **Theorem.** *Let M be a closed PL manifold with a free involution T. Then there is a basis $\{\mu_1, \cdots, \mu_r, \mu_1', \cdots, \mu_r'\}$ of $H^*(M)$ such that*

$$\langle \mu_i \smile T^* \mu_j, [M] \rangle = \langle \mu_i' \smile T^* \mu_j', [M] \rangle = 0\,,$$
$$\langle \mu_i \smile T^* \mu_j', [M] \rangle = \delta_{ij}\,,$$

and it holds

$$\vartheta(M) = \sum_{i=1}^{r} \phi^*(\mu_i \times \mu_i') \quad mod\ Ker\ \pi^*.$$

We call such a basis a *symplectic basis* of $H^*(M)$.

From (2.7) and (1.5) we have

(2.8) **Corollary.** *Let Σ be a mod 2 homology m-sphere with a free involution. Assume Σ is a PL manifold. Then we have*

$$\vartheta(\Sigma) = \phi^*(1 \times \omega)\,,$$

where $\omega \in H^m(\Sigma)$ is the generator.

REMARK. It is seen that the element $\Delta_\infty \in H^m(S^\infty \underset{T}{\times} M^2)$ in [7] is the image of ϑ under the homomorphism p^*: $H^m((M^2)_T, (DM)_T) \to H^m(S^\infty \underset{T}{\times} M^2)$ induced by the projection p: $S^\infty \underset{T}{\times} M^2 \to (M^2)_T$.

3. The equivariant point index $\hat{I}(f)$

Let N and M be m-dimensional closed PL manifolds on each of which an involution T is given. We assume that the involution of M is free, and the involution of N is nontrivial and PL. As in the preceding section we regard M^2 as a manifold with involution by the switching map T.

Given a continuous map f: $N \to M$, define an equivariant map $\hat{f}$: $N \to M^2$ by

$$\hat{f}(y) = (f(y), f(Ty))\,.$$

This induces a homomorphism $\hat{f}_T^*$: $H^*((M^2)_T, (DM)_T) \to H^*(N_T, F_T)$. We define a mod 2 integer $\hat{I}(f)$ by

$$\hat{f}_T^* \vartheta(M) = \hat{I}(f) \phi^*(\nu)\,,$$

where $\nu \in H^m(N)$ is the generator and ϕ^*: $H^m(N) \cong H^m(N_T, F_T)$ (see (1.3)). We call $\hat{I}(f)$ an *equivariant point index* of f.

We have the following (compare (1.4) of [7]).

(3.1) **Theorem.** *If $\hat{I}(f) \not\equiv 0$, there exists an equivariant point of f, i.e. a point $y \in N$ such that $f(Ty) = Tf(y)$.*

Proof. Put $A(f) = \{y \in N;\ fT(y) = Tf(y)\}$. Then the following diagram is commutative.

$$\begin{array}{ccc} H^m((M^2)_T, (M^2)_T-(\Delta M)_T) & \xrightarrow{k^*} & H^m((M^2)_T, (DM)_T) \\ \downarrow \hat{f}_T^* & & \downarrow \hat{f}_T^* \\ H^m(N_T, N_T-A(f)_T) & \xrightarrow{l^*} & H^m(N_T, F_T), \end{array}$$

where k^* and l^* are induced by the inclusions. Since $\vartheta(M)$ is the image of $i^{*-1}\gamma_V[(\Delta M)_T]\in H^m((M^2)_T, (M^2)_T-(\Delta M)_T)$ under k^* (see 2.4)), the assumption implies that $l^*\circ\hat{f}_T^*$ is non-trivial. Thus $H^m(N_T, N_T-A(f)_T)\neq 0$, and hence $A(f)$ is not empty. This completes the proof.

We have the following theorem which is more general than Theorem (5.2) in [7].

(3.2) **Theorem.** *Let N be an m-dimensional closed PL manifold with a non-trivial PL involution T, and let M be an m-dimensional closed PL manifold with a free involution T. Let F denote the fixed point set of the involution T of N, and let f: $N\to M$ be a continuous map satisfying a condition*:

(*) *$(f|F)^*$: $H^q(M)\to H^q(F)$ is trivial for $q\geqq m/2$.*

Taking a symplectic basis $\{\mu_1, \cdots, \mu_r, \mu_1', \cdots, \mu_r'\}$ of $H^*(M)$, put

$$\hat{\chi}(f)=\langle\sum_{i=1}^r f^*\mu_i\smile T^*f^*\mu_i', [N]\rangle\in \boldsymbol{Z}_2,$$

where f^*: $H^*(M)\to H^*(N)$. Then we have

$$\hat{I}(f)\equiv\hat{\chi}(f).$$

Consequently if $\hat{\chi}(f)\neq 0$ then f has an equivariant point.

Proof. Since the diagram

$$\begin{array}{ccccc} H^*(N^2) & \xleftarrow{(f\times f)^*} & H^*(M^2) & \xrightarrow{\phi^*} & H^*((M^2)_T, (DM)_T) \\ \downarrow (1\times T)^* & & \downarrow \hat{f}^* & & \downarrow \hat{f}_T^* \\ H^*(N^2) & \xrightarrow{D^*} & H^*(N) & \xrightarrow{\phi^*} & H^*(N_T, F_T) \end{array}$$

is commutative, it follows from (2.7) that

$$\hat{f}_T^*\vartheta(M)\equiv\sum_{i=1}^r \phi^*(f^*\mu_i\smile T^*f^*\mu_i') \quad \mathrm{mod}\ \hat{f}_T^*(\mathrm{Ker}\ \pi^*),$$

where π^*: $H^m((M^2)_T, (DM)_T)\to H^m(M^2, DM)$. Since the diagram

$$\begin{array}{ccc} H^q(M) & \xrightarrow{E_t} & H^{q+t}((M^2)_T, (DM)_T) \\ \downarrow (f|F)^* & & \downarrow \hat{f}_T^* \\ H^q(F) & \xrightarrow{E_t} & H^{q+t}(N_T, F_T) \end{array}$$

is commutative, it follows from (1.5) that $\hat{f}_T^*(\mathrm{Ker}\,\pi^*)$ is generated by the elements $E_{m-q}(f|F)^*\alpha$ ($\alpha\in H^q(M)$, $m/2\leqq q<m$). Therefore, by the assumption we have $\hat{f}_T^*(\mathrm{Ker}\,\pi^*)=0$. Consequently it holds

$$\hat{f}_T^*\vartheta(M)=\sum_{i=1}^{r}\phi^*(f^*\mu_i\smile T^*f^*\mu_i').$$

Since $\phi^*\alpha=\langle\alpha,[N]\rangle\phi^*\nu$ for the generator ν of $H^m(N)$ and any $\alpha\in H^m(N)$, we have

$$\hat{I}(f)\phi^*\nu=\hat{\chi}(f)\phi^*\nu$$

which proves the desired result.

REMARK 1. The above theorem shows that, if the condition (*) is satisfied, $\hat{\chi}(f)$ is independent of the choice of symplectic bases of $H^*(M)$. By making use of Corollary (1.11) of Bredon [3], we can prove the independency under a weaker condition that $(f|F)^*$: $H^{m/2}(M)\to H^{m/2}(F)$ is trivial.

REMARK 2. The condition (*) is satisfied for any continuous map f: $N\to M$ in the following cases:

i) The homomorphism i^*: $H^q(N)\to H^q(F)$ is trivial if $m/2\leqq q<m$. In particular, dim $F<m/2$ or N is a mod 2 homology m-sphere.

ii) M is a mod 2 homology m-sphere. (In this case, we have $\hat{\chi}(f)\equiv\deg f$ mod 2.)

(3.3) **Corollary.** *Let N be an m-dimensional closed PL manifold with a free involution T. Assume that the semi-characteristic $\hat{\chi}(N)$ of N is not zero. Let T' be a PL involution on N with the fixed point set F. Assume that i^*: $H^q(N)\to H^q(F)$ is trivial for $q\geqq m/2$, and that $T_*=T_*'$: $H_*(N)\to H_*(N)$. Then T and T' have a coincidence, i.e. there is a point $y\in N$ such that $T(y)=T'(y)$.*

As a special case of (3.3), we have the following result which answers a question rasied in [4].

(3.4) **Corollary.** *If Σ is a mod 2 homology sphere which is a PL manifold, then a free involution T on Σ and a non-trivial PL involution T' on Σ have a coincidence.*

4. Generalization of the Borsuk-Ulam theorem

R. Fenn has proved in [5] the following theorem of the Borsuk-Ulam type: Let T be a free PL involution on S^n. Then, for a continuous map g_1: $S^n\to S^n$ of odd degree and a continuous map g_2: $S^n\to \boldsymbol{R}^n$, there exists points y_1, y_2 in S^n such that $g_1(y_1)=Tg_1(y_2)$ and $g_2(y_1)=g_2(y_2)$. This theorem is generalized as follows, (4.1) **Theorem.** *Let Σ be a mod 2 homology n-sphere which is a PL manifold, and L be an n-dimensional closed PL manifold. Let $T: \Sigma \to \Sigma$ be a free involution (not necessarily PL). Then, for a continuous map $g_1: L \to \Sigma$ of odd degree and a continuous map $g_2: L \to \Sigma$ of even degree, there exist points y_1, y_2 in L such that $g_1(y_1) = Tg_1(y_2)$ and $g_2(y_1) = g_2(y_2)$.*

Proof. Put

$$M = \Sigma^2, \qquad N = L^2,$$

and regard M and N as manifolds with involution by T given as follows:

$$T(x_1, x_2) = (Tx_1, x_2) \qquad (x_i \in \Sigma),$$
$$T(y_1, y_2) = (y_2, y_1) \qquad (y_i \in L).$$

For any continuous maps $g_1, g_2: L \to \Sigma$, put $f = g_1 \times g_2: N \to M$. If $A(f)$ denotes the set of equivariant points of f, we have

$$A(f) = \{(y_1, y_2) \in L^2;\ g_1(y_1) = Tg_1(y_2),\ g_2(y_1) = g_2(y_2)\}.$$

Therefore, in virtue of (3.1), it suffices to prove that $\hat{I}(f) \neq 0$ if deg g_1 is odd and deg g_2 is even.

It follows from (1.5) that the kernel of $\pi^*: H^{2n}((M^2)_T, (DM)_T) \to H^{2n}(M^2, DM)$ is generated by $E_n(\omega \times 1)$, and $E_n(1 \times \omega)$, where $\omega \in H^n(\Sigma)$ is the generator. By (1.4) we have

$$E_n(\omega \times 1) = \phi^*(\omega \times 1 \times \omega \times 1),$$
$$E_n(1 \times \omega) = \phi^*(1 \times \omega \times 1 \times \omega).$$

Therefore, in virtue of (2.6) we may put

$$\begin{aligned}\vartheta(M) = {} & \phi^*(\omega \times \omega \times 1 \times 1) + \phi^*(\omega \times 1 \times 1 \times \omega) \\ & + \varepsilon_1 \phi^*(\omega \times 1 \times \omega \times 1) + \varepsilon_2 \phi^*(1 \times \omega \times 1 \times \omega)\end{aligned}$$

$(\varepsilon_1, \varepsilon_2 \in \boldsymbol{Z}_2)$. It follows that

$$\begin{aligned}& \hat{f}_T^* \phi^*(\alpha_1 \times \alpha_2 \times \beta_1 \times \beta_2) \\ & = \phi^*((g_1 \times g_2)^*(\alpha_1 \times \alpha_2) \smile T^*(g_1 \times g_2)^*(\beta_1 \times \beta_2)) \\ & = \phi^*((g_1^* \alpha_1 \times g_2^* \alpha_2) \smile (g_2^* \beta_2 \times g_1^* \beta_1)) \\ & = \phi^*((g_1^* \alpha_1 \smile g_2^* \beta_2) \times (g_2^* \alpha_2 \smile g_1^* \beta_1)).\end{aligned}$$

Consequently we have

$$\begin{aligned}& \hat{f}_T^* \vartheta(M) \\ & = \phi^*(g_1^* \omega \times g_2^* \omega) + \varepsilon_1 \phi^*(g_1^* \omega \times g_1^* \omega) + \varepsilon_2 \phi^*(g_2^* \omega \times g_2^* \omega) \\ & = (\deg g_1 \cdot \deg g_2 + \varepsilon_1 \deg g_1 + \varepsilon_2 \deg g_2) \phi^*(\sigma \times \sigma),\end{aligned}$$

where $\sigma \in H^n(L)$ is the generator. This shows

$$\hat{I}(g_1 \times g_2) \equiv \deg g_1 \cdot \deg g_2 + \varepsilon_1 \deg g_1 + \varepsilon_2 \deg g_2 \mod 2.$$

Consider a special case when $L=\Sigma$ and $g_2=$identity. We see that

$$A(g_1 \times 1) \text{ is empty,}$$
$$\hat{I}(g_1 \times 1) \equiv \deg g_1 + \varepsilon_1 \deg g_1 + \varepsilon_2 \mod 2$$

for any g_1: $\Sigma \to \Sigma$. Therefore, it follows from (3.1) that

$$\deg g_1 + \varepsilon_1 \deg g_1 + \varepsilon_2 \equiv 0 \mod 2$$

and hence $\varepsilon_2 \equiv 0$ and $\varepsilon_1 \equiv 1$.

Thus we have

$$\hat{I}(g_1 \times g_2) \equiv \deg g_1 \cdot \deg g_2 + \deg g_1 \mod 2,$$

which proves the desired result.

We have also the following theorem.

(4.2) **Theorem.** *Let Σ be a mod 2 homology n-sphere which is a PL manifold, and L be an n-dimensional closed PL manifold. Let T: $\Sigma \to \Sigma$ be a free involution. Then, for continuous maps g_1, g_2: $L \to \Sigma$ of odd degree, there exist points y_1, y_2 in L such that $g_1(y_1) = Tg_1(y_2)$, $g_2(y_1) = Tg_2(y_2)$.*

Proof. As in the proof of (4.1), we put $M=\Sigma^2$, $N=L^2$, and regard N as a manifold with involution by the switching map. However we regard M as a manifold with involution by the following T:

$$T(x_1, x_2) = (Tx_1, Tx_2) \qquad (x_i \in \Sigma).$$

By the same arguments as in the proof of (4.1), we see that

$$\hat{I}(g_1 \times g_2) \equiv \deg g_1 \cdot \deg g_2 + \varepsilon_1 \deg g_1 + \varepsilon_2 \deg g_2 \mod 2$$

for any continuous maps g_1, g_2: $L \to \Sigma$, where

$$\begin{aligned} \vartheta(M) = {} & \phi^*(\omega \times \omega \times 1 \times 1) + \phi^*(\omega \times 1 \times 1 \times \omega) \\ & + \varepsilon_1 \phi^*(\omega \times 1 \times \omega \times 1) + \varepsilon_2 \phi^*(1 \times \omega \times 1 \times \omega). \end{aligned}$$

We have

$$A(g_1 \times g_2) = \{(y_1, y_2) \in L^2; g_1(y_1) = Tg_1(y_2), g_2(y_1) = Tg_2(y_2)\}.$$

If g_1 or g_2 is trivial, then $A(g_1 \times g_2)$ is empty. Hence it follows from (3.1) that $\varepsilon_1 \deg g_1 \equiv 0$ and $\varepsilon_2 \deg g_2 \equiv 0$ for any g_1 and g_2. Thus we have

$$\varepsilon_1 \equiv 0 \quad \text{and} \quad \varepsilon_2 \equiv 0,$$

and so

$$\hat{I}(g_1 \times g_2) \equiv \deg g_1 \cdot \deg g_2 \mod 2,$$

which proves the desired result.

By applying (4.1), the following theorem can be proved as in the proof of Theorem 2 of [5].

(4.3) **Theorem.** *Let $\alpha \in \pi_n(RP^n \times S^n)$ $(n \geqq 2)$ be any element such that $p_{1*}\alpha \in \pi_n(RP^n)$ is an odd element and $p_{2*}\alpha \in \pi_n(S^n)$ is an even element, where p_1 and p_2 are the projections. Then α can not be realized by a topologically embedded sphere.*

OSAKA UNIVERSITY

References

[1] A. Aeppli-Y. Akiyama: *Cohomology operations in Smith theory,* Ann. Scuola Norm. Sup. Pisa. **24** (1970), 741–833.

[2] R. Bott: *On symmetric products and the Steenrod squares,* Ann. of Math. **57** (1953), 579–590.

[3] G.E. Bredon: *Cohomology aspects of transformation groups,* Proceedings of the Conference on Transformation Groups, 245–280, Springer, 1967.

[4] A. Dress: *Newman's theorems on transformation groups,* Topology **8** (1969), 203–207.

[5] R. Fenn: *Some generalizations of the Borsuk-Ulam theorem and applications to realizing homotopy classes by embedded spheres,* Proc. Cambridge. Philos. Soc. **74** (1973), 251–256.

[6] M. Nakaoka: *Cohomology theory of a complex with a transformation of prime period and its applications,* J. Inst. Polytech. Osaka City Univ. **7** (1956), 51–102.

[7] M. Nakaoka: *Continuous maps of manifolds with involution* I, Osaka J. Math. **11** (1974), 129–145.

[8] E.H. Spanier: Algebraic Topology, McGraw-Hill, New York, 1966.

[9] R. Thom: *Une théorie intrinsèque des puissances de Steenrod,* Colloque de Topologie de Strasbourg, 1951.

[10] W.T. Wu: A theory of Imbedding, Immersion and Isotopy of Polytopes in a Euclidean Space, Science Press, Peking, 1965.