Miyanishi, M. and Nakai, Y.
Osaka J. Math.
12 (1975), 1-17

SOME REMARKS ON STRONGLY INVARIANT RINGS

Masayoshi MIYANISHI and Yoshikazu NAKAI*)

(Received January 30, 1974)

Introduction. Adopting the terminology of [1][1)], a ring A is called *n-invariant* (or *strongly n-invariant*) if A satisfies the condition: Given a ring B and indeterminates $X_1, \cdots, X_n$; $Y_1, \cdots, Y_n$, if $A[X_1, \cdots, X_n]$ is isomorphic to $B[Y_1, \cdots, Y_n]$, then A is isomorphic to B (or if any ring B and any isomorphism φ: $A[X_1, \cdots, X_n] \rightarrow B(Y_1, \cdots, Y_n]$ are given, then $\varphi(A)=B$). If a ring A is n-invariant (or strongly n-invariant) for every integer $n \geqq 1$, A is said to be *invariant* (or *strongly invariant*).

Several types of rings are known to be invariant or strongly invariant respectively (cf. [1], [2], [8] etc.). However we have not any good criteria for a ring to be invariant or strongly invariant, and it is tempting to look for criteria of this kind. A purpose of the present paper is to give sufficient or necessary conditions for a ring to be strongly invariant in terms of locally finite (or locally finite iterative) higher derivations.

The present paper consists of three parts. In the first section, definitions of locally finite (or locally finite iterative) higher derivations are recalled, and several results which follow easily from definitions are given. In the second section, sufficient or necessary conditions for strong invariance are given. In the final section we shall see how well these conditions work in giving examples and counter-examples. In the appendix we shall prove a Lemma on a ring which has a locally finite iterative higher derivation.

Our terminology is essentially the same as that of [1].

1. Preliminaries

We shall begin with

Definition 1.1. *Let A be a ring (or an algebra over a ring R). A locally finite higher derivation on A is a set of endomorphisms $D=\{D_0, D_1, \cdots\}$ of the abelian group A satisfying the following conditions:*

(1) $D_0 = identity$, $D_i(ab) = \sum_{j+k=i} D_j(a)D_k(b)$ *for any a, b of A.*

(2) *For any a of A, there exists an integer $n>0$ such that $D_m(a)=0$ for every*

*) The senior author is supported by Takeda Science Foundation

1) The number in the bracket refers to the bibliography at the end of the paper.

$m \geq n$.

When A is an R-algebra and D_n is R-linear for all $n \geq 0$, D is called R-trivial. D is called iterative if D satisfies the additional conditions:

(3) $D_i D_j = \binom{i+j}{i} D_{i+j}$ *for all* $i, j \geq 0$.

The concept of locally finite (or locally finite iterative) higher derivations has its own geometric meanings. Namely we have

Lemma 1.2. *Let A be a ring (or an algebra over a ring R). Then the following conditions are equivalent to each other:*

(1) *D is a locally finite higher derivation (or a locally finite R-trivial higher derivation) on A.*

(2) *The mapping φ: $A \to A[t]$ given by $\varphi(a) = \sum_{n=0}^{\infty} D_n(a) t^n$ is a homomorphism of rings (or R-algebras), where t is an indeterminate.*

Similarly the following conditions are equivalent to each other:

(1′) *D is a locally finite iterative higher derivation (or a locally finite iterative R-trivial higher derivation) on A.*

(2′) *φ: $A \to A[t]$ defined in the above condition (2) is a homomorphism of rings (or R-algebras) such that $(\varphi \otimes id.)\varphi = (id. \otimes \Delta)\varphi$, where Δ: $\boldsymbol{Z}[t] \to \boldsymbol{Z}[t] \otimes_{\boldsymbol{Z}} \boldsymbol{Z}[t]$ (or Δ: $R[t] \to R[t] \otimes_R R[t]$) is a homomorphism of $\boldsymbol{Z}$-algebras (or R-algebras) defined by $\Delta(t) = t \otimes 1 + 1 \otimes t$ (cf. the following diagram);*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A[t] = A \otimes_R R[t] \\ \varphi \downarrow & & \downarrow \varphi \otimes id \\ A \otimes_R R[t] & \xrightarrow[id \otimes \Delta]{} & A \otimes_R R[t] \otimes_R R[t] \end{array}$$

(3′) *${}^a\varphi$: $\mathrm{Spec}(A) \times_{\boldsymbol{Z}} G_{a,\boldsymbol{Z}} \to \mathrm{Spec}(A)$ (or ${}^a\varphi$: $\mathrm{Spec}(A) \times_R G_{a,R} \to \mathrm{Spec}(A)$) is an action of the additive group scheme $G_{a,\boldsymbol{Z}}$ (or $G_{a,R}$) on* $\mathrm{Spec}(A)$.

Proof. The equivalence between (1) and (2) is nothing but a reformulation of the definition. The equivalence of (1′), (2′) and (3′) will be found in [7].

Let D be a locally finite higher derivation on a ring A. An element a of A is called a *D-constant* if $D_i(a) = 0$ for all $i > 0$. An easy remark on D-constants is the following

Lemma 3. *Let A be an integral domain and let D be a locally finite higher derivation on A. Then any invertible element of A is a D-constant.*

Proof. Let φ: $A \to A[t]$ be the homomorphism defined as above. If a is an invertible element of A, $\varphi(a)$ is invertible in $A[t]$. Hence $\varphi(a) \in A$. q.e.d.

The set of all D-constants form a subring A_0 of A, and it is easy to see that A_0 is algebraically closed in A. If D is iterative, we can say more about A_0. Namely,

Lemma 1.4. *Let A be a ring, let D be a locally finite iterative higher derivation on A and let A_0 be the subring of D-constants. If there is an element t of A such that $D_1(t)=1$ and $D_i(t)=0$ for all $i>1$, then $A=A_0[t]$ and t is algebraically independent over A_0. If A is an R-algebra and D is R-trivial, A_0 is an R-subalgebra of A.*

For the proof, we refer to [7].

When an integral domain A has a locally finite higher derivation D, we can ask if the quotient field K of A is a simple transcendental extension of a subfield of K. (If this is the case, we say that A is birationally ruled.) A partial answer is given in

Lemma 1.5. *Let k be a field, let A be a finitely generated k-domain, let K be the quotient field of A and let D be a non-trivial (i.e., $A\neq A_0$) locally finite higher derivation on A. Assume one of the following conditions*:

(1) *D is iterative.*

(2) $\dim A=1$ *and the set of k-rational points of* $\mathrm{Spec}(A)$, $\mathrm{Spec}(A)(k)$, *is a dense subset of* $\mathrm{Spec}(A)$.

(3) *The k-algebra homomorphism φ: $A\to A[t]$ defined by D gives $K(t)=\varphi(K)(t)$.*

Then A is birationally ruled.

Proof. (1) When D is iterative, the additive group scheme $G_{a,k}$ acts non-trivially on $\mathrm{Spec}(A)$. Our assertion then follows from ([7], Theorems 2.1, 2.2 and the remark at P. 205). Moreover, K is a simple transcendental extension of the quotient field K_0 of A_0 (*loc. cit.*).

(2) Assume the second condition. Then there is an element a of A such that $\varphi(a)=a+a_1t+\cdots+a_nt^n$ and $\varphi(a)\notin A$, where $a_i\in A$ $(1\leq i\leq n)$ and $a_n\neq 0$. Since $\mathrm{Spec}(A)(k)$ is dense in $\mathrm{Spec}(A)$, there is a k-algebra homomorphism $\pi: A\to k$ such that $\pi(a_n)\neq 0$. The composite $\rho=(\pi\otimes id)\varphi$: $A\to k[t]$ is a k-algebra homomorphism such that $\rho(A)\not\subset k$. Since $\dim A=1$, $\mathrm{Ker}\,\rho=0$. Hence by Luroth's theorem, A is rational.

(3) Consider the t^{-1}-adic valuation of $K(t)$ over K and its valuation ring $\mathfrak{v}^*$. Let us set $\mathfrak{v}=\mathfrak{v}^*\cap\varphi(K)$. Let $\mathfrak{p}^*$ and $\mathfrak{p}$ be the maximal ideals of $\mathfrak{v}^*$ and $\mathfrak{v}$ respectively. Since D is non-trivial, there is an element a of A such that $\varphi(a)\notin A$. Write $\varphi(a)=a+a_1t+\cdots+a_nt^n$ with $a_n\neq 0$. Then $\varphi(a)^{-1}=t^{-n}(a_n+a_{n-1}t^{-1}+\cdots+at^{-r})^{-1}\in\mathfrak{p}^*\cap\varphi(K)=\mathfrak{p}$. Hence $\mathfrak{p}\neq 0$. Since $\mathrm{tr.\,deg}_k\varphi(K)=\mathrm{tr.\,deg}_k(\mathfrak{v}/\mathfrak{p})+\mathrm{ht}\,\mathfrak{p}$, the residue field $\mathfrak{v}^*/\mathfrak{p}^*$, which is isomorphic to K, is not algebraic over the residue field $\mathfrak{v}/\mathfrak{p}$. Since $\mathfrak{v}^*$ is obviously a discrete valuation

ring, we can conclude by Nagata's theorem [9] that K is a simple transcendental extension of a finite algeraic extension of $\mathfrak{o}/\mathfrak{p}$. q.e.d.

An easy consequence of the first case of Lemma 1.5 is

Corallary 1.6. *Let k be a field and let A be a finitely generated k-domain of dimension* 1. *If A has a non-trivial locally finite iterative higher derivation D, then A is a polynomial ring over the algebraic closure k_0 of k in A.*

Proof. We have only to show the existence of a k_0-rational point in Spec(A). It is easy to see that A_0 is a subfield of A and is in fact the algebraic clusure of k in A. Then, Lemma 1.5 implies that Spec(A) is birational to the affine line A^1 over k_0. Hence Spec(A) has a k_0-rational point. q.e.d.

The assumption that A is a finitely generated k-algebra is not necessary in proving (1.6). For the proof we refer to the appendix.

2. Strongly invariant rings

In this section, we are interested in looking for sufficent or necessary conditions for a ring to be strongly invariant. A sufficient condition for strong 1-invariance is given, making use of Nagata's theorem [9], in the

Theorem 2.1. ([1]). *Let k be a field and let A be an affine k-domain. If A is not birationally ruled, then A is strongly* 1*-invariant.*

If A is not birationally ruled, it seems plausible that A is strongly invariant. However the authors could not determine whether this is true or not. Another sufficient condition for strong invariance is stated as follows:

Theorem 2.2. *Let A be a ring and assume that A contains a subdomain C of infinite cardinality, whose non-zero elements are non-zero divisors of A. If A has no non-trivial locally finite higher derivation, then A is strongly invariant.*

Proof. Assume that A is not strongly invariant. Then there is a ring B $(\neq A)$ such that $A[X_1, \cdots, X_n]=B[Y_1, \cdots, Y_n]$, where $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ are algebraically independent over A and B respectively. Since $A \neq B$, there is an element $a \in A$ such that, regarded as an element of $B[Y_1, \cdots, Y_n]$, a is written in the form $a=\sum b_{\alpha_1,\cdots,\alpha_n} Y_1^{\alpha_1} \cdots Y_n^{\alpha_n}$ with $b_{\alpha_1,\cdots,\alpha_n} \in B$. Let $w_1, \cdots, w_n$ be positive integers. Define a locally finite higher derivation D on $B[Y_1, \cdots, Y_n]$ by a homomorphism of rings φ: $B[Y_1, \cdots, Y_n] \rightarrow B[Y_1, \cdots, Y_n, t]$, which is given by $\varphi(b)=b$ for all $b \in B$ and $\varphi(Y_i)=Y_i+t^{w_i}$ for $1 \leq i \leq n$. We can choose $w_1, \cdots, w_n$ so that $\varphi(a)=bt^m+$(a polynomial in t of degree $<m$ with coefficients in $B[Y_1, \cdots, Y_n]$), where $b \in B$ and $m>0$. Since $B[Y_1, \cdots, Y_n]=A[X_1, \cdots, X_n]$, we can rewrite $\varphi(a)$ in the form $\varphi(a)=f(X_1, \cdots, X_n)t^m+$ (a polynomial in t of degree$<m$ and with coefficients in $A[X_1, \cdots, X_n]$). Then by virtue of the following easy sublemma, there are elements $c_1, \cdots, c_n$ in C such that $f(c_1, \cdots, c_n) \neq 0$. Let q: $A[X_1, \cdots, X_n, t] \rightarrow A[t]$ be the A-algebra homomorphism given by $q(t)=t$ and $q(X_i)=c_i$ for $1 \leq i \leq n$. Let ι be the canonical injection ι: $A \rightarrow A[X_1, \cdots, X_n]$ and let $\rho=q\varphi\iota$. Then ρ: $A \rightarrow A[t]$ is a homomorphism of ring with $\rho(A) \not\subset A$. Therefore ρ defines a non-trivial locally finite higher derivation on A and this is a contradiction. q.e.d.

Sublemma. *Let A and C be as in Theorem 2.2. Let $f(X_1, \cdots, X_n)$ be a polynomial over A in n-indeterminates $X_1, \cdots, X_n$. If $f(c_1, \cdots, c_n)=0$ for any set $(c_1, \cdots, c_n)$ of elements of C, then f is identically zero.*

The proof is standard and we therefore omit it.

The assumption on A in Theorem 2.2 is satisfied if either the characteristic of A is zero or A contains an infinite field.

A criterion of strong invariance given in Theorem 2.2 is rather difficult to use practically, and is well complemented by the following result, which can be proved by the same principle as in Theorem 2.2.

Theorem 2.3. *Let k be an infinite field and let A be a k-domain satisfying the conditions*:

(1) Spec$(A)(k)$ *is dense in* Spec(A).

(2) *There is no non-onstant k-morphism from the affine line to* Spec(A).

Then A is strongly invariant.

The converse of Theorem 2.2 does not hold as it is shown by the following

Theorem 2.4. *Let k be a field and let A be the affinie ring of the affine cone of a smooth projective variety U. Assume that there is no non-constant $\bar{k}$-rational mapping from the affine line $\mathbf{A}^1$ to U, $\bar{k}$ being an algebraic closure of k. Then A is strongly invariant, while A has a non-trivial locally finite higher derivation.*

Proof. (1) Write $A=k[Z_0, \cdots, Z_t]/(F_1, \cdots, F_m)$, where $F_1, \cdots, F_m$ are homogeneous polynomials of $k[Z_0, \cdots, Z_t]$. Define a higher derivation D on $k[Z_0, \cdots, Z_t]$ by setting $D_0=$id., $D_1(Z_i)=Z_i$ and $D_j(Z_i)=0$ for $0 \leq i \leq t$ and $j \geq 2$. It is then easy to see that D induces a non-trivial locally finite higher derivation $\bar{D}$ on A.

(2) We shall show next that A is strongly invariant. Assume that we are given a k-algebra B such that $A[X_1, \cdots, X_n]=B[Y_1, \cdots, Y_n]$, where $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ are algebraically independent over A and B respectively. By virtue of Lemma 2.7 below, we may assume that k is algebraically closed. Set $V=$Spec(A) and $W=$Spec(B). Noting that B is an affine k-domain as well, V and W are the affine varieties defined over k. Then we have $V \times \mathbf{A}^n=W \times \mathbf{A}^n$,

where $\boldsymbol{A}^n$ stands for the affine n-space. First of all we show that for any point w of W, the affine space $w=\boldsymbol{A}^n$ (which is a fibre of the canonical fibration $W\times\boldsymbol{A}^n\to W$) coincides with an affine space $v\times\boldsymbol{A}^n$ for some point v of V, $v\times\boldsymbol{A}^n$ being a fibre of the canonical fibration $q\colon V\times\boldsymbol{A}^n\to V$. Let π be the canonical projection $V-(0,\cdots,0)\to U$. The image $\pi q(w\times\boldsymbol{A}^n)$ should be one-point, since there is no non-constant rational mapping from $\boldsymbol{A}^1$ to U. Therefore the image $q(w\times\boldsymbol{A}^n)$ is either one-point or an affine rational curve with only one place at infinity. Assume the latter case. Then $q(w\times\boldsymbol{A}^n)$ coincides with a generator of the cone. Hence $q(w\times\boldsymbol{A}^n)$ passes through the vertex $\mathcal{O}=(0,\cdots,0)$ of the cone, which is the only singular point of V. This means that $w\times\boldsymbol{A}^n$ has a non-empty intersection with the singular locus $\mathcal{O}\times\boldsymbol{A}^n$ of $V\times\boldsymbol{A}^n$. However it is easy to see that W has the only one singular point w_0, and the singular locus of $W\times\boldsymbol{A}^n$ must be $w_0\times\boldsymbol{A}^n$. Since the affine space $w\times\boldsymbol{A}^n$ has no non-empty intersection with the singular locus $\mathcal{O}\times\boldsymbol{A}^n=w_0\times\boldsymbol{A}^n$ if $w\neq w_0$, the above observation leads to a contradiction. Thus $q(w\times\boldsymbol{A}^n)$ is a one-point v of V and $w\times\boldsymbol{A}^n$ is contained in the affine space $v\times\boldsymbol{A}^n$. Then we have obviously $w\times\boldsymbol{A}^n=v\times\boldsymbol{A}^n$. Actually this is the case for all points w of W, taking into account the fact that $\mathcal{O}\times\boldsymbol{A}^n=w_0\times\boldsymbol{A}^n$. This means that every maximal ideal of B is vertical relative to A. Then we conclude our proof owing to the result ([1], (1.13)). q.e.d.

To state Lemma 2.7 which we used in the above proof, we need

Definition 2.5. *Let R be a ring and let A be an R-algebra. A is called R-n-invariant (or strongly R-n-invariant) if A satisfies the condition: Given an R-algebra B and indeterminates $X_1,\cdots,X_n$; $Y_1,\cdots,Y_n$, if $A[X_1,\cdots,X_n]$ is R-isomorphic to $B[Y_1,\cdots,Y_n]$, then A is R-isomorphic to B (or if any R-algebra B and any R-isomorphism $\varphi\colon A[X_1,\cdots,X_n]\to B[Y_1,\cdots,Y_n]$ are given, then $\varphi(A)=B$). If an R-algebra A is R-n-invariant (or strongly R-n-invariant) for all $n\geq 1$, A is simply called R-invariant (or strongly R-invarinat).*

Lemma 2.6. *Let R be a ring and let A be a reduced R-algebra. Assume that A satisfies the condition: Given any R-algebra B and a relation $A[X_1,\cdots,X_n]=B[Y_1,\cdots,Y_n]$, $X_1,\cdots,X_n$ and $Y_1,\cdots,Y_n$ being algebraically independent over A and B respectively, we have $B[X_1,\cdots,X_n]=B[Y_1,\cdots,Y_n]$. Then A is strongly R-n-invariant.*

For the proof, we refer to [2].

Now the result used in the proof of Theorem 2.4 can be stated as follows:

Lemma 2.7. *Let R be a ring and let A be a reduced R-algebra. Let R' be an R-algebra which is a faithfully flat R-module. If $A\otimes_R R'$ is strongly R'-n-invariant, A is strongly R-n-invariant.*

Proof. Assume that we are are given an R-algebra B such that $A[X_1, \cdots, X_n]=B[Y_1, \cdots, Y_n]$. Then $(A\otimes_R R')[X_1, \cdots, X_n]=(B\otimes_R R')[Y_1, \cdots, Y_n]$. Since $A\otimes_R R'$ is strongly R'-n-invariant, we have $A\otimes_R R'=B\otimes_R R'$. On the other hand, we have the canonical inclusion $B[X_1, \cdots, X_n]\subseteq B[Y_1, \cdots, Y_n]$, and $(B\otimes_R R')[X_1, \cdots, X_n]=(B\otimes_R R')[Y_1, \cdots, Y_n]$. By the (fpqc)-descent theory ([5], IV (2.7.1)), we have $B[X_1, \cdots, X_n]=B[Y_1, \cdots, Y_n]$. Then Lemma 2.6 acomplishes our proof. q.e.d.

One of the concrete examples which satisfy the conditions of Theorem 2.4 is: $A=k[X, Y, Z]/(X^3+Y^3+Z^3)$.

Before going to further sufficient conditions for strong invariance, we shall give a necessary condition:

Theorem 2.8. *Let A be a ring. If A is strongly 1-invariant, A has no non-trivial locally finite iterative higher derivation.*

Proof. Assuming that A has a non-trivial locally finite iterative higher derivation D, we shall show that A is not strongly 1-invariant. Define a ring homomorphism φ: $A\to A[t]$ as usual, and let $B=\varphi(A)$. We claim that $A[t]=B[t]$. In fact, the inclusion $B[t]\subseteq A[t]$ is obvious. To show the converse inclusion, we need the notion of length of each element of A with respect to a given locally finite iterative higher derivation D. The length $l(a)$ of an element a of A is a non-negative integer n such that $D_n(a)\neq 0$ and $D_m(a)=0$ for all $m>n$. By induction on the length $l(a)$, we prove the inclusion: $A\subset B[t]$. If $l(a)=0$, then $a\in B$. Assume that all elements of A of length $<n$ belong to $B[t]$. Let a be an element with $l(a)=n$. Then $l(D_i(a))<n$ if $i>1$. Hence $D_i(a)\in B[t]$. By the way, $\varphi(a)=a+D_1(a)t+\cdots+D_n(a)t^n$. Hence $a\in B[t]$. Thus we have proved $A[t]=B[t]$. Since D is non-trivial, $B\neq A$. Therefore A is not strongly 1-invariant. q.e.d.

A question which arises from Theorem 2.8 is: Let A be a ring. If A has no non-trivial llocally finite iterative higher derivation, is A strongly invariant (or at least strongly 1-invariant)? When A is an affine domain of dimension 1 over a field, this is true and was essentially proved in [1], (3.4). We have in fact

Theorem 2.9. *Let k be a field and let A be an affine k-domain of dimension 1. Then the following conditions are equivalent to each other:*

(1) *A is strongly invariant.*

(2) *A is strongly 1-invariant.*

(3) *A has no non-trival locally finite iterative higher derivations.*

Proof. The implication (1)$\Rightarrow$(2) is clear, while (2)$\Rightarrow$(3) follows from (2.8). Therefore it remains to show (3)$\Rightarrow$(1). For this, we use the result, proved in [1], (3.4), that under the above assumptions A is either strongly invariant or A is a polynomial ring $k_0[x]$ over the algebraic closure k_0 of k in A. In the latter case, A has a non-trivial locally finite iterative higher derivation. Therefore, A should be strongly invariant under the condition (3). q.e.d.

When dim $A=2$, a partial answer to the above question is stated as follows:

Theorem 2.10. *Let k be an algebraically closed field of characteristic zero and let A be an irrational smooth k-affine domain of dimension* 2. *Then one of the following three cases takes place*:

(1) *A is strongly* 1*-invariant.*

(2) *A has a non-trivial locally finite iterative higher derivation.*

(3) *There is a surjective morphism from* Spec(A) *to a non-singular complete curve of genus*>0.

Proof. If A is not strongly 1-invariant, there exists a k-algebra B $(\neq A)$ such that $A[X]=B[Y]$, where X and Y are algebraically independent over A and B respectively. Then the ruling of Spec$(A[X])$ given by the fibration Spec$(A[X])=$Spec$(A)\times \boldsymbol{A}^1\rightarrow$Spec$(A)$ does not coincide with the ruling of the same variety given by the fibration Spec$(B[Y])=$Spec$(B)\times\boldsymbol{A}^1\rightarrow$Spec$(B)$. For, otherwise, all maximal ideals of B are vertical relative to A. Henec $A=B$ (cf. [1], (1.13)). Let $V=$Spec(A) and $W=$Spec(B). Then $V\times\boldsymbol{A}^1=W\times\boldsymbol{A}^1$. Let $\bar{V}$ be a non-singular completion of V, whose existence will be clear and which we may assume that $F=\bar{V}-V$ has no exceptional curves of the first kind. $\bar{V}$ is a ruled surface (cf. (2.1)). Let C be the base curve of $\bar{V}$ and let π: $\bar{V}\rightarrow C$ be the canonical projection. By the assumption that A is not rational. C is a non-singular complete curve of genus >0. Denote by l_w the affine line $w\times\boldsymbol{A}^1$ for any $w\in W$ and by l'_w the image of l_w by the canonical projection q: $V\times\boldsymbol{A}^1\rightarrow V$. For a general $w\in W$, l'_w is an affine rational curve with only one place at infinity. Since the genus of C is >0, l'_w must be contained in a fibre of π. Therefore, for a sufficiently general point $w\in W$, l'_w is isomorphic to $\boldsymbol{A}^1$ and is of the form: (a fibre of π)$-$(the point of infinity). Moreover, $l'_w\cap l'_{w'}=\phi$ if $l'_w\neq l'_{w'}$.

If $\pi(V)=C$. we are led to the third case in the statement of Theorem. Assume that $\pi(V)\neq C$. Then $\pi(V)$ is an affine open set of C. Let $\pi(V)=$ Spec(A_0). A_0 is a k-subalgebra of A. It is now easy to see that there exists an element s of A_0 such that setting $U=$(the set of points of $\pi(V)$ where s does not vanish), $\pi^{-1}(U)$ is a trivail P^1-bundle, i.e., $\pi^{-1}(U)\cong U\times\boldsymbol{P}^1$, and that for any $u\in U$, $V\cap\pi^{-1}(u)$ is of the form for l'_w some $w\in W$ and is isomorphic to $\boldsymbol{A}^1$. Let $F'=F\cap\pi^{-1}(U)$. Since $\pi|_{F'}$: $F'\rightarrow U$ is generically one to one and the characteristic of k is zero, $\pi|_{F'}$ is birational. Shrinking U to a smaller affine open set of C if necessary, we may assume that $\pi|_{F'}: F' \to U$ is an isomorphism. Then F' is a section of the trivial $\boldsymbol{P}^1$-bundle $\pi^{-1}(U)$. Therefore $V \cap \pi^{-1}(U) \cong \pi^{-1}(U) \times \boldsymbol{A}^1$. Let S be the multiplicatively closed set generated by s in A_0. Then $S^{-1}A = (S^{-1}A_0)[t]$ for some element t of A_0 which is algebraically independent over A. We can now define a non-trivial locally finite iterative higher derivation D on A as follows: $D_0 = id_A$, $D_1|_{A_0} = 0$ and $D_1(t) = s^\alpha$ with sufficiently large integer α and $D_i = (1/i!)D_1^i$. If $a_1, \cdots, a_n$ are generators of A over k and if $a_i = f_i(t)/s^{\alpha_i}$ with $f_i(t) \in A_0[t]$ for $1 \leq i \leq n$, it is enough to take α as large as $\alpha \geqslant \max_{1 \leq i \leq n} (\alpha_i)$. q.e.d.

The authors have an impression that the third case is contained in the first case. Further observation on the third case will be given in the final part of this paper. When dim $A > 2$, we have nothing to say.

3. Examples and counterexamples

1. One of the interesting examples of strongly invariant rings was first given in [6] and later discussed in [3]. This is the following:

Let $\boldsymbol{R}$ be the field of real numbers and let A_n be the affine ring of the real n-sphere, i.e., $A_n = \boldsymbol{R}[X_0, \cdots, X_n]/(X_0^2 + \cdots + X_n^2 - 1)$ for $n = 1, 2, \cdots$. Then A_n is strongly invariant, a one-parameter polynomial ring $A_n[t]$ is invariant and a polynomial ring $A_n[t_1, \cdots, t_n]$ of dimension n is not 1-invariant if $n \neq 1, 3, 7$.

Looking at this example one might ask if a one-parameter polynomial ring $A[t]$ is invariant (or A-invariant) provided that A is strongly invariant. However this is false as it is shown by the following example (cf. [2]): Let k be a field of characteristic $p > 0$ $(p \neq 2)$ and let $A = k[t^p, t^{p+1}]$, t being an indeterminate. Let $A[X]$ be a one-parameter polynomial ring. Then $A[X]$ is not invariant, while A is strongly invariant.

2. The converse of Lemma 2.7 is false. An example is a one-dimensional k-wound unipotent group: Let k be a non-perfect field of characteristic $p > 0$, and let $A = k[X, Y]/(Y^{p^n} - X - a_1X^p - \cdots - a_rX^{p^r})$, where $a_1, \cdots, a_r \in k$ and one of $a_1, \cdots, a_r \notin k^p$. Then A is strongly invariant (use (2.3)). (Note that A is not k-rational except when $p = 2$ and $Y^2 = X + aX^2$ with $a \in k$ and $a \notin k^2$.) However if k' is the perfect closure of k, $A \otimes_k k'$ is a polynomial ring over k'. Hence $A \otimes_k k'$ is not strongly invariant.

3. The remaining part of this paper is devoted to an investigation of the third case of Theorem 2.10. In the following k is an algebraically closed field of characteristic p and C is a non-singular complete curve of genus g>0. We use the term "vector bundle" synonymously with "locally free $\mathcal{O}_C$-module". A section of a projective bundle $\boldsymbol{P}(E)$ is a curve on $\boldsymbol{P}(E)$, on which the restriction of the canonical projection $\boldsymbol{P}(E) \to C$ is an isomorphism.

We have in the first place

Lemma 3.1. *Let L be an ample line bundle on C and let E be a non-trivial extension of L by $\mathcal{O}_C$. Let s be the section of $\boldsymbol{P}(E)$ corresponding to L and let $X=\boldsymbol{P}(E)-s$. Assume that the characteristic of k is zero or deg $L>2g$. Then X is an affine surface such that the restriction onto X of the projection $\boldsymbol{P}(E)\to C$ is a surjective morphism onto C. Conversely if an affine surface X is $\boldsymbol{P}^1$-bundle $\boldsymbol{P}(E)$ on C deleted a section, then X is isomorphic to an affine surface constructed in the above fashion.*

Proof. Let L be an ample line bundle on C and let E be a non-trivail extension of L by $\mathcal{O}_C$. Assume that the characteristic of k is zero or deg $L>2$g. Then it is known (cf. [4]) that E is an ample vector bundle on C and the tautological line bundle $\mathcal{O}_{\boldsymbol{P}(E)}(1)$ is isomorphic to $\mathcal{O}_{\boldsymbol{P}(E)}(s)$. Therefore s is an ample divisor on a non-singular projective surface $\boldsymbol{P}(E)$ and $X=\boldsymbol{P}(E)-s$ is affine. It is obvious that the restriction onto X of the projection $\boldsymbol{P}(E)\to C$ is a surjective morphism onto C.

Conversely, let E be a vector bundle on C, let $\boldsymbol{P}(E)$ be the $\boldsymbol{P}^1$-bundle associated with E and let X be the $\boldsymbol{P}(E)$ deleted a section s. Let L' be the quotient line bundle of E corresponding to s and let L be the kernel of $E\to L'$; $O\to L\to E\to L'\to O$. We shall show that $L'\otimes L^{-1}$ is ample and that X is isomorphic to $\boldsymbol{P}(E\otimes L^{-1})-$the section s' corresponding to $L'\otimes L^{-1}$. Since X is affine, s is irreducible and $\boldsymbol{P}(E)$ is non-singular, the section s regarded as a divisor on $\boldsymbol{P}(E)$ must be ample. Let i: $\boldsymbol{P}(E)\to\boldsymbol{P}(E\otimes L^{-1})$ be the canonical isomorphism. Then the section s is transformed to the section s' by i and X to the affine surface $\boldsymbol{P}(E\otimes L^{-1})-s'$. Hence s' is an ample divisor on $\boldsymbol{P}(E\otimes L^{-1})$. Let $j: C\to\boldsymbol{P}(E)$ be the isomorphism sending C to s. Then $i\cdot j$ is an imbedding. Taking account of the facts that $\mathcal{O}_{\boldsymbol{P}(E\otimes L^{-1})}(1)\cong\mathcal{O}_{\boldsymbol{P}(E\otimes L^{-1})}(s')$ and $(i\cdot j)^*(\mathcal{O}_{\boldsymbol{P}(E\otimes L^{-1})}(1))=j^*(\mathcal{O}_{\boldsymbol{P}(E)}(1)\otimes L^{-1})=L'\otimes L^{-1}$, we know that $L'\otimes L^{-1}$ is ample on C. q.e.d.

The affine ring of an affine surface constructed in Lemma 3.1 has no non-trivial locally finite iterative higher derivation, though it has a non-trivial locally finite higher derivation. This is an immediate consequence of

Lemma 3.2. *Let V be a variety defined over k, let L be a line bundle over V and let E be an extension of L by $\mathcal{O}_V$. Let X be the $\boldsymbol{P}(E)$ minus the section corresponding to L. If $H^0(V, L^{-1})\neq 0$, X has a non-trivial G_a-action. If there is no non-constant morphism from $\boldsymbol{A}^1$ to V, then $H^0(V, L^{-1})\neq 0$ provided that X has a non-trivial G_a-action.*

Proof. Let $\mathfrak{U}=\{U_i\}_{i\in I}$ be an affine open covering of V such that $E\mid U_i$ is trivial for any $i\in I$ and let $\left\{\begin{pmatrix} a_{ji} & b_{ji} \\ 0 & 1 \end{pmatrix}\right\}$ be the transition matrices of E relative to

$\mathfrak{U}$, where $\{a_{ji}\}$ is the transition functions of L. X is in fact an A^1-bundle on V with affine coordinates $\{x_i\}$ which are subject to $x_j=a_{ji}x_i+b_{ji}$ for any $i, j\in I$.

If $H(V^0, L^{-1})\neq 0$, we a may assume that there is a set of functions $\{s_i\}$ on V such that $s_i\in\Gamma(U_i, \mathcal{O}_V)$ and $s_j=a_{ji}s_i$ for any $i, j\in I$. Define a non-trivial locally finite iterative higher derivation $D=\{D_0, D_1, \cdots\}$ on $\Gamma(U_i, \mathcal{O}_V)[x_i]$, which is the affine ring of $\pi^{-1}(U_i)$, π being the canonical projection from X to V: $D_0=id.$, $D_n|_{\Gamma(U_i,\mathcal{O}_V)}=0$ for any $n>0$, $D_n(x_i^m)=\binom{m}{n}x_i^{m-n}s_i^n$ if $m\geq n$ and 0 otherwise. Then D gives rise to a non-trivial G_a-action on $\{\pi^{-1}(U_i)\}$, ${}^a\varphi_D$: $\pi^{-1}(U_i)\times G_a\rightarrow\pi^{-1}(U_i)$. It is now easy to show that the G_a-actions on $\{\pi^{-1}(U_i)\}$ patch each other on $\pi^{-1}(U_i\cap U_j)$ to give a nonstrivial G_a-action on X.

Assume next that there is no non-constant morphism from A^1 to V. If X has a non-trivial G_a-action, then G_a should act on X along fibres of π, and the G_a-invariant subfield in $k(X)$ is $k(V)$ (cf. [7]). For each $i\in I$, the G_a-action restricted on $\pi^{-1}(U_i)$ gives rise to a $\Gamma(U_i, \mathcal{O}_V)$-homomorphism φ_i: $\Gamma(U_i, \mathcal{O}_V)[x_i]\rightarrow\Gamma(U_i, \mathcal{O}_V)[x_i, t]$, t being an indeterminate. Write $\varphi_i(x_i)=s_it^n+$(terms of lower degree in t with coefficients in $\Gamma(U_i, \mathcal{O}_V)$), where $n\geq 1$ and $s_i\in\Gamma(U_i, \mathcal{O}_V)[x_i]$. Since the G_a-invariant subfield of $k(X)$ is $k(V)$ and since s_i is G_a-invariant, $s_i\in\Gamma(U_i, \mathcal{O}_V)[x_i]\cap k(V)=\Gamma(U_i, \mathcal{O}_V)$. Moreover it is easy to see that n is independent of i and $s_j=a_{ji}s_i$ for any $i, j\in I$. Then $\{s_i\}_{i\in I}$ give a non-zero section of $H^0(V, L^{-1})$. Hence $H^0(V, L^{-1})\neq 0$. q.e.d.

The affine ring of an affine surface constructed in Lemma 3.1 is strongly 1-invariant if the characteristic of k is zero. To prove this result, we need

Lemma 3.3. *Let k be a field of characteristic zero and let φ be a k-automorphism of the polynomial ring $k[x, y]$ in two variables given by $\varphi(x)=f$, $\varphi(y)=g$. Then f has the following form unless f is a polynomial in x or y alone.*

$$(*)\qquad f=ax^m+by^n+\sum_{\substack{m>i\\ n>j}}c_{ij}x^iy^j$$

where a, b and c's are elements of k and $ab\neq 0$. The same holds for g.

Proof. (I) First we shall treat the case where one of f and g, say f, is a polynomial in, say, y alone. By the assumption φ is an automorphism of $k[x, y]$. Hence we must have that $(\partial f/\partial y)(\partial g/\partial x)$ is a non-zero constant in k. Hence $\partial f/\partial y=a$ and $\partial g/\partial y=b$ are also non-zero constant in k. From this we get easily $f=ay+c$, $g=bx+h(y)$.

Now assume f has the form $(*)$. We shall show that if g is not a polynomial in x or y alone, then g has also the form of $(*)$. Let us set first

$$g=\alpha_0(y)x^u+\alpha_1(y)x^{u-1}+\cdots+\alpha_u(y)\qquad(\alpha_0(y)\neq 0,\ u>0)$$

where $\alpha_i(y)\in k[y]$. From the fact that $\partial(f, g)/\partial(x, y)$ is a non-zero constant we easily get $\alpha_0'(y)=0$. Hence $\alpha_0(y)$ is a non-zero element of k. Similarly if we rewrite g in the form

$$g = \beta_0(x)y^v+\beta_1(x)y^{v-1}+\cdots+\beta_v(x) \qquad (\beta_0(x)\neq 0,\ v>0)$$

We easily see that β_0 is an element of k. These two results imply that g has the form $(*)$.

(II) It is well known (cf. [10]) that any k-automorphism of $k[x, y]$ is written as a composite of linear automorphisms; $(x, y)\mapsto(\alpha x+\beta y+c, \gamma x+\delta y+d)$ with $\alpha\delta-\beta\gamma\neq 0$, and Jonquière automorphisms; $(x, y)\mapsto(x, y+h(x))$ with $h(x)\in k[x]$. Using this result, we shall show that any k-automorphism of $k[x, y]$ is a composite of automorphisms, each of which is an automorphism ρ such that $\rho(x)$ or $\rho(y)$ coincides with one of x and y. (We shall say such an automorphism to be of type (P)). Since Jonquière automorphisms are of type (P), it suffices to show that a linear automorphism is a composite of two linear automorphisms of type (P). In fact, a linear automorphism $(x, y)\mapsto(\alpha x+\beta y+c, \gamma x+\delta y+d)$ is decomposed as follows: If $\alpha\neq 0$, $(x, y)\mapsto(x', y')=(\alpha x+\beta y+c, y)$, $(x', y')\mapsto(x', (\gamma/\alpha')x'+((\alpha\delta-\beta\gamma)/\alpha)y'+(d-(\gamma c/\alpha)))$. If $\alpha=0$, $(x, y)\mapsto(x', y')=(y, \gamma x+\delta y+d)$, $(x', y')\mapsto(((\beta\gamma-\alpha\delta)/\gamma)x'+(\alpha/\gamma)y'+(c-(\alpha d/\gamma), y')$.

(III) Write the given automorphism φ as $\varphi=\varphi_r\cdot\varphi_{r-1}\cdots\varphi_1$, where $\varphi_1, \cdots, \varphi_r$ are automorphisms of type (P). We shall prove our assertion by induction on r. If $r=1$, φ is one of the following forms $(x, y)\mapsto(ax+h(y), y)$, $(x, y)\mapsto(y, a_1x+h_1(y))$, $(x, y)\mapsto(x, by+l(x))$ or $(x, y)\mapsto(b_1y+l_1(x), x)$, where a, a_1, b, $b_1\in k$ and $h(y)$, $h_1(y)\in k[y]$ and $l(x)$, $l_1(x)\in k[x]$. Hence the assertion is clear. Assuming next that the assertion is true when φ is a composite of less than r automorphisms of type (P), we consider the case where $\varphi=\varphi_r\cdot\varphi_{r-1}\cdot\cdots\cdot\varphi_1$. Let $\psi=\varphi_{r-1}\cdot\cdots\cdot\varphi_1$ and let $\psi(x, y)=(f_1, g_1)$ with $f_1, g_1\in k[x, y]$.

By induction assumption f_1, g_1 have the form $(*)$ unless they are polynomials in x or y alone. Since φ_r is an automorphism of type (P) we have one of the following four cases.

(i) $\varphi(x)=f_1$, (ii) $\varphi(x)=g_1$, (iii) $\varphi(y)=f_1$, (iv) $\varphi(y)=g_1$.
In any case we easily get the assertion from the results in the step (I). q.e.d.

Now we can prove

Theorem 3.4. *Let k be an algebraically closed field of characteristic zero, let C be a non-singular complete curve of genus $\geqq 0$ defined over k, let L be an ample line bundle over C and let E be a non-trivial extension of L by $\mathcal{O}_C$. Let X be the $\boldsymbol{P}^1$-bundle $\boldsymbol{P}(E)$ minus the section s corresponding to L and let A be the affine ring of X. Then A is strongly 1-invariant.*

Proof. Our proof consists of several steps.

(1) Let B be a k-algebra such that $A[T]=B[V]$, T and V being algebraically independent over A and B respectively. Let $Y=\mathrm{Spec}(B)$ and let $\pi: X\to C$ be the canonical projection. By the composite of projections $Y\times \boldsymbol{A}^1=X\times \boldsymbol{A}^1 \xrightarrow{p_1} X\to C$, each line $(y)\times \boldsymbol{A}^1$ with $y\in Y$ is sent to a point of C. Hence $\pi\cdot p_1$ factors as $Y\times \boldsymbol{A}^1 \xrightarrow{p_1'} Y \xrightarrow{q} C$ and Y is regarded as a C-scheme by q. Hence q is surjective. Let $\mathfrak{U}=\{U_i\}_{i\in I}$ be an affine open covering of C such that $E|_{U_i}$ is trivial for any $i\in I$. Let $\{x_i\}_{i\in I}$ be an affine coordinate system of X relative to $\mathfrak{U}$. Then $\{x_i\}_{i\in I}$ is subject to $x_j=a_{ji}x_i+b_{ji}$ for any $i, j\in I$. Let $U_i=\mathrm{Spec}(R_i)$ and let $q^{-1}(U_i)=\mathrm{Spec}(B_i)$. Then $R_i[x_i, T]=B_i[V]$. Since B_i is an R_i-algebra and R_i is regular, there is an element $y_i\in B_i$ such that $B_i=R_i[y_i]$ (cf. [1]). For any $i, j\in I$, we have: $y_j=a'_{ji}y_i+b'_{ji}$ with $a'_{ji}\in R^*_{ji}$ and $b'_{ji}\in R_{ji}$, where $U_i\cap U_j=\mathrm{Spec}(R_{ji})$. Then, it is easy to see that $\left\{\begin{pmatrix} a'_{ji} & b'_{ji} \\ 0 & 1\end{pmatrix}\right\}$ are transition matrices of a vector bundle E', which is an extension of a line bundle L' by $\mathcal{O}_C$, L' being defined by transition functions $\{a'_{ji}\}$ relative to $\mathfrak{U}$. Moreover Y is the $\boldsymbol{P}^1$-bundle $\boldsymbol{P}(E')$ deleted the section s' corresponding to L'.

(2) Let $\Omega^1_{X/C}$ be the $\mathcal{O}_X$-module of 1-differential forms of X over C. Since $\Omega^1_{X/C}|_{\pi^{-1}(U_i)}=(dx_i)\mathcal{O}_{\pi^{-1}(U_i)}$ and $dx_j=a_{ji}dx_i$, we have $\Omega^1_{X/C}\cong L\otimes_{\mathcal{O}_C}\mathcal{O}_X$. The relation $A[T]=B[V]$ implies that $L\otimes_{\mathcal{O}_C}\mathcal{O}_X[T]\oplus\mathcal{O}_X[T]\simeq L'\otimes_{\mathcal{O}_C}\mathcal{O}_Y[V]\oplus\mathcal{O}_Y[V]$. Hence $L\otimes_{\mathcal{O}_C}\mathcal{O}_X[T]\cong L'\otimes_{\mathcal{O}_C}\mathcal{O}_Y[V]$, and $(L\otimes_{\mathcal{O}_C}L'^{-1})\otimes_{\mathcal{O}_C}\mathcal{O}_X[T]\cong \mathcal{O}_X[T]$. Then reducing $\mathcal{O}_X[T]$ by $T\mathcal{O}_X[T]$, we have $(L\otimes_{\mathcal{O}_C}L'^{-1})\otimes_{\mathcal{O}_C}\mathcal{O}_X\cong\mathcal{O}_X$. Let D be a divisor on C such that $L\otimes L'^{-1}=\mathcal{O}_C(D)$. Then there exists a rational function h of $k(X)$ such that $\pi^{-1}(D)=(h)$. Let $\bar{\pi}$ be the canonical projection $\boldsymbol{P}(E)\to C$. Then, looking upon h as an element of $k(\boldsymbol{P}(E))$, we have $\bar{\pi}^{-1}(D)+ms=(h)$, where $m\in\boldsymbol{Z}$ and s is the section of $\boldsymbol{P}(E)$ coresponding to L. If $m\neq 0$, $(\bar{\pi}^{-1}(D)+ms)\cdot l=m$ for a general fibre l of $\bar{\pi}$. This is absurd unless $m=0$. Then restricting $\bar{\pi}^{-1}(D)=(h)$ on the section s, we know that $D\sim 0$ on C. Therefore $L\cong L'$. Shrinking the affine open covering to a finer one, we may assume that $a_{ji}=a'_{ji}$ for any $i, j\in I$.

(3) We have $R_i[x_i, T]=R_i[y_i, V]$ for any $i\in I$. Therefore we can write: $y_i=f_{i0}(x_i)+f_{i1}(x_i)T+\cdots+f_{in}(x_i)T^n$ with $f_{i0}(x_i),\cdots,f_{in}(x_i)\in R_i[x_i]$. We shall prove that $n=0$. Otherwise, y_i is a generator of the two-dimensional polynomial ring $K[x_i, T]$, K being the function field $k(C)$ of C. Then, Lemma 3.3 implies that $f_{in}(x_i)\in K$. Hence $f_{in}(x_i)\in R_i[x_i]\cap K=R_i$. On the other hand, it is easy to see that $f_{jn}(x_j)T^n$ is the term of the highest degree in T of y_j and that $f_{jn}(x_j)=a_{ji}f_{in}(x_i)$ for any $i, j\in I$. Let $\alpha_i=f_{in}(x_i)$. Then $\{\alpha_i\}_{i\in I}$ defines a non-zero section of $H^0(C, L^{-1})$. This is absurd since L is an ample divisor on C. Therefore we know that $n=0$. Then $y_i\in R_i[x_i]$ for any $i\in I$. It is easy to see that $B\subseteq A$. Changing the roles of x_i and y_i in the above argument, we have $A\subseteq B$. Therefore $A=B$, and A is thus strongly 1-invarinat. q.e.d.

We shall next give a brief indication of constructing more complicated affine surfaces from the affine surface given in Lemma 3.1. We use the same notations as in Lemma 3.1. Let l be a general fibre of the projection $\tilde{\pi}$: $\boldsymbol{P}(E)\to C$, and let $P_1, \cdots, P_m$ be points on l other than the point of intersection $P_0=l\cdot s$. Blow up points $P_1, \cdots, P_m$ and let $E_{P_1}, \cdots, E_{P_m}$ be the exceptional curves. Let l' be the proper transform of l. Then it is not difficult to show that $ns+l'$ is an ample divisor on the surface $Dil_{P_1,\cdots,P_m}(\boldsymbol{P}(E))$, for sufficiently large n. Therefore $X'=Dil_{P_1,\cdots,P_m}(\boldsymbol{P}(E))-(s\cup l')$ is an affine surface which is isomorphic to X with m affine lines inserted in place of one fibre deleted. More complicatedly, we can blow up as many infinitely near points at $P_1, \cdots, P_m$ successively as we like. Let ρ: $S\to P(E)$ be the surface obtained by these blowings up. Then a divisor on S whose components are irreducible components of $\rho^{-1}(s\cup l)$ with appropriate multiplicities is an ample divisor if we require the following conditions satisfied:

(1) D contains $\rho^{-1}(s)$ with sufficiently high multiplicity.

(2) D is connected and $S-D$ contains no complete curves.

Then $S-D$ is affine. The operation of this kind can be made on a finite number of fibres of $\tilde{\pi}$. A question which arises naturally is to ask whether these affine surfaces are strongly invarinat or not. In simpler cases, we can prove that they are strongly 1-invariant.

An affine version of our Theorem 3.4 was given in [3]. To state the results, we need

DEFINITION 3.5. *Let R be a ring and let A be an R-algebra. A is said a local polynomial ring over R if for any prime ideal $\mathfrak{p}$ of R, $A_{\mathfrak{p}}$ is a polynomial ring over $R_{\mathfrak{p}}$.*

Then we can prove the following two results, for whose proofs the reader can refer to [3].

Theorem 3.6. *Let R be a reduced ring and let A be a finitely generated local polynomial ring over R of relative dimension* 1. *Then there exists a projective module P of rank* 1 *such that A is R-isomorphic to the symmetric R-algebra $S^{\cdot}(P)$ generated by P.*

Theorem 3.7. *Let R be a normal domain and let P be a projective R-module of rank* 1. *Then the symmetric R-algebra $S^{\cdot}(P)$ generated by P is R-invariant, but it is not strongly R-invariant.*

In the proof of the last theorem, we use the fact that if R is a normal domain, then a polynomial ring over R of dimension 1 is R-invariant (cf. [2]).

Appendix

In this appendix we shall determine the structure of an integral domain which has a locally finite iterative higher derivation (abbreviated as "lfihd").

Theorem. *Let A be an integral domain and let $D=\{D_i,\ i=0,1,\cdots\}$ be a lfihd on A and let A_0 be the ring of D-constant, i.e. the set of elements a such that $D_i(a)=0$ for $i\geq 1$. Then there exists an element u of A_0 such that we have $A[u^{-1}]=A_0[u^{-1}][x]$ where x is a variable element over A_0. Conversely assume that A is finitely generated over a subring A_0. Then the existence of an element u satisfying the above condition implies that A has a lfihd.*

Proof. We shall denote by A_i the set of elements defined by

$$A_i=\{a\in A \mid D_n(a)=0 \quad \text{for} \quad n>i\}.$$

A_0 is the ring of D-constants, A_i's are A_0-modules and we have $A=\cup A_i$. An integer n will be called a jump index if we have $A_{n-1}\subsetneqq A_n$. If the first jump index is 1 the proof is immediate. In fact let x be an element of A_1 not in A_0. Then $u=D(x)$ is a D-constant. Hence we can extend D uniquely to the quotient ring $A[u^{-1}]$ in which the element xu^{-1} satisties the condition in (1.4). Hence we have $A[u^{-1}]=A_0[u^{-1}][x]$. Since the first jump index is 1 if the characteristic of A is zero and D is not trivial, we shall hereafter be mainly interested in the case of positive characteristic p. Hence in the following we shall assume that the characteristic of A is a positive prime p and the first jump index of D is larger than 1. We shall prove first the followings:

(1) The first jump index n is a power of p, say, $n=p^s$.

(2) The m-th jump index is mp^s ($m=1,2,\cdots$).

(3) Let a be an arbitrary element of $A_n\backslash A_{n-1}$. Then $\mathrm{Supp}(a)$ consists of powers of p, where we mean by $\mathrm{Supp}(a)$ the set of integers k such that $D_k(a)\neq 0$. Moreover $D_k(a)$'s are D-constant for any $k\in\mathrm{Supp}(a)$.

Let n be the first jump index and let

$$n=n_0+n_1p+n_2p^2+\cdots+n_sp^s, \quad (0\leq n_i<p,\ n_s\neq 0)$$

be its p-adic expansion. Since D is iterative we can see easily that we have

$$D_n=(1/n!!)(D_1)^{n_0}(D_p)^{n_1}\cdots(D_{p^s})^{n_s}$$

where $n!!=n_0!n_1!\cdots n_s!$. Now assume n is not a power of p. Then we have either

(i) $n_0\geq 1$

or

(ii) $n_0=0$ and $n_1+\cdots+n_s\geq 2$.

In case (i) we have $n\not\equiv 0 \pmod{p}$. Then $c=D_{n-1}(a)$ belongs to A_1 but not to A_0 since $D_1(c)=D_1D_{n-1}(a)=nD_n(a)\neq 0$ where $a\in A_n\backslash A_{n-1}$. This is a contradiction to the assumption that the first jump index is >1. In case (ii) we shall set

$c=D_{p^s}(a)$. Then we can see immediately that any integer larger than $n-p^s$ $(<n-1)$ cannot be a support of a. Hence c belongs to $A_{n-1}=A_0$. On the other hand we see that $D_{n-p^s}D_{p^s}(a)=n_sD_n(a)\neq 0$ and $n-p^s\geq 1$. This is a contradiction. Thus we have proven that n is a power of p.

Next we shall prove (2). It is clear that $mp^s(m=1, 2, \cdots)$ are jump indices because if x is in A_{p^s} and not in A_0, then x^m is contained in A_{mp^s} but not in the preceding A's. Hence it remains to show that mp^s exhaust all jump indices. Let q be the least jump index which is not a multiple of p^s and assume that we have

$$mp^s<q<(m+1)p^s .$$

Let us set $q_0=q-mp^s<p^s$. Let x be an element of A_q not in A_{q-1} and let us set $c=D_{mp^s}(x)$. Then we have $D_{q_0}(c)=D_{q_0}D_{mp^s}(x)=D_q(x)\neq 0$. Since $p^s>q_0>0$, c is not a D-constant. On the other hand the supoprt of c consists of integers less than p^s because x is in A_q. This is a contradiction.

The proof of (3) will be carried out by a similar device. In fact if x is an element of A_{p^s} which is not a D-constant and let m be a support of x, i.e., $D_m(x)\neq 0$. If we set

$$m = m_0+m_1p+\cdots+m_tp^t \qquad (m_t\neq 0,\ o\leq m_i<p)$$

and if we assume either $m_t\geq 2$ or some of $m_i(i<t)$ is not zero then we shall set $c=D_{p^t}(x)$. Then we have $D_{m-p^t}(c)\neq 0$, and this will lead us to a contradiction since $m\leq p^s$ and $A_0=\cdots=A_{p^s-1}$. The rest of assertion in (3) is also immediate.

After these preparation we shall go to the proof of the Theorem. Let, as before, $n=p^s$ be the first jump index for a lfihd D and let x be an elements of A_n not in A_0. Let u be the product of non-zero $D_{p^i}(x)$'s $(0<i\leq s)$. Since u is a D-constant we can extend D to the quotient ring $A[u^{-1}]$ in a unique way. Let a be an arbitrary element of A. Then there exists an integer m such that a belongs to A_{mp^s}. Let us set

$$a_1 = a-D_{mp^s}(a)D_{p^s}(x)^{-m}x^m .$$

Then we have $D_{mp^s}(a_1)=0$. Hence a_1 is contained in $A_{(m-1)p^s}$. We can continue this process until we get a polynomial expression in x with coefficients in $A_0[u^{-1}]$. It is easy to see that x is a variable over A_0.

Conversely assume that there exist an element u in A such that we have $A[u^{-1}]=A_0[u^{-1}][x]$ where A_0 is a subring of A containing u and x is a variable element over A_0 contained in A and A is finitely generated over A_0. Then we can define a lfihd D on $A[u^{-1}]$ in a standard way. Since A is finitely generated over A_0, there is an index f such that u^fD transforms A into itself (see the proof of (2.9)). This proves the Theorem completely.

Corollary. *Let A be an integral domain over a field k of transcendence degree* 1. *Then A has a lfihd if and only if we have $A=k_0[x]$ where k_0 is the algebraic closure of k in A.*

Proof. Let u be an element of A_0 such that $A[u^{-1}]=A_0[u^{-1}][x]$. Since A is of transcendence degree 1 over k and x is variable over A_0, A_0 must be algebraic over k, i.e., A_0 is contained in k_0. Hence u must be a unit of A. This implies that A is contained in $k_0[x]$. The converse inclusion is trivial and we have $A=k_0[x]$.

OSAKA UNIVERSITY

Bibliography

[1] S. Abhyankar, P. Eakin and W. Heinzer: *On the uniqueness of the ring of coefficients in a polynomial ring*, J. Algebra, **23** (1972) 310–342.

[2] T. Asanuma: *On the strongly invariant coefficient rings*, Osaka J. Math. **11** (1974), 587–593.

[3] P. Eakin and W. Heinzer: *A cancellation problem for rings*, Springer Lecture Notes 311, (1972), 61–77.

[4] D. Gieseker: *P-ample bundles and their chern classes*, Nagoya Math. J. **43** (1971), 91–116.

[5] A. Grothendieck et J. Dieudonne: *Eléments de Géomêtrie Algébrique*, Inst. Hautes Études Sci. Publ. Math. **24** (1965).

[6] M. Hochster: *Non-uniqueness of the ring of coefficients in a polynomial ring*, Proc. Amer. Math. Soc. **34** (1972), 81–82.

[7] M. Miyanishi: *Some remarks on the actions of the additive group schemes*, J. Kyoto Univ. **10** (1970), 189–205.

[8] M. Miyanishi: *Some remarks on polynomial rings*, Osaka J. Math. **10** (1973), 617–624.

[9] M. Nagata: *A theorem on valuation rings and its applications*, Nagoya Math. J. **29** (1967), 85–91.

[10] M. Nagata: On Automorphism Group of $k[k,y]$, Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 5, Kinokuniya Book Store Co. Ltd. Tokyo, 1972.