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A FEW REMARKS ON CLASS NUMBERS OF IMAGINARY QUADRATIC NUMBER FIELDS

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1. Let K be an imaginary quadratic number field with discriminant $-d$. As is well known, the class number $h(d)$ of K is given by the formula

$$h(d) = -\frac{1}{d}\sum_{n=1}^{d} \chi(n)n\,, \tag{1}$$

where χ is the Jacobi symbol modulo d.

Let us consider the case where d is a prime number such that $p\equiv 3 \pmod 4$. Then

$$h(p) = -\frac{1}{p}\sum_{n=1}^{p-1} \chi(n)n\,, \tag{2}$$

where χ is the Legendre symbol. From (2) we get

$$\begin{aligned} h(p) &= -\frac{1}{p}\sum_{n=1}^{(p-1)/2}\{\chi(n)n+\chi(p-n)(p-n)\} \\ &= -\frac{1}{p}\sum_{n=1}^{(p-1)/2}\{2\chi(n)n-\chi(n)p\} \\ &= -\frac{2}{p}\sum_{n=1}^{(p-1)/2}\chi(n)n+\sum_{n=1}^{(p-1)/2}\chi(n) \end{aligned}$$

Here it is well-known that

$$\sum_{n=1}^{(p-1)/2}\chi(n) = \{2-\chi(2)\}h(p)\,.$$

Summing up, we get

$$\{1-\chi(2)\}h(p) = \frac{2}{p}\sum_{n=1}^{(p-1)/2}\chi(n)n\,.$$

So, if $p\equiv -1 \pmod 7$, it holds that

(3) $$h(p)=\frac{1}{p}\sum_{n=1}^{(q-1/2}\chi(n)n\,.$$

Now denote by $\left\{\frac{n}{p}\right\}$ the frational part of n/p, i.e. $\left\{\frac{n}{p}\right\}=\frac{n}{p}-\left[\frac{n}{p}\right]$. Then we get

$$p\sum_{k=1}^{p-1}\left\{\frac{k^2}{p}\right\}=2\sum_{\substack{n=1\\ \chi(n)=1}}^{p-1}n\,,$$

which implies

$$\begin{aligned}h(p)&=-\frac{1}{p}\sum_{n=1}^{p-1}\chi(n)n\\ &=-\frac{1}{p}\{\sum_{\substack{n=1\\ \chi(n)=1}}^{p-1}n-\sum_{\substack{n=1\\ \chi(n)=-1}}^{p-1}n\}\\ &=-\frac{1}{p}\{\sum_{\substack{n=1\\ \chi(n)=1}}^{p-1}n-\sum_{n=1}^{p-1}n\}\\ &=-\sum_{k=)}^{p-1}\left\{\frac{k^2}{p}\right\}+\frac{p-1}{2}\,.\end{aligned}$$

Thus we have

(4) $$h(p)=\frac{p-1}{2}-\sum_{k=1}^{p-1}\left\{\frac{k^2}{p}\right\}.$$

Consider the area S; $0<x<p$, $0\leqq y<x^2/p$. Then (4) implies that $h(p)$ is the error term in estimating the lattice points in S. Since the hyperbola $y^2=x^2/p$ has no center, this implies the difficulty of estimation of $h(p)$ comparative with the circle problem and the divisor problem.

2. Let p be the prime such that $p\equiv 3 \pmod 4$ as before. Then we see easily $(p-1)/2!\equiv\pm 1 \pmod p$. Put $(p-1)/2!\equiv\varepsilon_p \pmod p$ with $\varepsilon_p=\pm 1$. Then $\varepsilon_p=+1$ iff the number of the set $\left\{1\leqq n\leqq(p-1)/2;\left(\frac{n}{p}\right)=1\right\}$ is even. From

$$h(p)=-\frac{1}{p}\sum_{n=1}^{p-1}\left(\frac{n}{p}\right)n$$

we have

$$h(p)=\frac{1}{2-\chi(2)}\sum_{n=1}^{(p-1)/2}\chi(n)\,,$$

as is well known. Therefore if $\varepsilon_p=+1$,

$$h(p)=\frac{1}{2-\chi(2)}\left(\frac{p-1}{2}-2s\right),$$

where s is the number of quadratic non-residue in $[1, (p-1)/2]$. Now

$$\frac{1}{2-\chi(2)}\cdot\frac{p-1}{2}\equiv 3 \pmod 4$$

as is easily verified. Therefore we have

$$h(p)\equiv -1 \pmod 4$$

regarding s is odd.

If $\varepsilon_p=-1$, we get

$$h(p)\equiv +1 \pmod 4$$

in the same way.

Summing up, we have

$$h(p)\equiv -\varepsilon_p \pmod 4.$$

It seems that the number of p with $\varepsilon_p=+1$ is asymptocally the same as that of p with $\varepsilon_p=-1$.

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Added in proof; The congruence

$$h(p) = -\varepsilon_p \pmod 4$$

is already known as the Jacobi-Mordell formula.