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# UNIFORM ALGEBRA GENERATED BY $z_1, \cdots, z_n, f_1(z), \cdots, f_s(z)$

Dedicated to Professor Yukinari Tôki on his 60th birthday

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## Introduction

Let $C^n$ be the complex Euclidean space with complex coordinates $z=(z_1, \cdots, z_n)$ and $K$ a compact subset of $C^n$. For any complex valued $C^\infty$-functions $f_1, \cdots, f_s$ defined on an open subset $U$ of $C^n$ containing $K$, we shall consider the uniform algebra $A$ consisting of uniform limits of polynomials of $z_1, \cdots, z_n, f_1, \cdots, f_s$ on $K$.

Hörmander-Wermer [1] proved that, if $s=n$ and if each $f_j$ is 'close' to $\bar{z}_j$ in some sense, then $A$ coincides with $C(K)$, the algebra of all complex valued continuous functions on $K$. In this paper, we shall deal with the case where $0<s<n$ and each $f_j$ is holomorphic in $z_{s+1}, \cdots, z_n$ near $K$. In Section 3, an approximation theorem on the graph of $f_1, \cdots, f_s$ will be proved. In Section 4, we shall give a sufficient condition on $f_j$ and $K$ assuring that every function holomorphic in $z_{s+1}, \cdots, z_n$ near $K$ belongs to $A$.

## 1. The graph of $f_1, \cdots, f_s$

Let $f_1, \cdots, f_s$ be $C^\infty$-functions defined on an open subset $U$ of $C^n$. The graph of $f_1, \cdots, f_s$

$$M = \{(z_1, \cdots, z_n, f_1(z), \cdots, f_s(z)) \in C^{n+s};\ z = (z_1, \cdots, z_n) \in U\}$$

is a real $2n$-dimensional submanifold of $C^{n+s}$. If $g$ is a $C^\infty$-function on $M$, then the function $g_0$ defined by

$$(1.1) \qquad g_0(z_1, \cdots, z_n) = g(z_1, \cdots, z_n, f_1(z_1, \cdots, z_n), \cdots, f_s(z_1, \cdots, z_n))$$

is a $C^\infty$-function on $U$.

We denote by $H_r(U)$, $r=n-s$, the class of functions of $C^\infty(U)$ which are holomorphic in $z_{s+1}, \cdots, z_n$.

We shall now consider the following assumptions on $f_1, \cdots, f_s$:

$$(1.2) \qquad f_1, \cdots, f_s \text{ belong to } H_r(U), \text{ and}$$

$$\det \left(\frac{\partial f_j}{\partial \bar{z}_k}\right)_{j,k=1,\cdots,s} \text{ has no zeros on } U. \tag{1.3}$$

These conditions imply that, for every point $p$ of $M$, the dimension of maximal complex submanifold of $\boldsymbol{C}^{n+s}$ through $p$ contained in $M$ is just $r$. It follows from the following lemma, which is easily proved by linear algebra.

**Lemma 1.** *The complex tangent space of $M$ at every point is of $r$-dimension if and only if*

$$\operatorname{rank} \left(\frac{\partial f_j}{\partial \bar{z}_k}\right)_{j=1,\cdots,s\,;\,k=1,\cdots,n} = n-r$$

*holds at every point of $U$.*

A $C^\infty$-function on $M$ which is holomorphic in complex coordinates of $M$ is called a *CR-function.* (1.1) gives an isomorphism of $H_r(U)$ and the algebra of $CR$-functions on $M$.

## 2. Holomorphic convexity of $M$

By a *region of holomorphy* we mean a disjoint sum of domains of holomorphy. We define

$$\phi(z) = \sum_{j=1}^{s} | f_j(z_1, \cdots, z_n) - z_{n+j} |^2, \qquad z \in U \times \boldsymbol{C}^s,$$

and

$$G_\varepsilon(V) = \{z \in V;\ \phi(z) < \varepsilon\},$$

for any open subset $V$ of $U \times \boldsymbol{C}^s$ and for any positive number $\varepsilon$.

**Lemma 2.** *Suppose that $f_j$ satisfy* (1.2) *and* (1.3). *Let $V$ be a region of holomorphy in $\boldsymbol{C}^{n+s}$ such that $\bar{V} \subset U \times \boldsymbol{C}^s$. Then there exists a positive number $\varepsilon_0$ such that $G_\varepsilon(V)$ is a region of holomorphy in $\boldsymbol{C}^{n+s}$ for any $\varepsilon$, $0 < \varepsilon < \varepsilon_0$.*

Proof. We consider the complex Hessian form

$$H(\xi, \bar{\xi})_z = \sum_{\nu,\mu=1}^{n+s} \frac{\partial^2 \phi}{\partial z_\nu \partial \bar{z}_\mu}(z) \xi_\nu \bar{\xi}_\mu, \quad z \in U \times \boldsymbol{C}^s.$$

Let $z = (z_1, \cdots, z_{n+s})$ be any point of $M$ and $z_0 = (z_1, \cdots, z_n)$ the corresponding point in $U$. Then we have

$$H(\xi, \bar{\xi})_z = \sum_{j=1}^{s} \left\{ \left| \sum_{\nu=1}^{n} \frac{\partial f_j}{\partial z_\nu}(z_0) \xi_\nu - \xi_{n+j} \right|^2 + \left| \sum_{\nu=1}^{s} \frac{\partial f_j}{\partial \bar{z}_\nu}(z_0) \bar{\xi}_\nu \right|^2 \right\}.$$

The right member can vanish only if

$$\sum_{\nu=1}^{s} \frac{\partial f_j}{\partial \bar{z}_\nu}(z_0)\bar{\xi}_\nu = 0 \quad \text{and} \quad \xi_{n+j} = \sum_{\nu=1}^{n} \frac{\partial f_j}{\partial z_\nu}(z_0)\xi_\nu, \quad j = 1, \cdots, s\,.$$

By (1.3), $H(\xi, \bar{\xi})_z$ can vanish only when $\xi$ is a complex tangent vector of $M$ at $z$:

$$\xi = \left(0, \cdots, 0, \xi_{s+1}, \cdots, \xi_n, \sum_{\nu=s+1}^{n} \frac{\partial f_1}{\partial z_\nu}(z_0)\xi_\nu, \cdots, \sum_{\nu=s+1}^{n} \frac{\partial f_s}{\partial z_\nu}(z_0)\xi_\nu\right).$$

Therefore the matrix $H_z = \left(\dfrac{\partial^2 \phi}{\partial z_\nu \partial \bar{z}_\mu}(z)\right)$ has $n+s-r$ non-zero eigenvalues for every point $z$ of $M$. Let $V_1$ be an open set such that $\bar{V} \subset V_1 \subset \bar{V}_1 \subset U \times C^s$. By continuity of $H_z$, there exists a positive number $\varepsilon_0$ such that $H_z$ has at least $n+s-r$ non zero eigen values for every $z$ in $G_{\varepsilon_0}(V_1)$.

Let $S_\varepsilon$ be the hypersurface $\{z \in V_1;\ \phi(z) = \varepsilon\}$. Fix an arbitrary point $\alpha = (\alpha_1, \cdots, \alpha_{n+s})$ on $S_\varepsilon$. We define a non-singular holomorphic map $z = \Phi(\zeta)$ of $U(\alpha) = \{\zeta \in C^r;\ (\alpha_1, \cdots, \alpha_s, \zeta_1, \cdots, \zeta_r) \in U\}$ into $C^{n+s}$ by

$$\Phi_j(\zeta) = \begin{cases} \alpha_j & j = 1, \cdots, s\,, \\ \zeta_{j-s} & j = s+1, \cdots, n\,, \\ f_{j-n}(\alpha_1, \cdots, \alpha_s, \zeta_1, \cdots, \zeta_r) - f_{j-n}(\alpha_1, \cdots, \alpha_n) + \alpha_j\,, & \\ & j = n+1, \cdots, n+s\,. \end{cases}$$

The $\Phi$-image of $U(\alpha)$ is an $r$-dimensional complex submanifold of $C^{n+s}$ containing $\alpha$. Since $\sum_{j=1}^{s} |f_j(\alpha_1, \cdots, \alpha_n) - \alpha_{n+j}|^2 = \varepsilon$, it is contained in $S_\varepsilon$. Hence the complex Hessian $H_\alpha$ evaluated at $\alpha$ has at least $r$ zero eigenvalues with complex eigenvectors tangent to $S_\varepsilon$ (see Wells [2], Lemma 2.5′). Thus, $H(\xi, \bar{\xi})_\alpha$ is non-negative for any tangent vector $\xi$ to $S_\varepsilon$. Since $V$ is a region of holomorphy in $C^{n+s}$, so is $G_\varepsilon(V)$. The lemma is proved.

A compact set $F$ of $C^n$ (or of $C^{n+s}$) is called an *H-convex set* in $C^n$ (or in $C^{n+s}$ resp.), if $F$ is the intersection of regions of holomorphy containing $F$ in $C^n$ (or in $C^{n+s}$ resp.). If $U_1$ is a region of holomorphy in $U$, then $U_1 \times C^s$ is a region of holomorphy in $C^{n+s}$. Therefore, we have

**Corollary.** *If $K$ is an H-convex compact seubset of $U$, then $K^* = \{(z_1, \cdots, z_{n+s}) \in M;\ (z_1, \cdots, z_n) \in K\}$ is H-convex in $C^{n+s}$.*

## 3. Holomorphic approximation on $M$

In this section, we suppose that $f_1, \cdots, f_s$ satisfy (1.2) and (1.3).

**Lemma 3.** *Suppose $g$ is a CR-function on $M$. Then for every positive integer $N$ and for every relatively compact open subset $U_0$ of $U$, there exist a function $\tilde{g} \in C^\infty(U \times C^s)$ and a positive constant $\gamma$ such that*

(i) $\tilde{g}|_M=g$, *and*

(ii) $\left|\frac{\partial g}{\partial \bar{z}_\nu}(z)\right| \leq \gamma \cdot d(z, M)^N, \quad z \in U_0 \times C^s, \quad \nu=1, \cdots, n+s,$

*where* $d(z, M)$ *is the Euclidean distance in* $C^{n+s}$ *between* $z$ *and* $M$.

Proof. We consider the system of linear equations at every point of $U$

$$\sum_{j=1}^{s} h_j \frac{\partial f_j}{\partial \bar{z}_\nu} = \frac{\partial g_0}{\partial \bar{z}_\nu}, \quad \nu = 1, \cdots, s, \tag{3.1}$$

where $g_0$ is the function defined by (1.1). By (1.3), there exist the uniquely determined solutions $h_j(z_1, \cdots, z_n)$, $j=1, \cdots, s$. Since $f_j$ and $g_0$ are of $H_r(U)$, so are $h_j$. We shall define the function $h_J$ inductively for every multi-index $J=(j_1, \cdots, j_k)$, $1 \leq i_i \leq s$. Suppose $h_J$ is given in $H_r(U)$. Then $h_{Jj}$, $j=1, \cdots, s$, will be defined as the solutions of the equations

$$\sum_{j=1}^{s} h_{Jj} \frac{\partial f_j}{\partial \bar{z}_\nu} = \frac{\partial h_J}{\partial \bar{z}_\nu}, \quad \nu = 1, \cdots, s. \tag{3.2}$$

The condition (1.3) guarantees the existence of the solutions $h_{Jj}$ in $H_r(U)$.

We shall prove that $h_J$ are symmetric with respect to $J$. By differentiating each equation of (3.1) by $\bar{z}_\mu$, we have

$$\sum_{j=1}^{s} \frac{\partial f_j}{\partial \bar{z}_\nu} \cdot \frac{\partial h_j}{\partial \bar{z}_\mu} = \frac{\partial^2 g_0}{\partial \bar{z}_\nu \partial \bar{z}_\mu} - \sum_{j=1}^{s} \frac{\partial^2 f_j}{\partial \bar{z}_\nu \partial \bar{z}_\mu} h_j .$$

Since the right member is symmetric in $\nu$ and $\mu$, we have

$$\sum_{j=1}^{s} \frac{\partial f_j}{\partial \bar{z}_\nu} \cdot \frac{\partial h_j}{\partial \bar{z}_\mu} = \sum_{j=1}^{s} \frac{\partial f_j}{\partial \bar{z}_\mu} \cdot \frac{\partial h_j}{\partial \bar{z}_\nu}, \quad \nu, \mu=1, \cdots, s. \tag{3.3}$$

Substituting (3.2) for $k=1$ to (3.3), we obtain

$$\sum_{j=1}^{s} \left( \sum_{i=1}^{s} h_{ji} \frac{\partial f_i}{\partial \bar{z}_\mu} \right) \frac{\partial f_j}{\partial \bar{z}_\nu} = \sum_{j=1}^{s} \left( \sum_{i=1}^{s} h_{ji} \frac{\partial f_i}{\partial \bar{z}_\nu} \right) \frac{\partial f_j}{\partial \bar{z}_\mu}, \quad \nu, \mu = 1, \cdots, s,$$

or equivalently

$$\sum_{i,j} (h_{ji} - h_{ij}) \frac{\partial f_j}{\partial \bar{z}_\mu} \cdot \frac{\partial f_i}{\partial \bar{z}_\nu} = 0, \quad \nu, \mu = 1, \cdots, s.$$

By using (1.3), we can find that $h_{ji}=h_{ij}$ for every $i$ and $j$.

General cases will be proved by induction. For simplicity, we write $J=(j_1, \cdots, j_k)$, $I=(j_1, \cdots, j_{i-1}, j_{i+1}, \cdots, j_k)$, $i=j_i$ and $J'=(j_1, \cdots, j_{i-1}, j, j_{i+1}, \cdots, j_k)$. Since $h_{J'}=h_{Ij}$ and $h_J=h_{Ii}$ by assumption of induction, we have

$$\sum_{i=1}^{s} h_J \frac{\partial f_i}{\partial \bar{z}_\nu} = \frac{\partial h_I}{\partial \bar{z}_\nu} \quad \text{and} \quad \sum_{j=1}^{s} h_{J'} \frac{\partial f_j}{\partial \bar{z}_\mu} = \frac{\partial h_I}{\partial \bar{z}_\mu}.$$

By differentiating the first identity by $\bar{z}_\mu$ and the second by $\bar{z}_\nu$, we obtain

$$\sum_{i=1}^{s} \frac{\partial h_J}{\partial \bar{z}_\mu} \cdot \frac{\partial f_i}{\partial \bar{z}_\nu} = \sum_{j=1}^{s} \frac{\partial h_{J'}}{\partial \bar{z}_\nu} \cdot \frac{\partial f_j}{\partial \bar{z}_\mu}, \quad \nu, \mu = 1, \cdots, s. \tag{3.4}$$

Substituting the equalities

$$\sum_{j=1}^{s} h_{Jj} \frac{\partial f_j}{\partial \bar{z}_\mu} = \frac{\partial h_J}{\partial \bar{z}_\mu} \quad \text{and} \quad \sum_{i=1}^{s} h_{J'i} \frac{\partial f_i}{\partial \bar{z}_\nu} = \frac{\partial h_{J'}}{\partial \bar{z}_\nu}$$

to (3.4), we have

$$\sum_{i,j} (h_{Jj} - h_{J'i}) \frac{\partial f_i}{\partial \bar{z}_\nu} \frac{\partial f_j}{\partial \bar{z}_\mu} = 0, \quad \nu, \mu = 1, \cdots, s.$$

By (1,3), we find that $h_{Jj} = h_{J'i}$, which implies the symmetry of $h_J$ for all $J$.

Now we define $\tilde{g}$ by

$$\tilde{g}(z_1, \cdots, z_{n+s}) = g_0(z_1, \cdots, z_n)$$
$$+ \sum_{k=1}^{N} \frac{1}{k!} \sum_{(j \cdots j_k)} h_{j_1 \cdots j_k}(z_1, \cdots, z_n) \, (z_{n+j_1} - f_{j_1}(z_1, \cdots, z_n)) \cdots (z_{n+j_k} - f_{j_k}(z_1, \cdots, z_n)).$$

If $\nu = s+1, \cdots, n+s$, we have $\dfrac{\partial \tilde{g}}{\partial \bar{z}_\nu} \equiv 0$. For $\nu = 1, \cdots, s$, we have

$$\frac{\partial \tilde{g}}{\partial \bar{z}_\nu} = \frac{1}{N!} \sum_{(j_1 \cdots j_N)} \frac{\partial h_{j_1 \cdots j_N}}{\partial \bar{z}_\nu} (z_{n+j_1} - f_{j_1}(z_1, \cdots, z_n)) \cdots (z_{n+j_N} - f_{j_N}(z_1, \cdots, z_n)),$$

which proves the lemma.

We consider two uniform algebras on a compact subset $K^*$ of $M$. $H(K^*)$ is the algebra of uniform limits on $K^*$ of functions each holomorphic in a neighborhood (in $C^{n+s}$) of $K^*$. $CR(K^*)$ is the algebra of uniform limits on $K^*$ of functions each of which is a $CR$-function on a neighborhood (in $M$) of $K^*$.

Suppose $K^*$ is $H$-convex in $C^{n+s}$. Let $g$ be a $CR$-function on a neighborhood $M_1$(in $M$) of $K^*$. We can find a region of holomorphy $V$ such that $K^* \subset V$ and $\bar{V} \cap M \subset M_1$. We denote by $K$ and $U_0$ the projections of $K^*$ and $M_0 = V \cap M$ respectively by the map $(z_1, \cdots, z_{n+s}) \to (z_1, \cdots, z_n)$. Let $d$ denote the distance between $K$ and $\partial U_0$. By the way of construction of $G_\varepsilon(V)$ in Lemma 2, we can find a positive constant $\eta$ such that, for every point $z^0$ of $K^*$, the ball $B_{\varepsilon\eta}(z^0) = \{z \in C^{n+s};\ |z - z^0| \leq \varepsilon\eta\}$ is contained in $G_\varepsilon(V)$, whenever $\varepsilon < d$. Therefore, by using Lemma 2 and Lemma 3 for $N = n+1$, and applying the same technique as one developed in [1], we obtain the following

**Theorem 1.** *If $K^*$ is a compact subset of $M$ which is $H$-convex in $\boldsymbol{C}^{n+s}$, then we have $H(K^*)=CR(K^*)$.*

## 4. Polynomial approximation.

We consider the following conditions for a compact subset $K$ of $\boldsymbol{C}^n$ and for functions $f_j$ of $C^\infty(\boldsymbol{C}^n)$;

(a) $f_1, \cdots, f_s$ are of $H_r(U)$ for some open set $U$ containing $K$,

(b) there exists a constant $k$, $0<k<1$, such that

$$\sum_{j=1}^{s} |f_j(z+\xi)-f_j(z)-\bar{\xi}_j|^2 \leq k \sum_{j=1}^{s} |\xi_j|^2$$

holds for any $z$ and $\xi=(\xi_1, \cdots, \xi_s, 0, \cdots, 0)$ in $\boldsymbol{C}^n$, and

(c) for any vector $\alpha'=(\alpha_1, \cdots, \alpha_s)$, $K \cap E_{\alpha'}$ is polynomially convex in $E_{\alpha'}$, where $E_{\alpha'}$ is the subspace $\{z \in \boldsymbol{C}^n;\ z_j=\alpha_j, j=1, \cdots, s\}$ of $\boldsymbol{C}^n$.

The condition (b) implies (1.3). In fact, we can find a constant $k_1$, $0<k_1<1$, such that

$$\sum_{j=1}^{s} \left| \sum_{\nu=1}^{s} \frac{\partial f_j}{\partial \bar{z}_\nu} \bar{\xi}_\nu + \bar{\xi}_j \right|^2 \leq k_1 \sum_{j=1}^{s} |\xi_j|^2,$$

and hence the system of linear equations

$$\sum_{\nu=1}^{s} \frac{\partial f_j}{\partial \bar{z}_\nu} \bar{\xi}_\nu = 0, \quad j = 1, \cdots, s$$

has only trivial solution.

We consider two uniform algebras on $K$. A is the algebra of uniform limits on $K$ of polynomials of $z_1, \cdots, z_n, f_1(z), \cdots, f_s(z)$. $H_r(K)$ is the algebra of uniform limits of functions each of which is holomorphic in $z_{s+1}, \cdots, z_n$, in a neighborhood of $K$.

**Theorem 2.** *Suppose the conditions* (*a*), (*b*) *and* (*c*) *are satisfied. Then we have $A=H_r(K)$.*

Proof. We shall first prove that $K^*$ is polynomially convex in $\boldsymbol{C}^{n+s}$. To do this, it is sufficient to show that the maximal ideal space of $P(K^*)$, the algebra of uniform limits of polynomials in $z_1, \cdots, z_{n+s}$ on $K^*$, ocincides with $K^*$, or equivalently that every complex homomorphism of $A$ is a point evaluation for some point of $K$. Let $\varphi$ be any complex homomorphism on $A$. Set $\alpha_j=\varphi(z_j)$, $j=1, \cdots, n$, and $\alpha=(\alpha_1, \cdots, \alpha_n)$. We consider the function

$$f(z) = \sum_{j=1}^{s} (z_j-\alpha_j)(f_j(z)-f_j(\alpha)).$$

Then $f(z)$ is in $A$. By the condition ($b$), we have

$$\operatorname{Re} f(z)>0 \qquad \text{for } z\notin E_{\alpha'},\ \alpha'=(\alpha_1, \cdots, \alpha_s)\,.$$

Let $m$ be a representing measure for $\varphi$ of $A$ supported on $K$. Then we have

$$0 = \operatorname{Re} \varphi(f) = \int \operatorname{Re} f\, dm\,.$$

Therefore, the support of $m$ must be contained in $K\cap E_{\alpha'}$ and, in particular, $K\cap E_{\alpha'}$ is not empty.

Let $h(z)$ be any polynomial of $z_1, \cdots, z_n, f_1(z), \cdots, f_s(z)$. For simplicity, we write $h_1(z_{s+1}, \cdots, z_n)=h(\alpha_1, \cdots, \alpha_s, z_{s+1}, \cdots, z_n)$. Then we have

$$\begin{aligned}\varphi(h) &= \int h(z)\, dm(z)\\ &= \int h_1(z_{s+1}, \cdots, z_n)\, dm(z_{s+1}, \cdots, z_n)\,.\end{aligned}$$

By the condition (a), $h_1$ is holomorphic in $U\cap E_{\alpha'}$. Since $K\cap E_{\alpha'}$ is polynomially convex, by Oka-Weil's theorem, $h_1$ is approximated uniformly on $K\cap E_{\alpha'}$ by polynomials of $z_{s+1}, \cdots, z_n$. Since every polynomial of $z_{s+1}, \cdots, z_n$ is considered as a polynomials of $z_1, \cdots, z_n$, $\varphi$ can be considered as a complex homomorphism $\psi$ of $P_0(K\cap E_{\alpha'})$, the algebra of uniform limits on $K\cap E_{\alpha'}$ of polynomials of $z_{s+1}, \cdots, z_n$. Polynomial convexity of $K\cap E_{\alpha'}$ implies that $\psi$ is a point evaluation at $\alpha$. Therefore we have

$$\varphi(h) = \psi(h_1) = h_1(\alpha_{s+1}, \cdots, \alpha_n) = h(\alpha)\,,$$

which proves the polynomial convexity of $K^*$.

By Oka-Weil's theorem, $H(K^*)$ coincides with $P(K^*)$. Since $K^*$ is the intersection of polynomial polyhedra containing $K^*$, it is $H$-convex, and therefore we have $H(K^*)=CR(K^*)$ by Theorem 1. A is isomorphic to $P(K^*)$ and $H_r(K)$ to $CR(K^*)$. Since $A\subset H_r(K)$, we obtain $A=H_r(K)$.

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## References


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