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ON THE HARDY CLASS OF HARMONIC SECTIONS AND VECTOR-VALUED POISSON INTEGRALS

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1. Introduction

Let $D=\{z\in \boldsymbol{C};\ |z|<1\}$ be the unit disc in $\boldsymbol{C}$ and $B=\{e^{it};\ 0\leq t\leq 2\pi\}$ the boundary of D. For a complex-valued continuous function F on D and $0\leq r\leq 1$, we define a continuous function F_r on B by $F_r(u)=F(ru)$ for $u\in B$. We denote by $||F_r||_2$ the usual L^2-norm of F_r with respect to the normalized rotation invariant measure $\frac{1}{2\pi}dt$ on B. Let Δ be the Laplace-Beltrami operator on C^∞ functions on D with respect to the Poincaré metric on D. We denote by $H^2(D)$ the class of all C^∞ functions F on D such that $\Delta F=0$ and $\sup_{0\leq r\leq 1}||F_r||_2$ is finite. The Poisson kernel $P(z,u)$ on $D\times B$ for Δ is given by

$$P(re^{i\theta}, e^{it})=\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}, \qquad 0\leq r<1 .$$

Then it is known (Zygmund [14]) that a function F on D belongs to $H^2(D)$ if and only if there exists a square integrable function f on B with respect to the measure $\frac{1}{2\pi}dt$ on B such that

$$F(z)=\frac{1}{2\pi}\int_0^{2\pi}f(e^{it})P(z, e^{it})dt$$

for $z\in D$. The integral is called the *Poisson integral* of f and $H^2(D)$ the *Hardy class* of harmonic functions on D. Our purpose is to extend these results to sections of a vector bundle on a symmetric space of non-compact type.

Now let G/K be a hermitian symmetric space of non-compact type. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G with respect to the Lie algebra $\mathfrak{k}$ of K. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$.

Now G/K can be holomorphically embedded (Harish-Chandra [2]) as a bounded domain $\mathscr{D}$ in a complex vector space $\mathfrak{p}^-$ and the Šilov boundary of $\mathscr{D}$ in $\mathfrak{p}^-$ is identified with the homogeneous space $G/B(E)$. Here the subgroup

$B(E)$ is a parabolic subgroup $M(E)AN$ where $M(E)$ is the centralizer in K of some element X^0 (cf. §3) in $\mathfrak{a}$, A is the analytic subgroup corresponding to $\mathfrak{a}$ and N is a nilpotent subgroup (cf. §2) of G. Let $\sum$ be the root system of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ with respect to the complexification $\mathfrak{h}^C$ of $\mathfrak{h}$. Fix an order on $\mathfrak{a}$ and choose an order (Satake [12]) of $\sum$ compatible with respect to the order on $\mathfrak{a}$. Let $2\rho_E$ be the restriction to $\mathfrak{a}$ of the sum of all positive roots α of $\sum$ with $\alpha(X^0)>0$.

Assume that G/K is holomorphically isomorphic with a tube domain. Let us consider a linear form $\lambda=z\rho_E$ on the complexification $\mathfrak{a}^C$ of $\mathfrak{a}$ where z is a complex number with the positive imaginary part. Let $\mathfrak{t}^C=Ad(u_1)^{-1}\mathfrak{h}^C$ be the Cartan subalgebra of the complexification $\mathfrak{k}^C$ of $\mathfrak{k}$, obtained from $\mathfrak{h}^C$ by the Cayley transform $Ad(u_1)$ (Moore [10]). Suppose that there exists an irreducible representation $(\tau_\Lambda, V_\Lambda)$ of K with the highest weight Λ on $\mathfrak{t}^C$ satisfying the condition

$$\text{(C)} \qquad {}^tAd(u_1^{-1})\Lambda = -(i\lambda+\rho_E) \qquad \text{on } \mathfrak{a}\,.$$

Let $\tau=\tau_\Lambda^*$ be the representation of K contragredient to τ_Λ. Let $L^2_{\tau,\lambda}(G/B(E))$ be the set of all measurable mapping ϕ of G into the dual space V_Λ^* of V_Λ satisfying

$$\text{(1)} \qquad \phi(\text{gman}) = e^{-(i\lambda+\rho_E)(\log a)}\tau(m^{-1})\phi(g)$$

for $m\in M(E)$, $a\in A$, $n\in N$, $g\in G$ where $\log a$ is the unique element in $\mathfrak{a}$ such that $\exp(\log a)=a$

$$\text{(2)} \qquad \int_K ||\phi(k)||^2 dk < \infty$$

where $||\cdot||$ is a $\tau(K)$-invariant norm on V_Λ^* and dk is the Haar measure of K, normalized by $\int_K dk=1$. G acts on $L^2_{\tau,\lambda}(G/B(E))$ by $U_{\tau,\lambda}(g)\phi(x)=\phi(g^{-1}x)$ for every g, $x\in G$.

Following K. Okamoto [11], we define the generalized Poisson integral $\mathcal{P}_{\tau,\lambda}$ as follows:

$$\mathcal{P}_{\tau,\lambda}\phi(g) = \int_K \tau(k)\phi(gk)dk(g\in G) \qquad \text{for } \phi\in L^2_{\tau,\lambda}(G/B(E))\,.$$

On the other hand, we define a norm $||\cdot||_2$, analogously in Knapp-Okamoto [6], for C^∞ sections of the vector bundle E_τ over G/K associated with the representation τ of K. We construct a representation $(U_\Lambda, \Gamma_2(\Lambda))$ of G on the completion of the space of all C^∞ sections f of E_τ satisfying the condition $||f||_2<\infty$ and certain boundary conditions (cf. §4). We define the *Hardy class* $H_2(\Lambda)$ as the space of all harmonic sections (cf. §5) in $\Gamma_2(\Lambda)$. Then we obtain the following results:

(i) The generalized Poisson integral $\mathcal{P}_{\tau,\lambda}$ maps $(U_{\tau,\lambda}, L^2_{\tau,\lambda}(G/B(E)))$ into $(U_\Lambda, \Gamma_2(\Lambda))$ G-equivariantly and strongly continuously (cf. Theorem 2 in §4).

(ii) The image of a certain G-submodule of $L^2_{\tau,\lambda}(G/B(E))$ under $\mathcal{P}_{\tau,\lambda}$ is contained in the Hardly class $H_2(\Lambda)$ (cf. Theorem 3 in §5).

The second result may be useful in proving the non-vanishingness of $H_2(\Lambda)$.

At the end of this paper we investigate the above condition (C) on the weight Λ for the unit disc D. If τ is the trivial representation, our Hardy class $H_2(\Lambda)$ is the usual Hardy class $H^2(D)$.

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2. Asymptotic behavior of Poisson integrals

In this section we investigate the asymptotic behavior of Poisson integrals of symmetric space G/K of non-compact type. The results obtained in this section are natural generalizations of those obtained by Korányi [7], [8].

Let G be a non-compact semi-simple Lie group with finite center and let K be a maximal compact subgroup of G. Then the homogeneous space G/K is a symmetric space of non-compact type. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G with respect to the Lie algebra $\mathfrak{k}$ of K. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$; then we can find a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{a}$. Let $\sum$ be the set of all non-zero roots of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ with respect the complexification $\mathfrak{h}^C$ of $\mathfrak{h}$. The conjugation σ of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$ preserves $\mathfrak{h}$, and induces the permutation σ of $\sum$ defined by

$$\sigma(\alpha)(H) = \overline{\alpha(\sigma(H))}$$

for $\alpha\in\sum$, $H\in\mathfrak{h}$. We fix a σ-order $>$ of $\sum$, that is a linear order of $\sum$ such that $\sigma(\alpha)>0$ if $\alpha>0$ and if the restriction of α to $\mathfrak{a}$ does not vanish. Let $\sum_0$ be the set of all elements of $\sum$ which vanish on $\mathfrak{a}$. The restriction to $\mathfrak{a}$ of a root of $\sum-\sum_0$ is called a restricted root. The order $>$ on $\sum$ induces a linear order $>$ on the set of restricted roots. Let F be the fundamental system of restricted roots with respect to the order $>$.

Following Satake [12] and Moore [10], if E is a subset of F, let

$$\begin{aligned} \mathfrak{a}(E) &= \{H\in\mathfrak{a};\ \gamma(H)=0 \qquad \text{for all } \gamma\in E\}\ , \\ \textstyle\sum_0(E) &= \{\alpha\in\textstyle\sum;\ \pi(\alpha)=\sum n_\gamma\gamma(\gamma\in E,\ n_\gamma \text{ integers})\} \end{aligned}$$

where π is the restriction map of linear forms on $\mathfrak{h}$ to $\mathfrak{a}$. Let $\sum_+(E)$ (resp. $\sum_-(E)$) be the set of all $\alpha\in\sum-\sum_0(E)$ such that $\alpha>0$ (resp. $\alpha<0$). Then the subalgebras $\sum_{\alpha\in\sum_+(E)} CE_\alpha$, $\sum_{\alpha\in\sum_-(E)} CE_\alpha$ of $\mathfrak{g}^C$ are both invariant under σ where E_α's are root vectors for $\alpha\in\sum_\pm(E)$. Their intersections $\mathfrak{n}(E)$, $\bar{\mathfrak{n}}(E)$ with $\mathfrak{g}$ are the real forms of these subalgebras. The analytic subgroups of G corresponding to $\mathfrak{n}(E)$, $\bar{\mathfrak{n}}(E)$ will be denoted by $N(E)$, $\bar{N}(E)$. Let $\mathfrak{b}(E)$ be the normalizer of $\mathfrak{n}(E)$ in $\mathfrak{g}$ and $\mathfrak{m}(E)$ be the centralizer of $\mathfrak{a}(E)$ in $\mathfrak{k}$. Let $B(E)$ be the normalizer of $\mathfrak{n}(E)$ in G, and $M(E)$ the centralizer of $\mathfrak{a}(E)$ in K. Let $A(E)$ be the analytic subgroup of G corresponding to $\mathfrak{a}(E)$. If $E=\phi$, we write $\mathfrak{a}$, A, $\mathfrak{n}$, $\bar{\mathfrak{n}}$, N, $\bar{N}$, $\mathfrak{b}$, $\mathfrak{m}$, B and M instead of $\mathfrak{a}(E)$, $A(E)$, $\cdots$ respectively. We denote by $2\rho_E \in \mathfrak{a}^*$ the sum of all restrictions of roots in $\sum_+(E)$ with multiplicity counted where $\mathfrak{a}^*$ is the dual space of $\mathfrak{a}$. We also write ρ instead of ρ_E if $E=\phi$. We obtain the Iwasawa decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ and $G=KAN$. So for $g \in G$, it can be uniquely decomposed as $g=\kappa(g)\exp H(g)n(g)$ where $\kappa(g)\in K$, $H(g)\in \mathfrak{a}$ and $n(g)\in N$.

DEFINITION. For a complex number $z \in \boldsymbol{C}$, we put $\lambda = z\rho_E \in \mathfrak{a}_C^*$ where $\mathfrak{a}_C^*$ is the complexification of the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$. For a finite dimensional unitary representation τ of K on a complex vector space V_τ, we denote by $C_{\tau,\lambda}(G/B(E))$ the set of all continuous mappings ϕ of G into V_τ satisfying the following condition:

$$\phi(gman) = e^{-(i\lambda+\rho_E)(\log a)}\tau(m^{-1})\phi(g) \tag{1}$$

for $m \in M(E)$, $a \in A$, $n \in N$ where $\log a$ denotes the unique element of $\mathfrak{a}$ such that $a=\exp(\log a)$. For a real number $p \geq 1$, we also denote by $L^p_{\tau,\lambda}(G/B(E))$ the set of all measurable mappings ϕ of G into V_τ satisfying (1) and

$$\|\phi\|_p^p = \int_K \|\phi(k)\|^p dk < \infty \tag{2}$$

where $\|\cdot\|$ is a $\tau(K)$-invariant norm of V_τ and dk is the Haar measure of K, normalized by $\int_K dk=1$.

Following Okamoto [11], for every element ϕ of $C_{\tau,\lambda}(G/B(E))$ or $L^p_{\tau,\lambda}(G/B(E))$, we define a Poisson integral of ϕ by

$$\mathcal{P}_{\tau,\lambda}\phi(g) = \int_K \tau(k)\phi(gk)dk\,. \tag{3}$$

Then $\mathcal{P}_{\tau,\lambda}\phi$ is a section of the vector bundle E_τ over G/K associated to the representation τ of K. Before investigating the asymptotic behavior of $\mathcal{P}_{\tau,\lambda}\phi$ we prepare the following Lemma.

Lemma 1. *Let G/K be a symmetric space of non-compace type. Then*

$$e^{2\rho_E(H(gm))} = e^{2\rho_E(H(g))}$$

for every $g \in G$, $m \in M(E)$.

Proof. For the proof, we notice (Korányi [8] Lemma 1.1) the fact that

$e^{2\rho_E(H(b))}=|\det(Ad(b))|$ for $b\in B(E)$ where $Ad(b)$ is the adjoint representation of $B(E)$ on $\mathfrak{b}(E)$. Then for $g=\kappa(g)\exp H(g)n(g)$ and $m\in M(E)$, using the decomposition $B(E)=M(E)AN$ (cf. Moore [10]), we may write

$$\exp H(g)n(g)m = m'a'n'$$

where $m'\in M(E)$, $a'\in A'$, $n'\in N$. Then $\exp H(gm)=a'$. Hence we obtain

$$\begin{aligned} e^{2\rho_E(H(gm))} &= |\det(Ad(H(gm)))| \\ &= |\det(Ad(m'a'n'))| \end{aligned}$$

since $M(E)$ is compact and N is nilpotent. Therefore we have

$$\begin{aligned} e^{2\rho_E(H(gm))} &= |\det(Ad(\exp(H(g))n(g)m))| \\ &= |\det(Ad(\exp H(g)))| \\ &= e^{2\rho_E(H(g))} \end{aligned}$$

Q.E.D.

From Lemma 1, the right translation by $b\in B(E)$ of the measure $e^{-2\rho_E(H(g))}dg$ on G is equal to $e^{-2\rho_E(H(b))}e^{-2\rho_E(H(g))}dg$. Therefore the measure $e^{-2\rho_E(H(g))}dg$ on G induces (Bourbaki [1]) the measure $d\mu_E$ on $G/B(E)$ unique up to the constant factor such that

$$\int_{G/B(E)}\int_{B(E)} f(gb)db\, d\mu_E(gB(E)) = \int_G f(g)e^{-2\rho_E(H(g))}dg$$

for every continuous function f on G with compact support. Let $d\mu_E(guB(E))$ be the transform of the measure $d\mu_E$ under the mapping $G/B(E)\ni uB(E)\mapsto guB(E)\in G/B(E)$, then it follows (Korányi [8]) that

$$(4) \qquad d\mu_E(guB(E)) = e^{2\rho_E(H(gu)-H(u))}d\mu_E(uB(E)).$$

Thus $d\mu_E$ is a K-invariant measure on $G/B(E)$. Let $d\mu_E$ be normalized by $\int_{G/B(E)} d\mu_E=1$. And let us identify $K/M(E)$ and $G/B(E)$ under the mapping $K/M(E)\ni kM(E)\mapsto kB(E)\in G/B(E)$, then the measure $d\mu_E$ corresponds to the measure $dk_{M(E)}$ on $K/M(E)$ induced from the measure dk on K. And then the mapping $G/B(E)\ni uB(E)\mapsto guB(E)\in G/B(E)$ induces the transformation $K/M(E)\ni kM(E)\mapsto\kappa(gk)M(E)\in K/M(E)$. Put $h=\kappa(gk)$. Then we have, from the above equality (4), that $k=\kappa(g^{-1}h)$, $H(gk)=-H(g^{-1}h)$ and

$$(4') \qquad dh_{M(E)} = e^{2\rho_E(H(gk))}dk_{M(E)}.$$

In the case $E=\phi$, $M(E)$ is the centralizer M of $\mathfrak{a}$ in K and the equality (4′) is obtained in Harish-Chandra [3].

Corollary *G acts on $L^p_{\tau,\lambda}(G/B(E))$ by $U_{\tau,\lambda}(g)\phi(x)=\phi(g^{-1}x)$ for every g, $x\in G$. Then $U_{\tau,\lambda}(g)$ is a bounded operator on $L^p_{\tau,\lambda}(G/B(E))$ with respect to* *the norm* $\|\cdot\|_p$ *and* $C_{\tau,\lambda}(G/B(E))$ *is a* G*-invariant subspace of* $L^p_{\tau,\lambda}(G/B(E))$.

Indeed, we have

$$\int_K \|\phi(g^{-1}k)\|^p dk \leq \sup_{k\in K} |e^{-(i\lambda+\rho_E)(H(g^{-1}k))}|^p \int_K \|\phi(\kappa(g^{-1}k))\|^p dk$$
$$\leq \sup_{k\in K} |e^{-(i\lambda+\rho_E)(H(g^{-1}k))}|^p \sup_{k\in K} e^{-2\rho_E(H(gk))} \int_K \|\phi(k)\|^p dk$$

since the mapping $k\mapsto \|\phi(\kappa(g^{-1}k))\|$ is right $M(E)$-invariant, i.e. it is invariant under the right translation by elements of $M(E)$.

Proposition 1. *Let* $\mathfrak{a}^+(E)=\{H\in\mathfrak{a}(E);\ \alpha(H)>0$ *for all* $\alpha\in\sum_+(E)\}$. *For* $H\in\mathfrak{a}^+(E)$, *we put* $a_t=\exp tH$. *Then we have*

$$(5)\quad \lim_{t\to\infty} e^{(i\lambda+\rho_E)(\log a_t)}\mathcal{P}_{\tau,\lambda}\phi(ga_t) = \int_{\bar{N}(E)} e^{(i\lambda-\rho_E)(H(\bar{n}))}\tau(\kappa(\bar{n}))\phi(g)d\bar{n}$$

for all $g\in G$ *and* $\phi\in C_{\tau,\lambda}(G/B(E))$. *Here the measure* $d\bar{n}$ *is the Haar measure on* $\bar{N}(E)$, *normalized by* $\int_{\bar{N}(E)} e^{-2\rho_E(H(\bar{n}))}d\bar{n}=1$.

Proof. For every integrable right $M(E)$-invariant function f on K, it follows (cf. Korányi [8] Lemma 1.3) that

$$\int_K f(k)dk = \int_{\bar{N}(E)} f(\kappa(\bar{n}))e^{-2\rho_E(H(\bar{n}))}d\bar{n}\ .$$

For $\phi\in C_{\tau,\lambda}(G/B(E))$, the V_τ-valued function $\tau(k)\phi(gk)$ on K is right $M(E)$-invariant for fixed $g\in G$. Hence it follows that

$$\mathcal{P}_{\tau,\lambda}\phi(ga_t) = \int_{\bar{N}(E)} e^{-2\rho_E(H(\bar{n}))}\tau(\kappa(\bar{n}))\phi(ga_t\kappa(\bar{n}))d\bar{n}\ .$$

Since we have

$$a_t\kappa(\bar{n}) = \kappa(a_t\bar{n}a_t^{-1})\exp(H(a_t\bar{n}a_t^{-1})-H(\bar{n})+tH)n\ ,\qquad n\in N$$

for $\bar{n}\in\bar{N}(E)$, it follows from (1) that

$$\mathcal{P}_{\tau,\lambda}\phi(ga_t) = \int_{\bar{N}(E)} e^{-(i\lambda+\rho_E)(H(a_t\bar{n}a_t^{-1})+H(n)+tH)}e^{-2\rho_E(H(\bar{n}))}\tau(\kappa(\bar{n}))\,\phi(g\kappa(a_t\bar{n}a_t^{-1}))d\bar{n}\ .$$

Then $a_t\bar{n}a_t^{-1}$ converges (cf. Korányi [8] Lemma 2.2) to the identity of G for every $H\in\mathfrak{a}^+(E)$ and $\bar{n}\in\bar{N}(E)$ as $t\to\infty$. Then we obtain the conclusion. Q.E.D.

If $\lambda,\ \mu\in\mathfrak{a}^*_C$, let $H_\lambda\in\mathfrak{a}^C$ be determined by $\lambda(H)=B(H_\lambda, H)$ for $H\in\mathfrak{a}$, where B is the Killing form of $\mathfrak{g}^C$. Put $\langle\lambda, \mu\rangle=B(H_\lambda, H_\mu)$. Then the integral of the right hand side of (5) converges if $\mathrm{Re}\langle i\lambda, \alpha\rangle<0$ for all $\alpha\in\sum_+(E)$.

From now on we shall assume $\lambda = z\rho_E$ satisfies the condition: $\mathrm{Re}\langle i\lambda, \alpha\rangle < 0$ for all $\alpha \in \sum_+(E)$, that is, $y > 0$ ($z = x + iy$).

Lemma 2. *For $\phi \in L^1_{\tau,\lambda}(G/B(E))$, we obtain*

$$(6)\qquad \mathcal{P}_{\tau,\lambda}\phi(g) = \int_{K/M(E)} e^{(i\lambda-\rho_E)(H(g^{-1}k))}\tau(\kappa(g^{-1}k))\phi(k)dk_{M(E)}$$

where $dk_{M(E)}$ is the K-invariant measure on $K/M(E)$ induced from the Haar measure dk on K. Therefore $\mathcal{P}_{\tau,\lambda}$ may be ragarded as an integral operator with the kernel $K_{\tau,\lambda}(g, k) = e^{(i\lambda-\rho_E)(H(g^{-1}k))}\tau(\kappa(g^{-1}k))$.

Proof. For $\phi \in L^1_{\tau,\lambda}(G/B(E))$,

$$\mathcal{P}_{\tau,\lambda}\phi(g) = \int_{K/M(E)} e^{-(i\lambda+\rho_E)(H(gk))}\tau(k)\phi(\kappa(gk))dk_{M(E)}$$

from the condition (1). Put $h = \kappa(gk)$. Substituting (4′) into the right hand of the above equality, Lemma 2 is obtained. Q.E.D.

Corollary *For every $g \in G$, $k \in K$, let $||K_{\tau,\lambda}(g, k)||$ be the operator norm $||\cdot||$ of the transformation $K_{\tau,\lambda}(g, k)$ of V_τ with respect to the norm $||\cdot||$ in V_τ. Then it follows that*

(i) $\quad ||K_{\tau,\lambda}(g, k)|| = |e^{(i\lambda-\rho_E)(H(g^{-1}k))}|$

(ii) $\quad ||K_{\tau,\lambda}(g, km)|| = ||K_{\tau,\lambda}(g, k)|| \qquad$ *for all $g \in G$, $k \in K$, $m \in M(E)$.*

Indeed, (i) is clear. Since we assume $\lambda = z\rho_E$ and z is a complex number, (ii) follows from Lemma 1.

Lemma 3. *For $H \in \mathfrak{a}^+(E)$, we put $a_t = \exp tH$. For every neighborhood V of $eM(E)$ in $K/M(E)$, we have*

$$\lim_{t\to\infty} |e^{(i\lambda+\rho_E)(\log a_t)}| \int_{K/M(E)-V} ||K_{\tau,\lambda}(a_t, k)||dk_{M(E)} = 0\,.$$

Proof. For every continuous function ϕ on $K/M(E)$, we have

$$(7)\qquad \int_{K/M(E)} ||K_{\tau,\lambda}(a_t, k)||\,|\phi(kM(E))|\,dk_{M(E)}$$
$$= \int_{\bar{N}(E)} |e^{(i\lambda-\rho_E)(a_t^{-1}\kappa(\bar{n}))}|\,|\phi(\kappa(\bar{n})M)|\,e^{-2\rho_E(H(\bar{n}))}d\bar{n}\,.$$

Put $\bar{n}' = a_t^{-1}\bar{n}a_t$. Then it follows that $d\bar{n} = e^{-2\rho_E(H(\bar{n}'))}d\bar{n}'$ and $a_t^{-1}\kappa(\bar{n}) = \kappa(\bar{n}')\exp(H(\bar{n}') - H(a_t\bar{n}'a_t^{-1}) - tH)n$ for some $n \in N$. Then, substituting these into (7), we have

$$(7) = |e^{-(i\lambda+\rho_E)(\log a_t)}| \int_{\bar{N}(E)} |e^{-(i\lambda+\rho_E)(H(a_t\bar{n}a_t^{-1}))}e^{(i\lambda-\rho_E)(H(\bar{n}))}\phi(\kappa(a_t\bar{n}a_t^{-1})M(E))|\,d\bar{n}\,.$$

Therefore we obtain

$$(8)\quad \lim_{t\to\infty} |e^{(i\lambda+\rho_E)(\log a_t)}| \int_{K/M(E)} \|K_{\tau,\lambda}(a_t, k)\| \, |\phi(kM(E))| \, dk_{M(E)}$$
$$= |\phi(eM(E))| \int_{\bar{N}(E)} |e^{(i\lambda-\rho_E)(H(\bar{n}))}| \, d\bar{n} = |\phi(eM(E))| \, C_E(\lambda) < \infty$$

where $C_E(\lambda) = \int_{\bar{N}(E)} |e^{(i\lambda-\rho_E)(H(\bar{n}))}| \, d\bar{n} = \int_{\bar{N}(E)} e^{-(y+1)\rho_E(H(\bar{n}))} d\bar{n} < \infty$ because of $\lambda = z\rho_E$ ($z = x+iy$, $y>0$). In particular, we have

$$(9)\quad \lim_{t\to\infty} |e^{(i\lambda+\rho_E)(\log a_t)}| \int_{K/M(E)} \|K_{\tau,\lambda}(a_t, k)\| dk_{M(E)} = C_E(\lambda) \,.$$

For every neighborhood V of $eM(E)$, there exists a continuous function ϕ on $K/M(E)$ such that $|\phi| \leq 1$, $\phi(eM(E)) = 1$ and $\sup_{kM(E)\notin V} |\phi(kM(E))| = m < 1$. Then we have

$$(10)\quad \lim_{t\to\infty} |e^{(i\lambda+\rho_E)(\log a_t)}| \int_{K/M(E)} \|K_{\tau,\lambda}(a_t, k)\| \, |\phi(kM(E))| \, dk_{M(E)} = C_E(\lambda) \,.$$

On the other hand, we obtain

$$(11)\quad \int_{K/M(E)} \|K_{\tau,\lambda}(a_t, k)\| \, |\phi(kM(E))| \, dk_{M(E)}$$
$$\leqq \int_{K/M(E)} \|K_{\tau,\lambda}(a_t, k)\| dk_{M(E)} + (m-1) \int_{K/M(E)-V} \|K_{\tau,\lambda}(a_t, k)\| dk_{M(E)} \,.$$

Hence from $m-1<0$, (9), (10), (11), the proof is complete. Q.E.D.

Proposition 2. *Let $1<p<\infty$. For $H \in \mathfrak{a}^+(E)$, we put $a_t = \exp tH$. Then for every $\phi \in L^p_{\tau,\lambda}(G/B(E))$, we have*

$$\lim_{t\to\infty} \int_K \left\{ \left\| e^{(i\lambda+\rho_E)(\log a_t)} \mathcal{P}_{\tau,\lambda}\phi(ka_t) - \int_{\bar{N}(E)} e^{(i\lambda-\rho_E)(H(\bar{n}))} \tau(\kappa(\bar{n}))\phi(k) d\bar{n} \right\| \right\}^p dk = 0 \,.$$

Proof. From Lemma 1, for every $\phi \in L^p_{\tau,\lambda}(G/B(E))$,

$$(12)\quad \mathcal{P}_{\tau,\lambda}\phi(ka_t) - \int_K K_{\tau,\lambda}(a_t, h)\phi(k) dh = \int_K K_{\tau,\lambda}(a_t, h)(\phi(kh) - \phi(k)) dh \,.$$

For every function $g \in L^q(K/M(E))$ where $\frac{1}{p} + \frac{1}{q} = 1$, we put $\tilde{g}(k) = g(kM(E))$. Then we have

$$\int_K \left\{ \int_K \|K_{\tau,\lambda}(a_t, h)\| \, \|\phi(kh) - \phi(k)\| dh \right\} \tilde{g}(k) dk$$
$$\leq \int_K \left\{ \left| \int_K \|\phi_h(k) - \phi(k)\| \tilde{g}(k) dk \right| \right\} \|K_{\tau,\lambda}(a_t, h)\| dh$$
$$\leq \|\tilde{g}\|_{L^p(K)} \int_K \|\phi_h - \phi\|_{L^p(K)} \|K_{\tau,\lambda}(a_t, h)\| dh$$

where $\phi_h(k)=\phi(kh)$ and $||\phi_h-\phi||_{L^p(K)}$ is the usual L^p-norm of the function $||\phi_h(k)-\phi(k)||$ on K with respect to the Haar measure dk on K. Hence together with (12), we obtain

$$\left\{\int_K\left\|\mathcal{P}_{\tau,\lambda}\phi(ka_t)-\int_K K_{\tau,\lambda}(a_t, h)\phi(k)dh\right\|^p dk\right\}^{1/p} \leqq \int_K ||\phi_h-\phi||_{L^p(K)}||K_{\tau,\lambda}(a_t, h)||dh\,. \tag{13}$$

Here for every neighborhood V of $eM(E)$ in $K/M(E)$, the right hand side of

$$(13)\leq \sup_{hM(E)\in V}||\phi_h-\phi||_{L^p(K)}\int_V||K_{\tau,\lambda}(a_t, h)||dh_{M(E)} +2||\phi||_{L^p(K/M(E))}\int_{K/M(E)-V}||K_{\tau,\lambda}(a_t, h)||dh_{M(E)}\,.$$

Therefore by Lemma 3 and its proof, we get

$$\lim_{t\to\infty}|e^{(i\lambda+\rho_E)(\log a_t)}|\left\{\int_K\left\{\left\|\mathcal{P}_{\tau,\lambda}\phi(ka_t)-\int_K K_{\tau,\lambda}(a_t, h)\phi(k)dh\right\|\right\}^p dk\right\}^{1/p}=0\,. \tag{14}$$

On the other hand, since $e^{(i\lambda+\rho_E)(\log a_t)}\int_K K_{\tau,\lambda}(a_t, h)\phi(k)dh$ is equal to $\int_{\overline{N}(E)}e^{(i\lambda-\rho_E)(H(\bar{n}))}e^{-(i\lambda+\rho_E)(H(a_t\bar{n}a_t^{-1}))}\tau(\kappa(\bar{n}))\phi(k)d\bar{n}$, it follows that

$$\left\|e^{(i\lambda+\rho_E)(\log a_t)}\int_K K_{\tau,\lambda}(a_t, h)\phi(k)dh-\int_{\overline{N}(E)}e^{(i\lambda-\rho_E)(H(\bar{n}))}\tau(\kappa(\bar{n}))\phi(k)d\bar{n}\right\| \leq ||\phi(k)||\left\{\int_{\overline{N}(E)}|e^{-(i\lambda+\rho_E)(H(a_t\bar{n}a_t^{-1}))}-1|\,|e^{(i\lambda-\rho_E)(H(\bar{n}))}|d\bar{n}\,.\right. \tag{15}$$

From the fact that $C_E(\lambda)=\int_{\overline{N}(E)}|e^{(i\lambda-\rho_E)(H(\bar{n}))}|d\bar{n}<\infty$ and $a_t\bar{n}a_t^{-1}$ converges to the identity as $t\to\infty$, the right hand side converges to zero as $t\to\infty$. So, together with (14), Proposition 2 is proved. Q.E.D.

REMARK From Proposition 2, it follows that for every $\phi\in L^p_{\tau,\lambda}(G/B(E))$

$$\lim_{t\to\infty}\int_K||e^{(i\lambda+\rho_E)(\log a_t)}\mathcal{P}_{\tau,\lambda}\phi(ka_t)||^p dk=\int_K\left\{\left\|\int_{\overline{N}(E)}e^{(i\lambda-\rho_E)(H(\bar{n}))}\tau(\kappa(\bar{n}))\phi(k)d\bar{n}\right\|\right\}^p dk$$

if $1<p<\infty$ and $\mathrm{Re}\langle i\lambda, \alpha\rangle<0$ for all $\alpha\in\sum_+(E)$. Now we denote the above limit by $(||\mathcal{P}_{\tau,\lambda}\phi||_{p,H})^p$. Then we have $||\mathcal{P}_{\tau,\lambda}\phi||_{p,H}\leq C_E(\lambda)||\phi||^p_{L^p(K/M(E))}$ where $||\phi||_{L^p(K/M(E))}$ is the usual L^p-norm of the function $||\phi(k)||$ on K.

3. Properties of hermitian symmetric spaces

From now on we shall assume G/K is an irreducible hermitian symmetric space of tube type; let G be a non-compact connected simple Lie group with a faithful matrix representation, K a maximal compact subgroup and we shall assume the homogeneous space G/K is an irreducible hermitian symmetric space holomorphically diffeomorphic with a tube domain. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ corresponding to $\mathfrak{k}$. For any subspace $\mathfrak{m}$ of $\mathfrak{g}$, we denote by $\mathfrak{m}^C$ the complexification of $\mathfrak{m}$. Since G has a faithful matrix representation, we can regard G as a subgroup of a connected Lie group G^C with Lie algebra $\mathfrak{g}^C$. Let K^C be the analytic subgroup of G^C with Lie algebra $\mathfrak{k}^C$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$, T the corresponding analytic subgroup of G, and let T^C be the analytic subgroup of G^C with Lie algebra $\mathfrak{t}^C$. Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$ and of $\mathfrak{g}$. And T is also a Cartan subgroup of K and of G.

Let R be the set of all non-zero roots of $(\mathfrak{g}^C, \mathfrak{t}^C)$. For $\alpha\in R$, let $\mathfrak{g}_\alpha$ be the root space for α, then $\mathfrak{g}_\alpha\subset\mathfrak{k}^C$ or $\mathfrak{g}_\alpha\subset\mathfrak{p}^C$, and α is called a compact root or a non-compact root according to the respective cases. Let $R_\mathfrak{k}$ and $R_\mathfrak{n}$ be the set of compact and non-compact roots respectively.

We identify $\mathfrak{p}$ and $\mathfrak{p}^C$ with the tangent space $T_{eK}(G/K)$ of G/K at eK and its complefixication $T^C_{eK}(G/K)$, respectively, under the natural projection of G onto G/K. Let $\mathfrak{p}_-$ (resp. $\mathfrak{p}_+$) be the subspace of $\mathfrak{p}^C$ corresponding to the set of all holomorphic (anti-holomorphic) tangent vectors of $T^C_{eK}(G/K)$ respectively. Then $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are $ad(\mathfrak{k}^C)$-invariant abelian subalgebras of $\mathfrak{p}^C$ such that $\mathfrak{p}^C=\mathfrak{p}_++\mathfrak{p}_-$. Let P_+, P_- be the corresponding analytic subgroups of G^C. Moreover there exists a subset $P_\mathfrak{n}$ of R such that $\mathfrak{p}_+=\sum_{\alpha\in P_\mathfrak{n}}\mathfrak{g}_\alpha$. We can define a linear order $\succ$ on R such that the set P of all positive roots includes $P_\mathfrak{n}$. We put $P_\mathfrak{k}=P\cap R_\mathfrak{k}$.

Let τ be the conjugation of $\mathfrak{g}^C$ with respect to the compact real form $\mathfrak{g}_u=\mathfrak{k}+\sqrt{-1}\,\mathfrak{p}$ of $\mathfrak{g}^C$, and we choose root vectors $\{E_\alpha\}$ such that $\tau E_\alpha=-E_{-\alpha}$ for $\alpha\in R$. Let $\Delta=\{\gamma_1\succ\cdots\succ\gamma_m\}$ be the maximal set of strongly orthogonal non-compact positive roots of Harish-Chandra [2]. For $\alpha\in R$, let H_α be the unique element of $\sqrt{-1}\,\mathfrak{t}$ satisfying $B(H_\alpha, H)=\alpha(H)$ for all $H\in\mathfrak{t}^C$. For $\alpha\in\Delta$, we put $X^0_\alpha=E_\alpha+E_{-\alpha}$, $Y^0_\alpha=(-\sqrt{-1})(E_\alpha-E_{-\alpha})$ and $H'_\alpha=\dfrac{2}{\langle\alpha, \alpha\rangle}H_\alpha$. Moreover we put $X^0=\sum_{\alpha\in\Delta}X^0_\alpha$ and $Z^0=-\dfrac{\sqrt{-1}}{2}\sum_{\alpha\in\Delta}H'_\alpha$. Let $\mathfrak{t}^-=\sqrt{-1}\sum_{\alpha\in\Delta}\boldsymbol{R}H'_\alpha$ be the subalgebra of $\mathfrak{t}$ spanned by $\sqrt{-1}\,H'_\alpha$, $\alpha\in\Delta$ over the real number field $\boldsymbol{R}$. Let $\mathfrak{t}^+$ be the orthocomplement of $\mathfrak{t}^-$ in $\mathfrak{t}$ with respect to the Killing form B, and let T^-, T^+ be the analytic subgroups of T corresponding to $\mathfrak{t}^-$, $\mathfrak{t}^+$ respectively. We have the decomposition $\mathfrak{t}^C=(\mathfrak{t}^+)^C+(\mathfrak{t}^-)^C$, and corresponding to this, we can decompose each element μ of the complexification $\mathfrak{t}^*_C$ of the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$, as

$$\mu=\mu_++\mu_- \tag{16}$$

where μ_+ (resp. μ_-) is the same as the restriction of μ on $(\mathfrak{t}^+)^C$ (resp. $(\mathfrak{t}^-)^C$) and vanishes identically on $(\mathfrak{t}^-)^C$ (resp. $(\mathfrak{t}^+)^C$). The vectors X_α^0, $\alpha\in\Delta$ span a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ and $\mathfrak{h}=\mathfrak{t}^+ + \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Let A, H be the analytic subgroups of G corresponding to $\mathfrak{a}$, $\mathfrak{h}$ respectively.

Now we define, analogously in Knapp-Okamoto [6]

$$u_t = \exp\left(\frac{\pi t}{4}\sum_{\alpha\in\Delta}(-\sqrt{-1})Y_\alpha^0\right)\in G^C \qquad \text{for} \quad 0\leq t\leq 1\,.$$

We have the following lemma:

Lemma 4. *Let G/K be an irreducible hermitian symmetric space, not necessarily of tube type. Then we have the following decomposition of u_t:*

$$(17) \qquad u_t = \zeta_t k_t z_t \qquad \text{for} \quad 0\leq t\leq 1$$

where $\zeta_t=\exp\left(\tan\frac{\pi t}{4}\sum_{\alpha\in\Delta}E_{-\alpha}\right)\in P_-$, $k_t=\exp\left(\log\left(\cos\frac{\pi t}{4}\right)\sum_{\alpha\in\Delta}H'_\alpha\right)\in T^C$ *and* $z_t=\exp\left(-\tan\frac{\pi t}{4}\sum_{\alpha\in\Delta}E_\alpha\right)\in P_+$. *Moreover for* $0<t<1$,

$$(18) \qquad \zeta_t = a_s h_r \eta_t$$

where $a_s=\exp(sX^0)\in A\left(\tanh(s)=\tan\frac{\pi t}{4}\right)$, $h_r=\exp(r\sum_{\alpha\in\Delta}H'_\alpha)\in T^C\left(e^r=\frac{1}{\cosh(s)}\right)$ *and* $\eta_t=\exp(-\tanh(s)e^{-2r}\sum_{\alpha\in\Delta}E_\alpha)\in P_+$.

The proof follows from a straightforward calculation in $SL(2, \boldsymbol{C})$, analogously in Knapp-Okamoto [6].

Now it is well-known that

$$(19) \qquad Ad(u_1) = id \text{ on } \mathfrak{t}^+ \text{ and } Ad(u_1)(H'_\alpha) = X_\alpha^0\,, \qquad \alpha\in\Delta\,.$$

Hence we obtain $Ad(u_1)(\mathfrak{t}^C)=\mathfrak{h}^C$. $Ad(u_1)$ is called a *Cayley transform* (cf. Moore [10]). Let Σ be the set of all non-zero roots of $(\mathfrak{g}^C, \mathfrak{h}^C)$. For $\lambda\in R$, we put ${}^tAd(u_1^{-1})\lambda(X)=\lambda(Ad(u_1^{-1})X)$, $X\in\mathfrak{h}$. Then ${}^tAd(u_1^{-1})\lambda$ belongs to Σ if $\lambda\in R$; ${}^tAd(u_1^{-1})$ sends R onto Σ. We can define a linear order $>$ on Σ such that the set of all positive roots in Σ coincides with ${}^tAd(u_1^{-1})P$.

Let Π be a fundamental system of R with respect to the order $\succ$. Then under the assumption of *tube type*, it follows (Moore [10]) that

$$\pi(\Pi)-\{0\} = \left\{\frac{1}{2}(\gamma_2-\gamma_1), \cdots, \frac{1}{2}(\gamma_m-\gamma_{m-1}), \gamma_1\right\}$$

where for a linear form λ on $\mathfrak{t}^C$, $\pi(\lambda)$ means the restriction of λ to $(\mathfrak{t}^-)^C$. Therefore it follows immediately that the above linear order $>$ on Σ is a σ-order. Then, as in §2, we can consider Σ_0, $\Sigma_\pm$ and F. The σ-invariantness of Σ_+ implies the following equality:

$$(20) \qquad \sum_{\alpha' \in \Sigma_+} H_{\alpha'} = 2H_\rho ,$$

where H_ρ is an element of $\mathfrak{a}$ defined by $\rho(H)=B(H, H_\rho)$ for all $H \in \mathfrak{a}$ and $H_{\alpha'}$ is an element of $\mathfrak{h}^C$ defined by $\alpha'(H)=B(H, H_{\alpha'})$ for all $H \in \mathfrak{h}$.

4. Construction of Hardly class (I)

We shall always assume that G/K is an irreducible hermitian symmetric space of tube type. We take $\{\alpha \in F: \alpha(X^0)=0\}$ as the subset E of F in §2. Then, under the notation in §2, $\mathfrak{a}(E)$ is spanned by X^0, and $M(E)$ is the centralizer of X^0 in K. Let 2δ be the sum of all roots in P. Then we obtain

$$(21) \qquad \begin{aligned} \rho &= {}^t Ad(u_1^{-1})\delta \qquad \text{on } \mathfrak{a} , \\ \rho_E(X^0) &= \rho(X^0) = \delta(\sum_{\alpha \in \Delta} H'_\alpha) . \end{aligned}$$

Let Λ be an integral linear form on $\mathfrak{t}^C$, dominant with respect to $\mathfrak{k}$, that is, Λ satisfies

(i) $\quad \Lambda(H) \in 2\pi\sqrt{-1}\,\boldsymbol{Z} \qquad$ for every $H \in \mathfrak{t}$, $\exp(H)=e$

(ii) $\quad \langle \Lambda, \alpha \rangle \geqq 0 \qquad$ for every $\alpha \in P_{\mathfrak{k}}$.

Let τ_Λ be the irreducible unitary representation of K with the highest weight Λ on the complex vector space V_Λ. Then τ_Λ is uniquely extended to a holomorphic representation of K^C. Since P_+ is a normal subgroup in the subgroup $K^C P_+$ of G^C, we can extend τ_Λ uniquely to a holomorphic representation of $K^C P_+$ which is trivial on P_+. We denote by the same notation τ_Λ this extended representation. Let $\tau = \tau_\Lambda^*$ be the representation contragredient to τ_Λ on the dual space V_Λ^* of V_Λ. Let $\tilde{E}_\Lambda$ be the vector bundle over $G^C/K^C P_+$ associated to the representation τ of $K^C P_+$. We notice that $G \cap K^C P_+ = K$. Then, as is well-known, G/K can be identified with the open G-orbit of the origin in $G^C/K^C P_+$. We denote by E_Λ the restriction of $\tilde{E}_\Lambda$ to the open submanifold G/K of $G^C/K^C P_+$.

DEFINITION. Let $\Gamma(\Lambda)$ be the set of all C^∞ mappings f of $GK^C P_+$ into V_Λ^* satisfying

$$(22) \qquad f(gb) = \tau(b^{-1}) f(g) , \quad g \in GK^C P_+ , \quad b \in K^C P_+$$

$$(23) \qquad \|f\|_2^2 = \lim_{t \uparrow 1} \int_K \|f(ku_t)\|^2 dk < \infty$$

where $\|\cdot\|$ is the operator norm in V_Λ^* with respect to $\tau_\Lambda(K)$ invariant norm $\|\cdot\|$ in V_Λ. From Lemma 4, $f(ku_t)$ is well-defined. We remark that the space $\Gamma(\Lambda)$ can be regarded as a space of C^∞ sections of E_Λ. For an element $\phi \in L^2_{\tau,\lambda}(G/B(E))$, a Poisson integral $\mathcal{P}_{\tau,\lambda}\phi$ of ϕ can be considered as a C^∞ section of E_Λ since $\mathcal{P}_{\tau,\lambda}$ is an integral operator with the kernel $K_{\tau,\lambda}$. Moreover from the results in §2, we have the following theorem.

Theorem 1. *Let G/K be an irreducible hermitian symmetric space of tube type. Suppose that $\lambda = z\rho_E \in \mathfrak{a}_C^*$, $z = x+iy \in \boldsymbol{C}$, $y>0$ satisfies the following condition*:

$$\text{(C)} \qquad {}^t Ad(u_1^{-1})\Lambda = -(i\lambda + \rho_E) \text{ on } \mathfrak{a}\,.$$

Then we have

$$\mathcal{P}_{\tau,\lambda} L^2_{\tau,\lambda}(G/B(E)) \subset \Gamma(\Lambda)\,.$$

Before proving the Theorem, we prepare the following Lemma.

Lemma 5. *Let G/K be an irreducible hermitian symmetric space of tube type. Under the above notation, for $a = \exp X$, $X \in Cl(\mathfrak{a}^+)$, we have*

$$\text{(24)} \qquad \|\tau(u_1^{-1}au_1)^{-1}v\| \leqq e^{\Lambda(Ad(u_1^{-1})X)}\|v\| \qquad \text{for all} \quad v \in V_\Lambda^*$$

where $\mathfrak{a}^+ = \{H \in \mathfrak{a};\ \alpha'(H) > 0$ for all $\alpha' \in \sum_+\}$ and $Cl(\mathfrak{a}^+)$ is the closure of $\mathfrak{a}^+$ in $\mathfrak{a}$. In particular, for $a_t = \exp tX^0$, we have

$$\text{(25)} \qquad \tau(u_1^{-1}a_t u_1)^{-1}v = e^{t\Lambda(\sum_{\alpha\in\Delta} H_\alpha')}v \qquad \text{for all} \quad v \in V_\Lambda^*$$

where $u_1^{-1}a_t u_1 = \exp(t\sum_{\alpha\in\Delta} H_\alpha')$.

Proof. From C. Moore [10],

$$Cl(\mathfrak{a}^+) = \{\sum_{i=1}^m a_i X_{\gamma_i}^0;\ 0 \leq a_1 \leq \cdots \leq a_m\}\,.$$

Let $a = \exp(\sum_{i=1}^m a_i X_{\gamma_i}^0)$, $0 \leq a_1 \leq \cdots \leq a_m$. Then by means of (19), we have $u_1^{-1}au_1 = \exp(\sum_{i=1}^m a_i H_{\gamma_i})$. On the other hand, all the weights of τ_Λ are of the form $\Lambda - \sum_{i=1}^p m_i\alpha_i$ when $D = \{\alpha_i\}_{i=1}^p$ is the set of all simple roots in $R_\mathfrak{k}$ with respect to the order $\succ$ in R and $m_i \geq 0$ are integers. Let $V_{\Lambda-\sum m_i\alpha_i}$ be the weight space for $\Lambda - \sum m_i\alpha_i$, and let $V^*_{\Lambda-\sum m_i\alpha_i}$ be the dual space of $V_{\Lambda-\sum m_i\alpha_i}$ which is identified with the subspace of all elements in V_Λ^* vanishing on the orthocomplement of $V_{\Lambda-\sum m_i\alpha_i}$ in V_Λ. Let $\{v^j_{m_1\cdots m_p};\ j=1, \cdots, \dim V_{\Lambda-\sum m_i\alpha_i}\}$ be an orthonomal base in $V_{\Lambda-\sum m_i\alpha_i}$, and let $\omega^j_{m_1\cdots m_p}$ be its dual base in $V^*_{\Lambda-\sum m_i\alpha_i}$. For $v \in V_\Lambda^*$, we put $v = \sum a^j_{m_1\cdots m_p}\omega^j_{m_1\cdots m_p}$, $a^j_{m_1\cdots m_p} \in \boldsymbol{C}$. Then we have

$$\begin{aligned} \tau(u_1^{-1}au_1)^{-1}v &= \sum a^j_{m_1\cdots m_p}\tau(u_1^{-1}au_1)^{-1}\omega^j_{m_1\cdots m_p} \\ &= \sum a^j_{m_1\cdots m_p} e^{(\Lambda - \sum m_i\alpha_i)(\sum_{k=1} a_k H_{\gamma_k})}\omega^j_{m_1\cdots m_p}\,. \end{aligned}$$

From C. Moore [10], the non-zero vectors in $\pi(D)$ are of the form

$$\frac{1}{2}(\gamma_{j+1}-\gamma_j), \qquad j=1, \cdots, m-1$$

if G/K is of tube type. Then we have

$$e^{(\Lambda-\sum m_i\alpha_i)(\sum_{k=1}^m a_k H'_{\gamma_k})} = e^{\Lambda(\sum_{k=1}^m a_k H'_{\gamma_k})} e^{-\sum_{k=1}^{m-1}\frac{d}{2}(a_{k+1}-a_k)n_k}$$
$$\leqq e^{\Lambda(\sum_{k=1}^m a_k H'_{\gamma_k})}$$

for some non-negative integers n_k $(k=1, \cdots, m-1)$ and the equality holds if $a_1=\cdots=a_m$. It follows that

$$||\tau(u_1^{-1}au_1)^{-1}v||^2 = \sum |a^j_{m_1\cdots m_p}|^2 ||\tau(u_1^{-1}au_1)^{-1}\omega^j_{m_1\cdots m_p}||^2$$
$$\leqq \left\{e^{\Lambda(\sum_{i=1}^m a_i H'_{\gamma_i})}\right\}^2 \sum |a^j_{m_1\cdots m_p}|^2$$
$$= \left\{e^{\Lambda(\sum_{i=1}^m a_i H'_{\gamma_i})}\right\}^2 ||v||^2 .$$

In partiqular, if $a=\exp tX^0$,

$$\tau(u_1^{-1}au_1)^{-1}v = e^{t\Lambda(\sum_{k=1}^m H'_{\gamma_k})} v . \qquad \text{Q.E.D.}$$

Proof of Theorem 1. For $\phi \in L^2_{\tau,\lambda}(G/B(E))$, from Lemma 4,

$$(26) \qquad \mathcal{P}_{\tau,\lambda}\phi(gu_t) = \tau(\exp(r\sum_{\alpha\in\Delta}H'_\alpha))^{-1}\tau(k_t)^{-1}\mathcal{P}_{\tau,\lambda}\phi(ga_s)$$
$$= e^{(r+\log(\cos\frac{\pi}{4}t))\Lambda(\sum_{\alpha\in\Delta}H'_\alpha)} \mathcal{P}_{\tau,\lambda}\phi(ga_s)$$
$$= e^{-(r+\log(\cos\frac{\pi}{4}t))(i\lambda+\rho_B)(X^0)} \mathcal{P}_{\tau,\lambda}\phi(ga_s)$$

under the notations in Lemma 4. We put $C=\lim_{t\to 1} e^{\log(\cos\frac{\pi}{4}t)(i\lambda+\rho_B)(X^0)} < \infty$. We notice that $e^{-r(i\lambda+\rho_B)(X^0)}=(\cosh(s))^{(i\lambda+\rho_B)(sX^0)} \sim \frac{1}{2}e^{(i\lambda+\rho_B)(sX^0)}$ as $s\to\infty$. Hence we obtain

$$\lim_{t\uparrow 1}\int_K ||\mathcal{P}_{\tau,\lambda}\phi(ku_t)||^2 dk = \frac{C}{2}\lim_{s\to\infty}|e^{(i\lambda+\rho_B)(sX^0)}|\int_K ||\mathcal{P}_{\tau,\lambda}\phi(ka_s)||dk$$
$$\leq \frac{C}{2}C_E(\lambda)||\phi||_{L^2(K/M(E))} < \infty .$$

from Remark of Proposition 2. Q.E.D.

Moreover, by considering the subspace $\Gamma_0(\Lambda)$ of $\Gamma(\Lambda)$ consisting of all elements of $\Gamma(\Lambda)$ which satisfies the following boundary conditions (iii), (iv), we construct a representation of G.

DEFINITION. Let $\Gamma_0(\Lambda)$ be the set of all $f\in\Gamma(\Lambda)$ satisfying

(iii) for every $g\in G$, there exists a limit $\lim_{t\uparrow 1} f(gu_t)$, say $f(gu_1)$, and the boundary value $f(gu_1)$ satisfies

$$(27)\qquad f(gmanu_t)=e^{t\,Ad(u_1^{-1})\Lambda(\log a)}\tau(m^{-1})f(gu_1)$$

for $g\in G$, $m\in M$, $a\in A$ and $n\in N$ where M is the centralizer of $\mathfrak{a}$ in K.

(iv) $G\ni g\mapsto \|f(gu_1)\|$ is continuous.

Then we can apply Theorem of bounded convergence to the sequence of functions $k\mapsto\|f(ku_t)\|$ $(0\leq t\leq 1)$ by means of the conditions (iii), (iv), and then it follows that

$$(28)\qquad \|f\|_2^2=\lim_{t\uparrow 1}\int_K\|f(ku_t)\|dk=\int_K\|f(ku_1)\|dk \qquad \text{for } f\in\Gamma_0(\Lambda).$$

Let us define the action $U_\Lambda(g)$ of G on $\Gamma_0(\Lambda)$ by $U_\Lambda(g)f(x)=f(g^{-1}x)$. Let us consider the factor space of $\Gamma_0(\Lambda)$ by the subspace $\{f\in\Gamma_0(\Lambda);\ \|f\|_2=0\}$, and let $\Gamma_2(\Lambda)$ be its completion with respect to the norm induced from the norm $\|\cdot\|_2$. Then we have the following Proposition.

Proposition 3. *Let us preserve the assumption in Theorem* 1. *Then* $\Gamma_0(\Lambda)$ *is stable under* $U_\Lambda(g)$ *and* $U_\Lambda(g)$ *acts by a bounded operator on it with respect to the norm* $\|\cdot\|_2$. *Moreover* $U_\Lambda(g)$ *acts on* $\Gamma_2(\Lambda)$ *by a bounded representation of* G.

Proof. For $g\in G$,

$$\int_K\|f(g^{-1}ku_1)\|^2dk\leqq\sup_{k\in K}|e^{t\,Ad(u_1^{-1})\Lambda(H(g^{-1}k))}|^2\int_K\|f(\kappa(g^{-1}k)u_1\|^2dk\,.$$

The function $k\to\|f(\kappa(g^{-1}k)u_1)\|^2$ is a right M-invariant because of $\kappa(g^{-1}km)M=\kappa(g^{-1}k)M$ in K/M and the condition (27). Put $h=\kappa(g^{-1}k)$. Then it follows from (4′) that

$$k=\kappa(gh)\,,\ H(g^{-1}k)=-H(gh)\quad\text{and}\quad dk_M=e^{-2\rho(H(gh))}dh_M.$$

Therefore $\int_K\|f(\kappa(g^{-1}k)u_1)\|^2dk\leqq\sup_{h\in K}e^{-2\rho(H(gh))}\int_K\|f(hu_1)\|^2dh$. Hence $\Gamma_0(\Lambda)$ is stable under $U_\Lambda(g)$ and $U_\Lambda(g)$ acts by a bounded operator on it with respect to the norm $\|\ \|_2$.

For the proof of the last statement, let $L^2_\lambda(G/MAN)$ be the set of all measurable mappings ϕ of G into $\boldsymbol{C}$ satisfying $\phi(\text{gman})=e^{(-\nu+1)\rho_E(\log a)}\phi(g)$ and $\|\phi\|_2^2=\int_K|\phi(k)|^2dk$ is finite. Then G acts on $L^2_\lambda(G/MAN)$ by $U_\lambda(g)\phi(x)=\phi(g^{-1}x)$. Then $U_\lambda(g)$ is a bounded operator on $L^2_\lambda(G/MAN)$ with respect to the above norm $\|\ \|_2$. Now we define the linear map $\mathcal{L}$ of $\Gamma_0(\Lambda)$ into $L^2_\lambda(G/MAN)$ by $(\mathcal{L}f)(g)=f(gu_1)$ for $f\in\Gamma_0(\Lambda)$. Then $\mathcal{L}$ is a G-equivariant isometry of $\Gamma_0(\Lambda)$ into $L^2_\lambda(G/MAN)$, that is, $\mathcal{L}U_\Lambda(g)=U_\lambda(g)\mathcal{L}$ and $\|\mathcal{L}f=f\|_2$, i.e. $\int_K \|f(ku_2)\|dk$ $=\lim_{t\to1}\int_K \|f(ku_t)\|^2dk$, for $f\in\Gamma_0(\Lambda)$ because of (28). Therefore $U_\Lambda(g)$ can be extended to a bounded operator on $\Gamma_2(\Lambda)$. Q.E.D.

Summing up the above results, we have the following theorem as a Corallary of Theorem 1.

Theorem 2. *Let G/K be an irreducible hermitian symmetric space of tube type. Suppose that $\lambda=z\rho_E$, $z=x+iy$, $y>0$ and Λ satisfy the condition (C). Then $\mathcal{P}_{\tau,\lambda}$ is a G-equivariant bounded operator from $L^2_{\tau,\lambda}(G/B(E))$ into $\Gamma_2(\Lambda)$, that is,*

$$(29)\qquad U_\Lambda(g)\circ\mathcal{P}_{\tau,\lambda}=\mathcal{P}_{\tau,\lambda}\circ U_{\tau,\lambda}(g)\qquad \text{on}\quad L^2_{\tau,\lambda}(G/B(E))\,.$$

Proof. The boundedness of $\mathcal{P}_{\tau,\lambda}$ has been proved in Theorem 1 and, by the definition of Poisson integrals, we have the G-equivariantness (29) of $\mathcal{P}_{\tau,\lambda}$. Since $C_{\tau,\lambda}(G/B(E))$ is dense in $L^2(G/B(E))$, it suffices to prove that $\mathcal{P}_{\tau,\lambda}C_{\tau,\lambda}(G/B(E))\subset\Gamma_0(\Lambda)$.

For $\phi\in C_{\tau,\lambda}(G/B(E))$, we have

$$(26)\qquad \mathcal{P}_{\tau,\lambda}\phi(gu_t)=e^{-(r+\log(\cos\frac{\pi}{4}t))(i\lambda+\rho_E)(X^0)}\mathcal{P}_{\tau,\lambda}\phi(ga_s)\,.$$

Then, from Proposition 1, we obtain

$$\begin{aligned}\lim_{t\to1}\mathcal{P}_{\tau,\lambda}\phi(gu_t)&=\frac{C}{2}\lim_{s\to\infty}e^{(i\lambda+\rho_E)(sX^0)}\mathcal{P}_{\tau,\lambda}\phi(ga_s)\\&=\frac{C}{2}\int_{\bar N(E)}e^{(i\lambda-\rho_E)(H(\bar n))}\tau(\kappa(\bar n))\phi(g)d\bar n\,,\end{aligned}$$

that is, $\mathcal{P}_{\tau,\lambda}\phi(gu_1)=\frac{C}{2}\int_{\bar N(E)}e^{(i\lambda-\rho_E)(H(\bar n))}\tau(\kappa(\bar n))\phi(g)d\bar n$. From the condition (1), we have, for $m\in M$, $a\in A$, $n\in N$,

$$\mathcal{P}_{\tau,\lambda}\phi(gmanu_1)=\frac{C}{2}e^{-(i\lambda+\rho_E)(\log a)}\int_{\bar N(E)}e^{(i\lambda-\rho_E)(H(\bar n))}\tau(\kappa(\bar n)m^{-1})\phi(g)d\bar n\,.$$

Here $\kappa(\bar n)m^{-1}=m^{-1}\kappa(m\bar nm^{-1})$ for $m\in M$. We put $\bar n'=m\bar nm^{-1}$, then $H(\bar n')=H(\bar n)$ and $d\bar n'=d\bar n$. Therefore we have $\mathcal{P}_{\tau,\lambda}\phi(gmanu_1)=\frac{C}{2}e^{-(i\lambda+\rho_E)(\log a)}\tau(m^{-1})\mathcal{P}_{\tau,\lambda}\times$ $\phi(gu_1)$. It follows from the assumption (C) that the condition (27) is satisfied. Q.E.D.

5. Construction of Hardy class (II)

We preserve the notation and the assumption in §4. Let $C^\infty(G, V_\Lambda^*)$ be the set of all C^∞ mappings of G into V_Λ^*. Let ν be the left regular representation of G on $C^\infty(G, V_\Lambda^*)$. We define a representation ν of $\mathfrak{g}^C$ on $C^\infty(G, V_\Lambda^*)$ by

$$\nu(X)f(g) = \left[\frac{d}{dt} f(\exp(-tx)g)\right]_{t=0}$$

for $g \in G$, $f \in C^\infty(G, V_\Lambda^*)$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}^C$. Then ν defines a representation ν of $U(\mathfrak{g})$ on $C^\infty(G, V_\Lambda^*)$. Let $\nu(C)$ be the Casimir operator of ν with respect to the Killing form B on $C^\infty(G, V_\Lambda^*)$.

We put $C_{\tau,\lambda}^\infty(G/B(E)) = C_{\tau,\lambda}(G/B(E)) \cap C^\infty(G, V_\Lambda^*)$. Then the representations $(\Gamma_0(\Lambda), U_\Lambda)$ and $(C_{\tau,\lambda}^\infty(G/B(E)), U_{\tau,\lambda})$ are subrepresentations of the left regular representation of $(C^\infty(G, V_\Lambda^*), \nu)$ of G.

DEFINITION. Let $H_0(\Lambda)$ be the set of elements f in $\Gamma_0(\Lambda)$ satisfying

$$(\nu(C) - \langle \Lambda + 2\delta, \Lambda \rangle) f = 0 . \tag{30}$$

Let us consider the factor space of $H_0(\Lambda)$ by the subspace $\{f \in H_0(\Lambda);\ \|f\|_2 = 0\}$ and let $H_2(\Lambda)$ be its completion with respect to the norm $\|\cdot\|_2$. Then, for $g \in G$, $U_\Lambda(g)$ acts on $H_2(\Lambda)$ as a bounded operator with respect to this norm. $H_2(\Lambda)$ is called the *Hardy class* of the vector bundle E_Λ over G/K.

Now we can write Λ and δ as $\Lambda = \Lambda_+ + \Lambda_-$, $\delta = \delta_+ + \delta_-$ according to (16). Let M_0 be the connected component of the centralizer M of $\mathfrak{a}$ in K. Then $\mathfrak{t}^+$ is a Cartan subalgebra of the Lie algebra of M, M_0 and Λ_+ satisfies the following conditions:

(i) $\Lambda_+(H) = \Lambda(H) \in 2\pi\sqrt{-1}\,\boldsymbol{Z}$ for all $H \in \mathfrak{t}_+ \subset \mathfrak{t}$, $\exp H = e$

(ii) $\langle \Lambda_+, \alpha \rangle \geq 0$ for all $\alpha \in P\mathfrak{t}$ such that $\pi(\alpha) = 0$.

Hence there exists an irreducible unitary representation π_{Λ_+} of M_0 with the highest weight Λ_+ on a representation space V_{Λ_+}. We define the projection operator e_{Λ_+} of $C_{\tau,\lambda}^\infty(G/B(E))$ as follows:

$$e_{\Lambda_+}\phi(g) = d_{\Lambda_+} \int_{M_0} \bar{\theta}_{\Lambda_+}(m)\phi(gm)dm \qquad \text{for} \quad \phi \in C_{\tau,\lambda}^\infty(G/B(E))$$

where $d_{\Lambda_+} = \dim V_{\Lambda_+}$, θ_{Λ_+} the character of τ_{Λ_+} and $\bar{\theta}_{\Lambda_+}(m)$ is the complex conjugate of $\theta_{\Lambda_+}(m)$.

Then $e_{\Lambda_+} C_{\tau,\lambda}^\infty(G/B(E))$ is a G-invariant subspace of $C_{\tau,\lambda}^\infty(G/B(E))$. Moreover we have the following theorem.

Theorem 3. *Under the assumption of theorem 2, we have*

$$\mathcal{P}_{\tau,\lambda} e_{\Lambda_+} C_{\tau,\lambda}^\infty(G/B(E)) \subset H_2(\Lambda) .$$

Proof. We will prove that $\nu(C)\mathcal{P}_{\tau,\lambda}e_{\Lambda_+}\phi=\langle\Lambda+2\delta,\Lambda\rangle\mathcal{P}_{\tau,\lambda}e_{\Lambda_+}\phi$ for $\phi\in C_{\tau,\lambda}(G/B(E))$. Since $U_\Lambda(g)\circ\mathcal{P}_{\tau,\lambda}=\mathcal{P}_{\tau,\lambda}U_{\tau,\lambda}(g)$, it suffices to prove that

$$\nu(C)e_{\Lambda_+}\phi=\langle\Lambda+2\delta,\Lambda\rangle e_{\Lambda_+}\phi \qquad \text{for} \quad \phi\in C^\infty_{\tau,\lambda}(G/B(E)) .$$

Now let $\tilde{\nu}$ be the right regular representation of G on $C^\infty(G, V^*_\Lambda)$. We define a representation $\tilde{\nu}$ of $\mathfrak{g}^C$ on $C^\infty(G, V^*_\Lambda)$ by

$$\tilde{\nu}(X)f(g)=\left[\frac{d}{dt}f(g\exp tX)\right]_{t=0}$$

for $g\in G$, $X\in\mathfrak{g}$ and $f\in C^\infty(G, V^*_\Lambda)$. $\tilde{\nu}$ defines a representation $\tilde{\nu}$ of $U(\mathfrak{g})$ on $C^\infty(G, V^*_\Lambda)$. Then it follows (Harish-Chandra [4]) that

$$\nu(C)\phi=\tilde{\nu}(C)\phi \qquad \text{for every} \quad \phi\in C^\infty(G, V^*_\Lambda) .$$

So we will show that $\tilde{\nu}(C)e_{\Lambda_+}\phi=\langle\Lambda+2\delta,\Lambda\rangle e_{\Lambda_+}\phi$ for $\phi\in C^\infty_{\tau,\lambda}(G/B(E))$.

Following Harish-Chandra [3], let $\{X_{\alpha'}\}$ be the root vector for $\alpha'\in\sum$ such that $\tau X_{\alpha'}=-X_{-\alpha'}$ and $B(X_{\alpha'}, X_{-\alpha'})=1$, and let $H_{\alpha'}$ be an element of $\mathfrak{t}^C$ such that $B(H, H_{\alpha'})=\alpha'(H)$, for $H\in\mathfrak{h}$. Then $[X_{\alpha'}, X_{-\alpha'}]=H_{\alpha'}$. Let $\{H_i\}^l_{i=1}$ be a base of $\mathfrak{h}^C$ such that $H_1,\cdots,H_m$ is an orthonormal base of $\mathfrak{a}$ with respect to the Killing form B of $\mathfrak{g}^C$ and $H_{m+1},\cdots,H_l$ is that of $\sqrt{-1}\mathfrak{t}^+$ with respect to B. Then $\{H_1,\cdots,H_l, X_{\alpha'}, X_{-\alpha'};\alpha'\in\sum,\alpha'>0\}$ is a base of $\mathfrak{g}^C$. Then we have

$$\begin{aligned}\tilde{\nu}(C)&=\sum_{i=1}^{l}\tilde{\nu}(H_i)^2+\sum_{\substack{\alpha'\in\Sigma\\ \alpha'>0}}(\tilde{\nu}(X_{\alpha'})\tilde{\nu}(X_{-\alpha'})+\tilde{\nu}(X_{-\alpha'})\tilde{\nu}(X_{\alpha'}))\\ &=D_1+D_2+D_2\end{aligned}$$

where
$$D_1=\sum_{i=m+1}^{l}\tilde{\nu}(H_i)^2+\sum_{\substack{\alpha'\in\Sigma_0\\ \alpha'>0}}(\tilde{\nu}(X_{\alpha'})\tilde{\nu}(X_{-\alpha'})+\tilde{\nu}(X_{-\alpha'})\tilde{\nu}(X_{\alpha'}))$$

$$D_2=\sum_{i=1}^{m}\tilde{\nu}(H_i)^2+\sum_{\alpha'\in\Sigma_+}\tilde{\nu}(H_{\alpha'})^2$$

and
$$D_3=2\sum_{\alpha'\in\Sigma_+}\tilde{\nu}(X_{-\alpha'})\tilde{\nu}(X_{\alpha'}) .$$

Since $e_{\Lambda_+}\phi$ belongs to $C^\infty_{\tau,\lambda}(G/B(E))$, we have

$$(31) \qquad D_3e_{\Lambda_+}\phi=0$$

because of $e_{\Lambda_+}\phi(gn)=e_{\Lambda_+}\phi(g)$, $n\in N$.
We note (20) $\sum_{\alpha'\in\Sigma_+}H_{\alpha'}=2H_\rho\in\mathfrak{a}$. Then since we have

$$\phi(g\exp H)=e^{-(i\lambda+\rho_B)(H(g)+H)}\phi(\kappa(g))$$

for every $\phi\in C^\infty_{\tau,\lambda}(G/B(E))$, $H\in\mathfrak{a}$, it follows that

$$(32)\qquad D_2 e_{\Lambda_+}\phi = (\langle i\lambda+\rho_E,\, i\lambda+\rho_E\rangle - \langle i\lambda+\rho_E,\, 2\rho\rangle) e_{\Lambda_+}\phi\,.$$

On the other hand, let $\tau_{\Lambda_+}(C_M)$ be the Casimir operator of the representation τ_{Λ_+} of M_0 with respect to the form B. Then we have

$$\tau_{\Lambda_+}(C_M) = \langle \Lambda_+ + 2\delta_+,\, \Lambda_+\rangle \boldsymbol{I}$$

where $\boldsymbol{I}$ is the identity operator on V_{Λ_+}. And we have (cf. Harish-Chandra [4])

$$\begin{aligned} D_1 \xi_{\Lambda_+}(m) &= \sum_i (v_i,\, \tau_{\Lambda_+}(m)\tau_{\Lambda_+}(C_M)v_i) \\ &= \langle \Lambda_+ + 2\delta_+,\, \Lambda_+\rangle \xi_{\Lambda_+}(m) \end{aligned}$$

where $\{v_i\}$ is an orthonormal basis of V_{Λ_+} with respect to the inner product (,) on V_{Λ_+}. Then we have (cf. Harish-Chandra [4])

$$(33)\qquad \begin{aligned} D_1 e_{\Lambda_+}\phi(g) &= d_{\Lambda_+}\int_{M_0} \overline{\xi_{\Lambda_+}}(m)(D_1\phi)(gm)dm \\ &= d_{\Lambda_+}\int_{M_0} D_1\overline{\xi}_{\Lambda_+}(m)\phi(gm)dm \\ &= \langle \Lambda_+ + 2\delta_+,\, \Lambda_+\rangle e_{\Lambda_+}\phi(g)\,. \end{aligned}$$

Hence together with (31), (32), (33), we have

$$(34)\quad \mathfrak{D}(C)e_{\Lambda_+}\phi = \{\langle i\lambda+\rho_E,\, i\lambda+\rho_E\rangle - \langle i\lambda+\rho_E,\, 2\rho\rangle + \langle \Lambda_+ + 2\rho_+,\, \Lambda_+\rangle\} e_{\Lambda_+}\phi.$$

Since we have ${}^tAd(u_1^{-1})\Lambda_- = -(i\lambda+\rho_E)$ and (21) $\rho = {}^tAd(u_1^{-1})\delta$ on $\mathfrak{a}$, it follows that

$$\begin{aligned} (34) &= \{\langle \Lambda_-,\, \Lambda_-\rangle + \langle \Lambda_-,\, 2\delta\rangle + \langle \Lambda_+ + 2\delta_+,\, \Lambda_+\rangle\} e_{\Lambda_+}\phi \\ &= \langle \Lambda + 2\delta,\, \Lambda\rangle e_{\Lambda_+}\phi\,. \end{aligned}$$

Q.E.D.

EXAMPLE. Let $G=SU(1,1)$, $K=T=S(U(1)\times U(1))=\left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix} : \theta\in\boldsymbol{R}\right\}$, and so G/K is the unit disc D. Then $G^C = SL(2, \boldsymbol{C})$, $K^C = T^C = \left\{\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1}\end{pmatrix} : \gamma\in\boldsymbol{C}-(0)\right\}$. Then $\mathfrak{g}=\mathfrak{su}(1,1)$, $\mathfrak{k}=\mathfrak{t}=\left\{\begin{pmatrix} i\theta & 0 \\ 0 & -i\theta\end{pmatrix} : \theta\in\boldsymbol{R}\right\}$, $\mathfrak{g}^C=\mathfrak{sl}(2,\boldsymbol{C})$, $\mathfrak{k}^C=\mathfrak{t}^C=\left\{\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha\end{pmatrix} ;\ \alpha\in\boldsymbol{C}\right\}$ and the set R of roots of $(\mathfrak{g}^C, \mathfrak{t}^C)$ is given by

$$R = \{\pm\gamma\}\,,\quad \text{where}\quad \gamma\colon \mathfrak{t}^C \ni \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha\end{pmatrix} \mapsto -2\alpha\,.$$

A linear order $\succ$ on R is defined as $\gamma\succ 0$. Let $E_\gamma=\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}$, $E_{-\gamma}=\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$. We have

$$X_\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y_\gamma^0 = -\sqrt{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$u_t = \exp\left(-\frac{\pi t}{4}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} \cos\frac{\pi t}{4} & \sin\frac{\pi t}{4} \\ -\sin\frac{\pi t}{4} & \cos\frac{\pi t}{4} \end{pmatrix},$$

$$\mathfrak{t}^- = \mathfrak{t}, \ \mathfrak{t}^+ = (0), \quad M = M(E) = \left\{\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}; \varepsilon = \pm 1\right\},$$

$$\delta : \mathfrak{t} \ni \begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix} \mapsto -i\theta \quad \text{and} \quad \rho : \mathfrak{a} \ni \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} \mapsto t .$$

Let $\Lambda = -n\delta$, $n \in \boldsymbol{Z}$. Then we obtain a holomorphic representation $\tau = \tau_\Lambda^*$ of $K^C P_+$ given by

$$K^C P_+ \ni \begin{pmatrix} \gamma & 0 \\ \alpha & \gamma^{-1} \end{pmatrix} \mapsto \gamma^{-n} \in \boldsymbol{C} - (0) .$$

Now our conditions "Re$\langle i\lambda, \alpha \rangle < 0$, $\alpha = 2\rho$ and ${}^t Ad(u_1^{-1})\Lambda = -(i\lambda + \rho)$ on $\mathfrak{a}$" coincide with (cf. Okamoto [11])

$$i\lambda = (n-1)\rho , \qquad n < 1 , \ n \in \boldsymbol{Z} .$$

If $n=0$ i.e., $\Lambda = 0$, then $i\lambda = -\rho$ and τ_Λ is the trivial representation of K. Then our Hardy class $H_2(\Lambda)$ is the usual Hardy class $H^2(D)$ given in the introduction.

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