Hirashima, Y.
Osaka J. Math.
12 (1975), 187-196

ON THE BP_*-HOPF INVARIANT

YASUMASA HIRASHIMA

(Received December 13, 1973)
(Revised June 28, 1974)

In this paper we will consider the BP_*-Hopf invariant, $\pi_*(S^0)\to \mathrm{Ext}^{1,*}_{BP_*(BP)}(BP_*, BP_*)$, i.e. the Hopf invariant defined by making use of the homology theory of the Brown-Peterson spectrum BP. The BP_*-Hopf invariant is essentially "the functional coaction character". Similarly we will define the BP_*-e invariant ("the functional Chern-Dold character") and show that the BP_*-Hopf invariant coincides with the BP_*-e invariant by the BP-analogue of Buhstaber-Panov's theorem ([6], [7]). As applications we give a proof of the non-existence of elements of Hopf invariant 1, and detect α-series.

We will use freely notations of Adams [2], [3], [4]. For example, S, H, HZ_p and $HZ_{(p)}$ denote the sphere spectrum, the Eilenberg-MacLane spectrum, Z_p coefficient Eilenberg-MacLane spectrum and $Z_{(p)}$ coeffitient Eilenberg-MacLane spectrum respectively, where $Z_{(p)}$ is the ring of integers localized at the fixed prime p.

We list some well known facts:

$$\pi_*(BP)=BP_*(S^0)=BP_*=Z_{(p)}[v_1, v_2, \cdots], \quad \deg v_k=|v_k|=2(p^k-1).$$
$$H_*(BP)=HZ_{(p)*}(BP)=Z_{(p)}[n_1, n_2, \cdots], \quad \deg n_k=|n_k|=2(p^k-1).$$

The Hurewicz map

$$h^H=(i^H\wedge 1_{BP})_*:\pi_*(BP)\to H_*(BP)$$

is decided by the formula [5]

$$h^H(v_k)=pn_k-\sum_{0<s<k}h^H(v_{k-s})^{p^s}n_s.$$
$$BP_*(BP)=BP_*[t_1, t_2, \cdots], \quad \deg t_k=|t_k|=2(p^k-1).$$

The Thom map $BP\overset{\mu}{\to}HZ$ induces

$$BP_*(BP)\overset{\mu}{\to}HZ_{(p)*}(BP)=H_*(BP), \quad \mu(t_k)=n_k; \quad \mu(v_k\cdot 1)=0$$

($k>0$) and ([10])

$$HZ_{(p)*}(BP) \overset{\mu_*}{\to} (HZ_p)_*(HZ_p)\,, \quad \mu_*(n_k) = c(\xi_k)\,,$$

where c is the conjugation map of the Hopf algebra $(HZ_p)_*(HZ_p)$ and ξ_k $(k=1, 2, \cdots)$ are Milnor's basis of a polynomial subalgebra $Z_p[\xi_1, \xi_2, \cdots] \subset (HZ_p)_*(HZ_p)$. $BP^*(BP)=BP_*\hat{\otimes}Z_{(p)}\{r_E\}$, where E runs through sequences of non-negative integers $E=(e_1, e_2, \cdots)$ in which all but finite number of terms are zero and $\deg r_E=|E|=|r_E|=2(\sum_{k\geq 1} e_k(p^k-1))$.

1. *BP*-analogue of Panov's theorem

To compute $\mathrm{Ext}^{1,*}_{BP_*(BP)}(BP_*, BP_*)$ we define some subquotient group of $H_*(BP)$ and compute this group and next relate this with $\mathrm{Ext}^{1,*}_{BP_*(BP)}(BP_*, BP_*)$.

We may regard $\pi_*(BP)$ as a submodule of $H_*(BP)$ by the Hurewicz map h^H. Cohomology operations r_E act on $H_*(BP)$ so that we define

$$N = \bigcap_{E\neq 0} r_E^{-1}(\mathrm{Im}\, h^H) \quad \text{and} \quad N/\mathrm{Im}\, h^H\,.$$

We fix a prime p and discuss the Brown-Peterson spectrum associated with this prime, then for $n\neq 2k(p-1)$ $(N/\mathrm{Im}\, h^H)_n=0$ as $H_n(BP)=0$, thus it remains to decide the groups $(N/\mathrm{Im}\, h^H)_{2k(p-1)}$.

Theorem 1.1. *For odd prime p $(N/\mathrm{Im}\, h^H)_{2k(p-1)}=Z_{p^{\nu_p(k)+1}}$ with generator $v_1{}^k/p^{\nu_p(k)+1}$ where $\nu_p(k)$ denotes the exponent of highest power of p dividing k. For $p=2$ $(N/\mathrm{Im}\, h^H)_{2k}=Z_2$ $(k$: odd$)$, Z_4 $(k=2)$ and $Z_{2^{\nu_2(k)+2}}$ $(k>2$, even$)$ with generators $v_1{}^k/2$, $v_1{}^2/4$ and $v_1{}^k/2^{\nu_2(k)+2}+v_1{}^{k-3}v_2/2$ respectively.*

Similar theorem for MU spectrum was first computed by Panov [7], and Landweber [6] gave a shortened proof of which BP-analogue we follow faithfully.

Exponent sequences $E=(e_1, e_2, \cdots)$, $F=(f_1, f_2, \cdots)$ are ordered as follows: $E>F$ if

(1) $|E|>|F|$, or
(2) $|E|=|F|$, and $n(E)=\sum_{k\geq 1} e_k<n(F)$, or
(3) $E=F$, $n(E)=n(F)$ and there exist a k such that $e_k>f_k$, $e_i=f_i$ $(i>k)$.

We have that if $E>E'$ and $F>F'$ then $E+F>E'+F'$, where the sum is componentwise. We say that an element a of N has type E if $r_E(a)\notin(p)=p\cdot\mathrm{Im}\, h^H$ and $r_F(a)\in(p)$ for any $F>E$, especially a has type 0 if a has type $(0, 0, \cdots)$. If a has type E, such a E is denoted by $t(a)$.

Lemma 1.2.

(1) *v_{k+1} has type $p\Delta_k$ $(k\geq 1)$ and v_1 has type* 0 (i.e. $t(v_{k+1})=p\Delta_k$, $(t(v_1)=0)$.

(2) $r_{\Delta_{k+1}}(v_{k+1})=p$.
(3) $t(v^E)=(pe_1, pe_2, \cdots)$ *where* $E=(e_1, e_2, \cdots)$

and v^E *means* $v_1^{e_1}v_2^{e_2}\cdots$.

Using the formula ([10])

$$r_E(n_k) = \begin{cases} n_i, & E = p^i\Delta_j\ (i+j=k); \\ 0, & \text{otherwise,} \end{cases}$$

and

$$v_k = pn_k \sum_{0<s<k} v_{k-s}^{p^s} n_s,$$

the lemma can be proved by a routine induction on k, so we omit it.

By the above lemma we get $t(v^E)\neq t(v^F)$ for $E\neq F$, $|E|=|F|$.
Theorem 1.1 is divided into three lemmas as Landweber did in MU case.

Lemma 1.3. $(N/\mathrm{Im}\, h^H)_{2k(p-1)}$ *is cyclic (i.e., has one generator).*

Lemma 1.4.

(1) $v_1^k/p^{\nu_p(k)+1}\in N_{2k(p-1)}$, *and*
(2) *if* p *is odd, or* $p=2$ *and* k *is odd, or* $p=2$ *and* $k=2$, *then* $v_1^k/p^{\nu_p(k)+1}$ *represents the generator of* $(N/\mathrm{Im}\, h^H)_{2k(p-1)}$.

Lemma 1.5. *If* $p=2$ *and* $k>2$, *then* $v_1^k/2^{\nu_2(k)+2}+v_1^{k-3}v_2/2$ *represents the generator of* $(N/\mathrm{Im}\, h^H)_{2k}$.

Proof of Lemma 1.3. Let $a\in N_{2k(p-1)}$ represent an element of order p in $(N/\mathrm{Im}\, h^H)_{2k(p-1)}$, then $pa\in \mathrm{Im}\, h^H$. Write $pa=\lambda v_1^k+\lambda_1 v^{E_1}+\lambda_2 v^{E_2}+\cdots+\lambda_i v^{E_i}$ with $\lambda, \lambda_j\in Z_{(p)}$, $|E_j|=2k(p-1)$ and $t(v^{E_1})<t(v^{E_2})<\cdots<t(v^{E_i})$. Apply $r_{t(v^{E_i})}$ to the element pa. We get $\lambda_i\equiv 0 \bmod (p)$ since $\lambda_i r_{t(v^{E_i})}(v^{E_i})\equiv 0 \bmod (p)$. Next apply $r_{t(v^{E_{i-1}})}$. By the same argument we have $\lambda_{i-1}\equiv 0 \bmod (p)$. Continue these argument, then we get $\lambda_1\equiv\lambda_2\equiv\cdots\equiv\lambda_i\equiv 0 \bmod (p)$. So we conclude

$$pa \equiv \lambda v_1^k \bmod (p)$$

and hence

$$a = \lambda\cdot(v_1^k/p) \quad \text{in} \quad (N/\mathrm{Im}\, h^H)_{2k(p-1)}.$$

This implies Lemma 1.3.

Proof of Lemma 1.4. (1) We get by induction

$$r_E(v_1^k) = \begin{cases} \binom{k}{e} p^e v_1^{k-e}, & E = e\Delta_1; \\ 0, & \text{otherwise,} \end{cases}$$

Using the formula ([6], [7])

$$\nu_p\left(\binom{k}{e}\right) = \nu_p(k) - \nu_p(e) \quad \text{for} \quad e \leq p^{\nu_p(k)},$$

we have

$$\nu_p\left(\binom{k}{e}p^e\right) \geq \nu_p(k)+1\,.$$

The equality holds for $e=1$. Hence

$$v_1{}^k/p^{\nu_p(k)+1} \in N_{2k(p-1)} \quad \text{and} \quad v_1{}^k/p^{\nu_p(k)+2} \notin N_{2k(p-1)}.$$

(2) For odd prime p, or $p=2$ and k: odd, or $p=2$ and $k=2$, $v_1{}^k/p^{\nu_p(k)+1}$ has type $\Delta_1(<t(v^E),\ |E|=2k(p-1),\ E\neq k\Delta_1)$ by the above argument.

If $$a = \lambda\cdot v_1{}^k/p^{\nu_p(k)+2} + \sum_{\substack{E=2k(p-1)\\ E\neq k\Delta_1}} \lambda_E\cdot v^E \in N_{2k(p-1)},\ \lambda,\ p\lambda_E \in Z_{(p)}\,,$$

then pa has type 0 so that $p\,|\,\lambda$, $p\,|\,\lambda_E$ by the same type-argument as the proof of Lemma 1.3. This shows that there is no element a such that $v_1{}^k/p^{\nu_p(k)+1}=pa$ in $(N/\mathrm{Im}\,h^H)_{2k(p-1)}$. This implies Lemma 1.4.

Proof of Lemma 1.5. In case $p=2$ and $k>2$,

$$\nu_2\left(\binom{k}{e}2^e\right) = \nu_2(k)+1 \quad (e=1,2)$$

and

$$\nu_2\left(\binom{k}{e}2^e\right) > \nu_2(k)+1 \quad (e>2)\,.$$

These imply that $v_1{}^k/2^{\nu_2(k)+1}$ has type $2\Delta_1$. After routine computations we obtain

$$r_{\Delta_1}(v_1{}^k/2^{\nu_2(k)+1}) = v_1^{k-1} \equiv r_{\Delta_1}(v_1^{k-3}v_2) \bmod (2)$$
$$r_{2\Delta_1}(v_1{}^k/2^{\nu_2(k)+1}) = v_1^{k-2} \equiv r_{2\Delta_1}(v^{k-3}v_2) \bmod (2)\,.$$

So we conclude that $v_1{}^k/2^{\nu_2(k)+1}+v_1^{k-3}v_2$ has type 0, and thus $v_1{}^k/2^{\nu_2(k)+2}+v_1^{k-3}v_2/2$ $\in N_{2k}$. Put $P=v_1{}^k/2^{\nu_2(k)+2}+v_1^{k-3}v_2/2$. We decide the type of P in several steps. $t(P)=\Delta_2$ or $t(P)>\Delta_2$ by $r_{\Delta_2}(P)=v_1^{k-3}$. The Cartan formula implies $r_E(P)\equiv 0 \bmod (2)$ for $E>\Delta_2$ and $E\neq i\Delta_1$, so that $t(P)=\Delta_2$ or $i\Delta_1$ ($i\geqq 4$). For $i\geqq 4$

$$r_{i\Delta_1}(P) \equiv \left(\binom{k}{i}2^i/2^{\nu_2(k)+2}\right)v_1^{k-i} \bmod (2)\,,$$

thus

$$\binom{k}{i}2^i/2^{\nu_2(k)+2} \equiv 0 \bmod (2) \quad \text{if} \quad \nu_2(k)=1\,,$$

and if $\nu_2(k)\geq 2$

$$\binom{k}{i}2^i/2^{\nu_2(k)+2} \equiv \begin{cases} 2^{i-2-\nu_2(i)}, & i \leq \nu_2(k)+2; \\ 0 \bmod (2), & i > \nu_2(k)+2. \end{cases}$$

This implies that $t(P)=\Delta_2$ if $\nu_2(k)=1$ and $t(P)=4\Delta_1$ if $\nu_2(k)\geq 2$.

There is no element $a\in \operatorname{Im} h^H$ such that $r_E(P)\equiv r_E(a) \bmod (2)$ for any E. If not, a is represented by a linear combination of v_1^k, $v_1^{k-3}v_2$ and $v_1^{k-6}v_2^2$, then $r_{\Delta_2}(a)\equiv 0 \bmod (2)$ which contradicts the assumption. This implies Lemma 1.5 and also completes the proof of Theorem 1.1.

Next we lift the group $(N/\operatorname{Im} h^H)$ to a subquotient group of $BP_*(BP)$ by Thom map $BP_*(BP)\xrightarrow{\mu}H_*(BP)$. We denote by $(r_E)_*$ the right action of r_E on $BP_*(BP)$ which is compatible under Thom map μ with the action on $H_*(BP)$. We consider the groups

$$N^{BP} = \bigcap_{E\neq 0} (r_E)_*^{-1}(\operatorname{Im} h^{BP}) \text{ and } N^{BP}/\operatorname{Im} h^{BP}+BP_*\cdot 1,$$

on which Thom map induces the group homomorphism

$$N^{BP}/\operatorname{Im} h^{BP}+BP_*\cdot 1 \xrightarrow{\hat{\mu}} N/\operatorname{Im} h^H.$$

Theorem 1.6. *$\hat{\mu}$ is isomorphic.*

Proof. In $BP_*(BP)\otimes Q = BP_*\otimes Q[n_1, n_2, \cdots]$

$$\bigcap_{E\neq 0} (r_E)_*^{-1}(0) = BP_*\otimes Q$$

so that in $BP_*(BP)$

$$\bigcap_{E\neq 0} (r_E)_*^{-1}(0) = BP_*\cdot 1 \ (= BP_*\otimes 1).$$

We get easily

$$\operatorname{Ker} \mu \cap N^{BP} \subset \bigcap_{E\neq 0} (r_E)_*^{-1}(0) = BP_*\cdot 1$$

and

$$\operatorname{Ker} \hat{\mu} = \{(\operatorname{Im} h^{BP}+\operatorname{Ker} \mu)\cap N^{BP}+\operatorname{Im} h^{BP}+BP_*\cdot 1\}/\operatorname{Im} h^{BP}+BP_*\cdot 1$$
$$= 0$$

so that $\hat{\mu}$ is monomorphic.

For any prime p, $h^{BP}(v_1)=v_1=v_1\cdot 1+pt_1$, and thus

$$v_1^k - v_1^k\cdot 1 = \sum_{1\leq e\leq k} \binom{k}{e} p^e v_1^{k-e} t_1^e.$$

We get

$$(v_1^k - v_1^k\cdot 1)/p^{\nu_p(k)+1} \in BP_{2k(p-1)}(BP)$$

and

$$(v_1^k - v_1^k \cdot 1)/p^{\nu_p(k)+1} \in N^{BP}_{2k(p-1)} .$$

In case of $p=2$ and $k \geq 2$,

$$h^{BP}(v_2) = v_2 = v_2 \cdot 1 - 3v_1^2 t_1 - 5v_1 t_1^2 + 2t_2 - 4t_1^3 , \quad ([3]) .$$

We get

$$(v_1^k - v_1^k \cdot 1)/2^{\nu_2(k)+2} = (1/2)v_1^{k-1}t_1 + (1/2)v_1^{k-2}t_1^2 + A ,$$
$$(v_1^{k-3}v_2 - (v_1^{k-3}v_2) \cdot 1)/2 = (-1/2)v_1^{k-1}t_1 + (-1/2)v_1^{k-2}t_1^2 + B ,$$

where A, $B \in BP_*(BP)$, and thus

$$(v_1^k - v_1^k \cdot 1)/2^{\nu_2(k)+2} + (v_1^{k-3}v_2 - (v_1^{k-3}v_2 \cdot))1)/2 \in BP_{2k}(BP) .$$

We have easily

$$(v_1^k - v_1^k \cdot 1)/2^{\nu_2(k)+2} + (v_1^{k-3}v_2 - (v_1^{k-3}v_2) \cdot 1)/2 \in N^{BP}_{2k} .$$

These conclude that $\hat{\mu}$ is epimorphic and complete the proof of Theorem 1.6.

The conjugation map c of the Hopf algebra $BP_*(BP)$ induces the isomorphism

$$\hat{c}\colon\ N^{BP}/\mathrm{Im}\, h^{BP} + BP_* \cdot 1 \to \bigcap_{E \neq 0} r_E^{-1}(BP_* \cdot 1)/\mathrm{Im}\, h^{BP} + BP_* \cdot 1 ,$$

but $\hat{c}$ preserves the generators given in Theorem 1.6 up to sign, so that we obtain

Corollary 1.7.

$$N^{BP}/\mathrm{Im}\, h^{BP} + BP_* \cdot 1 = \bigcap_{E \neq 0} r_E^{-1}(BP_* \cdot 1)/\mathrm{Im}\, h^{BP} + BP_* \cdot 1 .$$

We next show that

$$\mathrm{Ext}^{1,*}_{BP_*(BP)}(BP_*, BP_*) = \bigcap_{E \neq 0} r_E^{-1}(BP_* \cdot 1)/\mathrm{Im}\, h^{BP} + BP_* \cdot 1$$
$$\cong N/\mathrm{Im}\, h^H .$$

Let $S \xrightarrow{i} BP \xrightarrow{p} I$ be the cofibration obtained from the unit $S \xrightarrow{i} BP$, $I^{(k)} = I \wedge I \wedge \cdots \wedge I$ (k-factors) and d_k be the composition $BP \wedge I^{(k)} \xrightarrow{p \wedge 1} I^{(k+1)} \xrightarrow{B} BP \wedge I^{(k+1)}$ (or equivalently $BP \wedge I^{(k)} \xrightarrow{B} BP \wedge BP \wedge^{(k)} \xrightarrow{1 \wedge p \wedge 1} BP \wedge I^{(k+1)}$). Then we obtain the geometric resolution of Adams [4]

$$BP \xrightarrow{d_0} BP \wedge I \xrightarrow{d_1} BP \wedge I^{(2)} \xrightarrow{d_2} BP \wedge I^{(3)} \xrightarrow{d_3} \cdots ,$$

which defines a chain complex of a spectrum X

$$BP_*(X) \xrightarrow{(d_0)_*} (BP \wedge I)_*(X) \xrightarrow{(d_1)_*} (BP \wedge I^{(2)})_*(X) \to \cdots,$$

and

$$\mathrm{Ext}_{BP_*(BP)}^{k,*}(BP_*, BP_*(X)) = \mathrm{Ker}\,(d_k)_*/\mathrm{Im}\,(d_{k-1})_* .$$

For $X=S^0$, $(d_0)_*=p_*h^{BP}$ and $(d_1)_*=(p_*\otimes 1)\Psi_I$ where $BP_*(BP)\xrightarrow{p_*} BP_*(I)= BP_*(BP)/BP_*\cdot 1$ is the canonical projection, $BP_*(I)\xrightarrow{\Psi_I} BP_*(BP)\otimes_{BP_*}BP_*(I)$ is the coaction map of I for which

$$\Psi_I(x) = \sum_E t^E\otimes r_E(x) .$$

REMARK. This coaction map is twisted by the conjugation map c of $BP_*(BP)$ from the one denfied by Adams [2].

$$\begin{aligned}\mathrm{Ker}\,(d_1)_* &= \{x\in BP_*(BP)/BP_*\cdot 1 \mid r_E(x) = 0\,,\ E\neq 0\} \\ &= \bigcap_{E\neq 0} r_E^{-1}(BP_*\cdot 1)/BP_*\cdot 1\,,\end{aligned}$$

$$\mathrm{Im}\,(d_0)_* = \mathrm{Im}\,h^{BP}/\mathrm{Im}\,h^{BP}\cap BP_*\cdot 1 = \mathrm{Im}\,h^{BP}+BP_*\cdot 1/BP_*\cdot 1 .$$

Hence we obtain

Theorem 1.8.

$$\mathrm{Ext}_{BP_*(BP)}^{1,*}(BP_*, BP_*) = \bigcap_{E\neq 0} r_E^{-1}(BP_*\cdot 1)/\mathrm{Im}\,h^{BP}+BP_*\cdot 1 .$$

Corollary 1.9.

$$\mathrm{Ext}_{BP_*(BP)}^{1,*}(BP_*, BP_*) \overset{\hat{\mu}}{\cong} N/\mathrm{Im}\,h^H .$$

2. The BP_*-Hopf invariant

Since $BP_*(BP)$ is flat over BP_*, $BP_*(BP)$ comodules and $BP_*(BP)$ comodule homomorphisms form a relative abelian category so that similar construction of Adams [1] is valid for BP_* homology theory. We review the construction of the BP_*-Hopf invariant quickly; for a morphism $f\colon X\to Y$ of CW-spectra in homotopy category such that $f_*=0$, we have a short exact sequence

$$E(f)\colon\ 0 \to BP_*(Y) \to BP_*(C_f) \to BP_*(SX) \to 0\,,$$

which is regarded as an element of $\mathrm{Ext}_{BP_*(BP)}^{1,*}(BP_*(X), BP_*(Y))$. This is the BP_*-Hopf invariant of f.

For $X=S^{kq-1}(q=2(p-1))$, $Y=S^0$ the BP_*-Hopf invariant is defined on the whole group $\pi_{kq-1}(S^0)$. For a short exact sequence $E(f)$ we apply the Adams resolution $BP \to BP\wedge I \to BP\wedge I^{(2)} \to \cdots$ then we obtain a short exact sequence of chain complexes

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & BP_*(S^0) & \to & (BP\wedge I)_*(S^0) & \to & (BP\wedge I^{(2)})_*(S^0) \to \cdots \\
 & & \downarrow & & \downarrow i_* & & \downarrow \\
0 & \to & BP_*(C_f) & \to & (BP\wedge I)_*(C_f) & \to & (BP\wedge I^{(2)})_*(C_f) \to \cdots \\
 & & \downarrow j_* & & \downarrow & & \downarrow \\
0 & \to & BP_*(S^{kq}) & \to & (BP\wedge I)_*(S^{kq}) & \to & (BP\wedge I^{(2)})_*(S^{kq}) \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

Let $\sigma_{kq} \in BP_{kq}(S^{kq})$ be a generator and $\delta(\sigma_{kq}) = [i_*^{-1}(d_0)_* j_*^{-1}(\sigma_{kq})] \in \mathrm{Ext}^{1,kq}_{BP_*(BP)}(BP_*, BP_*)$ then the element $\delta(\sigma_{kq}) = E(f)_*(\sigma_{kq})$ is just the element $E(f)$ by well known technique of homological algebra. This construction is considered as follows; let $\sigma_0 = i_*(\sigma_0)$, μ_{kq} are generators of $BP_*(C_f)$ of dimension 0, dimension kq respectively so that $j_*(\mu_{kq}) = \sigma_{kq}$, and let η_R: $BP \simeq S\wedge BP \to BP\wedge BP$ be the Boardman map and put $\eta_{R^*}(\mu_{kq}) = A_f\sigma_0 + \mu_{kq}$ $(A_f \in BP_*(BP))$. Then A_f represents $E(f)$. Replacing $BP \to BP\wedge BP$ by $BP \to H\wedge BP$, we have the $BP_* - e$ invariant (or the functional Chern-Dold character) and is equivalent to the BP_*-Hopf invariant by Corollary 1.9.

3. Applications

For an element $f \in \pi_{kq-1}(S^0)$ $(q = 2(p-1))$ we get a short exact sequence $0 \to (HZ_p)_*(S^0) \to (HZ_p)_*(C_f) \to (HZ_p)_*(S^{kq}) \to 0$ and can choose generators σ_0' and μ'_{kq} of $(HZ_p)_*(C_f)$such that $\sigma_0' = i_*(\sigma_0')$ and $j_*(\mu'_{kq}) = \sigma'_{kq}$ where σ_n' is a canonical generator of $(HZ_p)_n(S^n)$. Let Ψ: $(HZ_p)_*(C_f) \to A_* \otimes (HZ_p)_*(C_f)$ be the coaction, then the definition of the Hopf invariant in the sence of Steenrod is described as follows; $f (\in \pi_{kq-1}(S^0))$ is said to have mod p Hopf invariant 1 if $\langle P^k, H_f \rangle \neq 0$, where P^k is the Steenrod reduced power (interpreted as Sq^{2k} if $p=2$) and $\Psi(\mu'_{kq}) = H_f\sigma_0' + \mu'_{kq}$ $(H_f \in A_*)$.

Theorem 3.1 (Adams, Liulevicius, Shimada-Yamanoshita.) *If f has mod p Hopf invariant* 1 *then*

(2) $k=1, 2$ *or* 4 *for* $p=2$;

(2) $k=1$ *for odd prime* p.

Proof. Consider the following diagram

$$
\begin{array}{ccccc}
BP_*(C_f) & & \xrightarrow{\rho_p\mu} & & (HZ_p)_*(C_f) \\
\downarrow B_* & & \eta_{R*} \swarrow & & \downarrow \Psi \\
H_*(BP)\otimes H_*(C_f) & \xrightarrow{\mu_*\otimes 1} & A_*\otimes(HZ_p)_*(C_f) & \xrightarrow{c\otimes 1} & A_*\otimes(HZ_p)_*(C_f)
\end{array}
$$

then

$$
\begin{aligned}
\Psi(\mu'_{kq}) &= \Psi\mu(\mu_{kq}) = (c\otimes 1)(\mu_*\otimes 1)(A_f\sigma_0'+\mu'_{kq}) \\
&= H_f\sigma'_0+\mu'_{kq}\,.
\end{aligned}
$$

Since A_f is a multiple of $v_1^k/p^{\nu_p(k)+1}=p^{k-\nu_p(k)-1}n_1^k$ or $v_1^k/2^{\nu_2(k)+2}+v_1^{k-3}v_2\times 2^{k-\nu_2(k)-2}n_1^k \bmod 2\cdot H_{kq}(BP)$ by Theorem 1.1, H_f is a multiple of $p^{k-\nu_p(k)-1}\xi_1^k$ or $2^{k-\nu_2(k)-2}\xi_1^{2k}$. In case of an odd prime p $H_f=0$ for $k>1$, in case of $p=2$ and odd number k $H_f=0$ for $k>1$, in case of $p=2$ and even k $H_f=0$ for $k>4$. This completes the proof of Theorem 3.1.

Let $V(0)=S^0\cup_p e^1$ then there exists a map $\phi_k\colon S^{kq}\to S^{kq}V(0)\to V(0)$ such that $\phi_{k*}(\sigma_{kq})=v_1^k\cdot\gamma_0$ where $\sigma_{kq}\in BP_{kq}(S^{kq})$ and $\gamma_0\in BP_0(V(0))$ are generators ([8]). α-series elements $\alpha_k\,(k=1,2,\cdots)$ of $\pi_{kq-1}(S^0)$ are defined by $\alpha_k=j\phi_k$ where $j\colon V(0)\to S^1$ is the canonical projection. We detect these elements by means of the BP_*-Hopf invariant. We have the following diagram of cofibrations;

$$
\begin{array}{ccccccccc}
 & & S^0 & = & S^0 & & & & \\
 & & \downarrow i & & \vdots & & & & \\
S^{kq} & \xrightarrow{\phi_k} & V(0) & \xrightarrow{a} & C_{\phi_k} & \xrightarrow{b} & S^{kq+1} & \longrightarrow & SV(0) \\
\| & & \downarrow j & & \downarrow & & \| & & \downarrow \\
S^{kq} & \xrightarrow{\alpha_k} & S^1 & \xrightarrow{c} & C_{\alpha_k} & \xrightarrow{d} & S^{kq+1} & \longrightarrow & S^2 \\
 & & \downarrow & & \downarrow & & & & \\
 & & S^1 & = & S^1 & & & &
\end{array}
$$

Considering the above diagram following results are obtained;

$$
\begin{array}{ll}
BP_*(S^n) & \left\{\begin{array}{l}\text{generator; } \sigma_n \\ \text{relations; none,}\end{array}\right. \\
BP_*(V(0))=BP_*/(p) & \left\{\begin{array}{l}\text{generator; } \gamma_0 \\ \text{relations; } p\gamma_0=0\,,\end{array}\right. \\
BP_*(C_{\phi_k}) & \left\{\begin{array}{ll}\text{generators;} & a_*(\gamma_0),\ \lambda_{kq+1} \\ \text{relations;} & (p,\, v_1^k)\cdot a_*(\gamma_0)=0 \\ \text{formula;} & b_*(\lambda_{kq+1})=p\sigma_{kq+1}\,,\end{array}\right.
\end{array}
$$

$$BP_*(C_{\alpha_k}) \begin{cases} \text{generators;} & c_*(\sigma_1),\ \mu_{kq+1} \\ \text{relations;} & \text{none} \\ \text{formula;} & d_*(\mu_{kq+1}) = \sigma_{kq+1}\,, \end{cases}$$

and the formula

$$h_*(\lambda_{kq+1}) = p\mu_{kq+1} - v_1{}^k c_*(\sigma_1)\,.$$

The coefficient $v_1{}^k$ of $c_*(\sigma_1)$ is decided up to a multiple of a unit of $Z_{(p)}$. The image of these generators of Thom homomorphism are denoted by σ_n', γ_n', λ_n' and μ_n' respectively.

Theorem 3.2.

$$e(\alpha_k) = v_1{}^k/p \quad in \quad N/\mathrm{Im}\, h^H \cong \mathrm{Ext}^{1,kq}_{BP_*(BP)}(BP_*, BP_*)\,.$$

Proof. By applying the Chern-Dold character $BP_*(C_{\alpha_k}) \xrightarrow{B_*} (H\wedge BP)_*(C_{\alpha_k})$ to μ_{kq+1} we get $B_*(\mu_{kq+1}) = A_{\alpha_k} c_*(\sigma_1') + \mu'_{kq+1}$. A_{α_k} represents BP_*-e invariant of α_k in $N/\mathrm{Im}\, h^H$. The computation

$$\begin{aligned} p\mu'_{kq+1} &= h_* B_*(\lambda_{kq+1}) = B_*(p\mu_{kq+1} - v_1{}^k c_*(\sigma_1)) \\ &= p\mu'_{kq+1} + (pA_{\alpha_k} - v_1{}^k) c_*(\sigma_1') \end{aligned}$$

implies $pA_{\alpha_k} = v_1{}^k$ and this completes the proof.

OSAKA CITY UNIVERCITY

References

[1] J.F. Adams: *On the groups J(X)-IV*, Topology **5** (1966), 21–71.

[2] J.F Adams: Lectures on Generalized Cohomology, Lecture Note in Mathematics 99, Springer-Verlag, 1969, 1–138

[3] J.F. Adams: Quillen's Work on Formal Groups Lows and Complex Cobordism, University of Chicago, 1970.

[4] J.F. Adams: Stable Homotopy and Generalized Homology, University of Chicago, 1971.

[5] S. Araki: Typical Formal Groups in Complex Cobordism and K-theory, Kyoto University, 1973.

[6] P.S. Landweber: *On Panov's theorem*, (to appear).

[7] N.V. Panov: *Characteristic numbers in U-theory*, Izv. Akad. Nauk. SSSR Ser. Mat. **35** (1971), Math. USSR Izv. **5** (1971), 1365–1385.

[8] L. Smith: *On realizing complex bordism modules* I, Amer. J. Math. **92** (1970), 793–856.

[9] L. Smith and R. Zahler: *Detecting stable homotoyp classes by primary BP cohomology operations*, Math. Z. **129** (1972), 137–156.

[10] R. Zahler: *The Adams-Novikov spectral sequence for spheres*, Ann. of Math. **96** (1972), 480–504.