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REGULARITY THEOREMS OF BOUNDARY VALUE PROBLEMS FOR FIRST ORDER SYSTEMS

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1. Introduction

In this article, we consider the regularity theorems of weak solutions to boundary value problems for first order systems of partial differential equations which satisfy some L^2 *a priori* inequality. Let L be a first order system of partial differential operators

$$L\equiv L(x, D)=\sum_{j=1}^{n} A_j(x)D_j+A_0(x)\,,\quad D_j=-\sqrt{-1}\frac{\partial}{\partial x_j}$$

with smooth $p\times p$ matrix coefficients, which are defined in a domain $\Omega\subset R^n$ having the smooth, compact boundary Γ. We write the formal adjoint of L

$$L^*v\equiv L^*(x, D)v=\sum_{j=1}^{n} D_j(A_j^*(x)v)+A_0^*(x)v(x)\,,$$

where A_j^* is the conjugate transpose of $A_j(x)$, $j=0, \cdots, n$. Throughout this paper we assume that Γ is non-characteristic for L, i.e., for the exterior unit normal vector $\nu(x)$ on Γ, the matrix $\beta(x)=\sum_{j=1}^{n} A_j(x)\nu_j(x)$ is non-singular on Γ. We associate with L the following two function spaces

$$B=\{u(x)\in C^\infty(\bar{\Omega})\cap H_1(\Omega);\, M(x)u(x)=0 \text{ on } \Gamma\}\,,$$
$$B^*=\{v(x)\in C^\infty(\bar{\Omega})\cap H_1(\Omega);\, M^*\beta^*v=\beta^*v \text{ on } \Gamma\}\,.$$

Here the boundary operator $M(x)$ is a $p\times p$ idempotent matrix (i.e. $M^2(x)=M(x)$ on Γ), β^* is the conjugate transpose of $\beta(x)$ and $H_1(\Omega)$ is a Sobolev space defined in §2.

We shall call $u\in L^2(\Omega)$ a weak solution of inhomogeneous boundary value problem, $Lu=f$ in Ω, $Mu=g$ on Γ, if there exists $f\in L^2(\Omega)$ and $g\in L^2(\Gamma)\cap$ Range M such that

$$(u, L^*v)=(f, v)+i\langle\beta g, v\rangle,\ v\in B^*$$

where $(\ ,\ )$, $\langle\ ,\ \rangle$ are $L^2(\Omega)$, $L^2(\Gamma)$ inner products respectively. Under this situation, we suppose the following inequality (P. 1) holds

(P. 1) $$q\|v\|^2_{0,\Omega}\leq\|Lv\|^2_{0,\Omega}+C_q\|v\|^2_{-1,\Omega}\,,\qquad v\in B\,,$$

here $\|\cdot\|_{s,\Omega}$ is a suitable Sobolev norm over Ω defined in §2.

Theorem 1. *Let r be a given non-negative integer. Let u be a weak solution of $Lu=f$ and $Mu=g$ with $f\in H_r(\Omega)$, $g\in H_{r+1/2}(\Gamma)$. Furthermore we assume that* (P. 1) *is valid with sufficiently large positive q and some constant C_q (how large q must be depends on r, certain derivatives of the coefficients $A_j(x)$, $j=1, \cdots, n$, and $M(x)$ but not on u, f and g). Then u belongs to $H_r(\Omega)$.*

REMARKS. 1. If the stronger estimate

$$q\|w\|^2_{0,\Omega}\leqq\|Lw\|^2_{0,\Omega}+C_q\|w\|^2_{-1,\Omega}+\langle Mw\rangle^2_{\frac{1}{2},\Gamma}\,,\qquad w\in C^\infty(\bar{\Omega})\cap H_1(\Omega)$$

holds, then this estimate implies (P. 1), so Theorem 1 holds in this case. Here $\langle\cdot\rangle_{s,\Gamma}$ indicates the Sobolev norm over Γ defined in §2.

2. It is well known that if a matrix M is idempotent then the trace of M is equal to the rank of M. Thus our matrix $M(x)$ is of constant rank over Γ. But we do not need this fact.

We suppose the following another estimate (P. 2) holds

(P. 2) $$q\|v\|^2_{0,\Omega}\leqq\|Lv\|^2_{0,\Omega}+C_q\|v\|^2_{-1,\Omega}+\langle Mv\rangle^2_{0,\Gamma}\,,\qquad v\in C^\infty(\bar{\Omega})\cap H_1(\Omega)\,.$$

In this case, we obtain the following regularity theorem.

Theorem 2. *Let r be a positive integer. Let u be a weak solution of $Lu=f$ and $Mu=g$ with $f\in H_r(\Omega)$, $g\in H_r(\Gamma)$. Suppose that the inequality* (P. 2) *holds with a sufficiently large number q (compared with derivatives of order $\leqq r$ of the leading coefficients of L and M). Then the vector u belongs to $H_r(\Omega)$.*

D.S. Tartakoff [6] considered the regularity theorems under the same situation. He obtains the same theorems. However, in the second theorem case he assumes (*P.* 2) and the dual estimate of (P. 1). Using a mollifier method, he first obtains the regularity theorem of homogeneous boundary value problem (i.e. $g=0$) supposing the inequality (P. 1). In the inhomogeneous case, he applies the regularity and existence theorems of homogenous case and the technique of functional analysis. But our method is a more unified one.

To verify the theorems, we use the function space $H_{m,s}(\bar{R}^n_+)$ described in [3]. Since Γ is non-characteristic for L, the trace $u|_\Gamma$ on Γ of the weak solution u has a meaning in the distribution sence, and $Mu|_\Gamma$ coincides with the date g as a distribution on Γ. By the transformation from a part of $\bar{\Omega}$ near each boundary point of Ω to some neighbourhood ω of the origin in $\bar{R}^n_+$, the inequality (P. 1) implies the following inequality,

(P. 1)′ $$q\|w\|^2_0\leqq C\|Lw\|^2_0+C_q'\|w\|^2_{1,-2}+C'\langle Mw\rangle^2_{\frac{1}{2}}\,,\qquad w\in C^\infty_0(\bar{R}^n_+\cap\omega)\,.$$

Thus, Theorm 1 and 2 will be proved by the same method, using the mollifier technique. K. Kubota [5] suggests the use of the function space mentioned above and its trace theorem, which is essential in our proof.

As an application we mention that (P. 1) is valid with given sufficiently large q for positive symmetrizable systems in the sense of Friedrichs and Lax [1], [2]. Hörmander [4] and others consider "subelliptic" case and obtain hypoellipticity results. Our estimate (P. 1) is weaker than subelliptic estimate, so our result gives an extension of subelliptic case in a certain sense.

For the higher order single equations with normal boundary operators, we can also prove the analogous regularity theorems.

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2. Analytic preliminaries

a) *Function spaces and families of norms.*

We use some function spaces which are slight modifications of L. Hörmander's [3].

DEFINITION 2.1.

i) For real s and $\delta\in(0, 1]$, we define $H_{(s,\delta)}(R^n)$ as the completion of $C_0^\infty(R^n)$ under the norm

$$||u||^2_{(s,\delta)} = (2\pi)^{-n}\int|\hat{u}(\xi)|^2(1+|\xi|^2)^{s+1}(1+|\delta\xi|^2)^{-1}d\xi\ ,$$

where $\hat{u}(\xi)$ denotes the Fourier transform

$$\hat{u}(\xi) = \int e^{-i\langle x,\xi\rangle}u(x)dx\ .$$

When $\delta=1$, we write merely $H_{(s,1)}(R^n)=H_s(R^n)$ and $||u||_{(s,1)}=||u||_s$.

ii) For real s and $\delta\in(0, 1]$, by $H_{(s,\delta)}(\Omega)$ we mean the set of all $u\in\mathcal{D}'(\Omega)$ such that there exists $U\in H_{(s,\delta)}(R^n)$ with $U=u$ in $\mathcal{D}'(\Omega)$. The norm of $H_{(s,\delta)}(\Omega)$ is defined by

$$||u||_{(s,\delta)} = \inf ||U||_{(s,\delta)}\ ,$$

the infimum being taken over all such U. Similarly, $H_{(s,1)}(\Omega)=H_s(\Omega)$, $||u||_{(s,1)} =||u||_s$.

iii) For real m, s and $\delta\in(0, 1]$, we define $H_{m,(s,\delta)}(R^n)$ as the completion of $C_0^\infty(R^n)$ under the norm

$$||u||^2_{m,(s,\delta)} = (2\pi)^{-n}\int|\hat{u}(\xi)|^2(1+|\xi|^2)^m(1+|\xi'|^2)^{s+1}(1+|\delta\xi'|^2)^{-1}d\xi$$

where ξ' is the co-variable of $x'=(x_1, \cdots, x_{n-1})$.

iv) For real, m, s and $\delta\in(0,1]$, by $H_{m,(s,\delta)}(\bar{R}^n_+)$ we mean the set of all $u\in\mathcal{D}'(R^n_+)$ such that there exists a distribution $U\in H_{m,(s,\delta)}(R^n)$ with $U=u$ in R^n_+. Here $R^n_+=\{x;\ x_n>0\}$, $\bar{R}^n_+=\{x;\ x_n\geq 0\}$. The norm of u is defined by

$$||u||_{m,(s,\delta)}=\inf||U||_{m,(s,\delta)}\,,$$

the infimum being taken over all such U. For simplicity, we write $H_{m,(s,1)}(\bar{R}^n_+)$ $H_{m,s}(\bar{R}^n_+)$ and $||u||_{m,(s,1)}=||u||_{m,s}$.

In the following, we state several properties of these norms and spaces, whose the greater part are described in chapter II of [3].

To prove our regularity theorms, we must consider that the norms of $H_{(s,\delta)}(R^n)$ and $H_{m,(s,\delta)}(\bar{R}^n_+)$ are invariant under the C^∞-local transformations. Let Ω_X, Ω_Y be bounded open sets in R^n, $Y=(y_1,\cdots,y_n)$ be a diffeomorphism from Ω_X to Ω_Y and X be the inverse transformation.

Proposition 2.1.

i) *Let K be a given compact subset of Ω_X. For real s, there exists C_s such that if supp $u\subset K$, $u(x)$ belongs to $H_s(R^n)$, then $\tilde{u}(y)=u(X(y))$ also belongs to $H_s(R^n)$ and*

$$||\tilde{u}(y)||_{(s,\delta)}\leqq C_s||u(x)||_{(s,\delta)}\qquad\text{for}\quad 0<\delta\leq 1\,.$$

ii) *If $\dfrac{\partial x_i}{\partial y_j}=\dfrac{\partial y_i}{\partial x_j}=\delta_{ij}$ for $i=n$ or $j=n$, and K is a compact subset of Ω_X, then for a non-negative integar m and a real number s, there exists $C_{m,s}$ such that if $u(x)\in H_{m,(s,\delta)}(\bar{R}^n_+)$ and supp $u\subset K$, then $\tilde{u}(y)=u(X(y))\in H_{m,(s,\delta)}(\bar{R}^n_+)$ and*

$$||\tilde{u}(y)||_{m,(s,\delta)}\leq C_{m,s}||u(x)||_{m,(s,\delta)}\,,\qquad \textit{for}\quad 0<\delta\leqq 1\,.$$

Here C_s and $C_{m,s}$ are independent of δ.

The proof of i) is denoted in D. S. Tartakoff [6]. By Proposition 2.7 stated below, a similar fact holds for $||u||_{m,(s,\delta)}$.

Proposition 2.2.

i) *$u(x)\in H_{s+1}(R^n)$ iff $u(x)\in H_s(R^n)$ and $\sup_\delta ||u||_{(s,\delta)}<\infty$.*

ii) *For a non-negative integer s, $u(x)\in H_s(\Omega)$ iff $D^\alpha u\in L^2(\Omega)$ for $|\alpha|\leq s$.*

Proposition 2.3. *The subspace $C_0^\infty(\bar{R}^n_+)$ is dense set in $H_{m,(s,\delta)}(\bar{R}^n_+)$.*

Proposition 2.4. *In order that $H_{m_1,(s_1,\delta)}(\bar{R}^n_+)\subset H_{m_2,(s_2,\delta)}(\bar{R}^n_+)$, it is necessary and sufficient that $m_2\leqq m_1$ and $m_2+s_2\leqq m_1+s_1$.*

Proposition 2.5. *In order that $u\in H_{m,(s,\delta)}(\bar{R}^n_+)$ iff $u\in H_{m-1,(s+1,\delta)}(\bar{R}^n_+)$ and $D_nu\in H_{m-1,(s,\delta)}(\bar{R}^n_+)$. Moreover*

$$\begin{aligned}C_1||u||^2_{m,(s,\delta)}&\leqq||D_nu||^2_{m-1,(s,\delta)}+||u||^2_{m-1,(s+1,\delta)}\\&\leqq C_2||u||^2_{m,(s,\delta)}\,,\qquad u\in H_{m,(s,\delta)}(\bar{R}^n_+)\,.\end{aligned}$$

Proposition 2.6. *If m and s are non-negative integers, the space* $H_{m,s}(\bar{R}^n_+)$ *consists of all* $u \in L^2(R^n_+)$ *such that* $D^\alpha u \in L^2(R^n_+)$ *when* $|\alpha| \leqq s+m$ *and* $\alpha_n \leqq m$. *For the norm we have the etimate*

$$C_1 \|u\|^2_{m,s} \leqq \sum_{\substack{|\alpha| \leqq m+s \\ \alpha_n \leqq m}} \|D^\alpha u\|^2_0 \leqq \|u\|^2_{m,s}, \qquad u \in H_{m,s}(\bar{R}^n_+)$$

where C_1 *is a positive constant, depending on s and m but not on u.*

Proposition 2.7. *If m is a non-negative integer, the space* $H_{m,(s,\delta)}(\bar{R}^n_+)$ *consists of all* $u \in \mathcal{D}'(R^n_+)$ *such that* $D^j_n \hat{u}_n$ *is a measurable function when* $j \leqq m$ *and*

$$\sum_{j=0}^{m} (2\pi)^{1-n} \iint_0^\infty |D^j_n \hat{u}_n(\xi', x_n)|^2 (1+|\xi'|^2)^{s+m-j+1} (1+|\delta\xi'|^2)^{-1} dx_n d\xi' < \infty .$$

The left-hand side is a norm equivalent to $\|u\|^2_{m,(s,\delta)}$, *where* $\hat{u}_n(\xi', x_n)$ *denotes a partial Fourier transform*

$$\hat{u}_n(\xi', x_n) = \int u(x', x_n) e^{-i<x', \xi'>} dx' .$$

Proposition 2.8. *Let m be a non-negative integer, then* $u \in H_{m,s+1}(\bar{R}^n_+)$ *iff* $u(x) \in H_{m,s}(\bar{R}^n_+)$ *and* $\sup_\delta \|u\|_{m,(s,\delta)} < \infty$.

Proposition 2.9. *If m and j are integers,* $0 \leqq j < m$, *the mapping*

$$C^\infty_0(\bar{R}^n_+) \ni u \to D^j_n u(\cdot, x_n)$$

can for fixed $x_n \geqq 0$ *be extended in one and only one way to a continuous mapping of* $H_{m,(s,\delta)}(\bar{R}^n_+)$ *into* $H_{(s+m-j-\frac{1}{2},\delta)}(R^{n-1})$ *with the following inequality*

$$\|D^j_n u(\cdot, x_n)\|_{(s+m-j-\frac{1}{2},\delta)} \leqq \|u\|_{m,(s,\delta)} .$$

Proposition 2.10. *For an arbitrary non-negative integer m and* $f_k \in H_{(s+m-k-\frac{1}{2},\delta)}(R^{n-1})$, $k=0, \cdots, 1$, *there exists a function* $u \in H_{m,(s,\delta)}(\bar{R}^n_+)$ *with* $D^k_n u(\cdot, 0) = f_k$ $k=0, \cdots, m-1$ *and*

$$\|u\|_{m,(s,\delta)} \leqq C \sum_0^{m-1} \langle f_k \rangle_{(s+m-k-\frac{1}{2},\delta)} .$$

Where if the f_k *belongs to* $\mathcal{S}(R^{n-1})$ $(k=0, \cdots, m-1)$ *the choice of u is independent of m, s and C is not dependent on* δ *and* f_k $(k=0, \cdots, m-1)$.

REMARK. For the proof in L. Hörmander [3] (Theorem 2.5.7), if $f_k \in C^\infty_0(\tilde{\Omega})$ $k=0, \cdots, m-1$, then we can choose u such that $C^\infty_0(\Omega)$, where $\tilde{\Omega}$ is a bounded open set in R^{n-1} and Ω is an open set with $\Omega \cap \{x_n = 0\} = \tilde{\Omega}$. For by the following proposition we can cut the function u outside a neighbourhood of $\tilde{\Omega}$ in R^n.

Proposition 2.11. *If* $u \in H_{m,(s,\delta)}(\bar{R}^n_+)$ *and* $a \in \mathcal{S}(R^n)$, *it follows that* $au \in H_{m,(s,\delta)}(\bar{R}^n_+)$ *and*

$$||au||_{m,(s,\delta)} \leqq C||u||_{m,(s,\delta)} .$$

When Γ is a C^∞ compact, $n-1$ dimensional surface without boundary in R^n, we may also define $H_{(s,\delta)}(\Gamma)$. Let $\{\Omega_i\}_{i=1,\cdots,N}$ be an open covering of the neighbourhood of Γ, $\{\psi_i^{-1}\}_{i=1,\cdots,N}$ be a diffeomorphism defined on Ω_i such that $\Omega_i \cap \Gamma$ is mapped on an open set of R^{n-1}. Let $\{\lambda_i\}_{i=1,\cdots,N}$ be the partition of unity subordinated to $\{\Omega_i\}_{i=1,\cdots,N}$. We mean by $u \in H_{(s,\delta)}(\Gamma)$ that $(\lambda_i u)\circ\psi_i \in H_{(s,\delta)}(R^{n-1})$ $i=1,\cdots,N$ and denote $\sum_1^N ||(\lambda_i u)\circ\psi_i||^2_{(s,\delta)}$ by $\langle u\rangle^2_{(s,\delta)}$. We remark that different choices of $\{\Omega_i\}$ and $\{\lambda_i\}$ will yield equivalent norms.

b) *Mollifier.*

DEFINITION 2.2.

i) Let $\chi \in C_0^\infty(R^n)$ and assume that for some integer $k \geqq 0$

$$\hat{\chi}(\xi) = 0(|\xi|^k) \qquad \text{as } \xi \to 0 ,$$

but that $\hat{\chi}(t\xi)=0$ for all real t implies $\xi=0$ if $\xi \in R^n$, then the family $J=\{J_\varepsilon\}$ of operators, defined for $0<\varepsilon\leqslant 1$ by

$$(J_\varepsilon u)(x) = (\chi_\varepsilon * u)(x) , \quad \chi_\varepsilon(x) = \varepsilon^{-n}\chi(x/\varepsilon) , \quad u \in \mathcal{D}'(R^n)$$

is called a full mollifier of type k with kernel $\chi(x)$.

ii) Let $\chi'(x') \in C_0^\infty(R^{n-1})$ satisfy the corresponding conditions required of $\chi(x)$ in i). If we define for $0<\varepsilon\leqslant 1$ and $\phi \in C^\infty(R^n)$

$$(\chi_\varepsilon' dx')(\phi) = \int \chi_\varepsilon'(x')\phi(x', 0)dx'$$

then evidently $\chi_\varepsilon' dx'$ belongs to $\mathcal{E}'(R^n)$ and a family $J'=\{J_\varepsilon'\}$ with $J_\varepsilon' u = \chi_\varepsilon' dx' * u$ is called a tangential mollifier of type k with kernel $\chi'(x')$.

Proposition 2.12.

i) *Let J be a full mollifier of type k with kernel $\chi(x)$ and let s, p be real numbers with $s<k$. Then there exist positive constants C_1 and C_2 independent of δ and u such that*

$$C_1||u||^2_{(s+p-1,\delta)} \leqq \int_0^1 ||J_\varepsilon u||^2_p \varepsilon^{-2s}(1+\delta^2/\varepsilon^2)^{-1} d\varepsilon/\varepsilon + ||u||^2_{s+p-1}$$
$$\leqq C_2||u||^2_{(s+p-1,\delta)} , \qquad u \in H_{s+p-1}(R^n) .$$

ii) *Let J' be a tangential mollifier of type k with kernel $\chi'(x')$ and let m be a non-negative interger and s, p be real numbers with $k>s$. Then there exist positive constants C_1 and C_2 independent of δ and u such that*

$$C_1\|u\|^2_{m,(s+p-1,\delta)}\leqq\int_0^1\|J_\varepsilon' u\|^2_{m,p}\varepsilon^{-2s}(1+\delta^2/\varepsilon^2)^{-1}d\varepsilon/\varepsilon+\|u\|^2_{m,s+p-1}$$
$$\leqq C_2\|u\|^2_{m,(s+p-1,\delta)}\,,\qquad u\in H_{m,s+p-1}(\bar{R}^n_+)\,.$$

Proof. i) can be proved by applying Theorem 2.4.1 [3] ($p=0$) to the inverse Fourier transform of $(1+|\xi|^2)^{p/2}\hat{u}(\xi)$. In the case ii) applying above i) and Proposition 2.7, let us replace u by the inverse partial Fourier transform of $(1+|\xi'|^2)^{(m+j+p-1)/2}D_n^j\hat{u}_n(\xi', x_n)$ $(0\leqq j\leqq m)$.

Proposition 2.13.

i) *Let $a\in\mathcal{S}(R^n)$ and J be a full mollifier of type k with kernel $\chi(x)$ and let s, p be arbitrary real numbers with $k>s$. Then there exists a positive constant C independent of δ and u such that*

$$\int_0^1\|[a, J_\varepsilon]u\|^2_p\varepsilon^{-2s}(1+\delta^2/\varepsilon^2)^{-1}d\varepsilon/\varepsilon\leqq C\|u\|^2_{(s+p-2,\delta)}\,,$$
$$\textit{for all } u\in H_{s+p-2}(\bar{R}^n_+)\,.$$

ii) *Let us replace J in* i) *by J' and let m be a non-negative integer. Then there exists a positive constant C independent of δ and u such that*

$$\int_0^1\|[a, J_\varepsilon']u\|^2_{m,p}\varepsilon^{-2s}(1+\delta^2/\varepsilon^2)^{-1}d\varepsilon/\varepsilon\leqq C\|u\|^2_{m,(s+p-2,\delta)}\,,$$
$$\textit{for all } u\in H_{m,s+p-2}(\bar{R}^n_+)\,.$$

Here the commutator $[a, J_\varepsilon]$ means $aJ_\varepsilon-J_\varepsilon a$.

Proof. i) The proof for $p=0$ is given in L. Hörmander [3] (Theorem 2.4.2). For arbitrary p, let Λ^p be a pseudo-differential opeator such that

$$\widehat{\Lambda^p u}(\xi)=(1+|\xi|^2)^{p/2}\hat{u}(\xi)\,.$$

We remark that the operator Λ^p commutes with J_ε and

$$\|u\|_{(p,\delta)}=\|\Lambda^p u\|_{(0,\delta)}\,,\qquad u\in H_{(p,\delta)}(R^n)\,.$$

since $\Lambda^p[a, J_\varepsilon]=[\Lambda^p, a]J_\varepsilon+[a, J_\varepsilon]\Lambda^p+J_\varepsilon[a, \Lambda^p]$,

$$\int_0^1\|[a, J_\varepsilon]u\|^2_p\varepsilon^{-2s}(1+\delta^2/\varepsilon^2)^{-1}d\varepsilon/\varepsilon$$
$$=\int_0^1\|\Lambda^p[a, J_\varepsilon]u\|^2_0\varepsilon^{-2s}(1+\delta^2/\varepsilon^2)^{-1}d\varepsilon/\varepsilon$$
$$\leqq\int_0^1\|[\Lambda^p, a]J_\varepsilon u\|^2_0\varepsilon^{-2s}(1+\delta^2/\varepsilon^2)^{-1}d\varepsilon/\varepsilon\,,$$
$$+\int_0^1\|[a, J_\varepsilon]\Lambda^p u\|^2_0\varepsilon^{-2s}(1+\delta^2/\varepsilon^2)^{-1}d\varepsilon/\varepsilon$$
$$+\int_0^1\|J_\varepsilon[a, \Lambda^p]u\|^2_0\varepsilon^{-2s}(1+\delta^2/\varepsilon^2)^{-1}d\varepsilon/\varepsilon.$$

Here the commutator $[a, \Lambda^p]$ is of order $p-1$. Hence in view of Proposition 2.12 and Theorem 2.4.2 of L. Hörmander [3], we obtain following inequalities,

$$\int_0^1 \|[\Lambda^p, a]J_\varepsilon u\|_0^2 \varepsilon^{-2s}(1+\delta^2/\varepsilon^2)^{-1}d\varepsilon/\varepsilon$$

$$\leq C\int_0^1 \|J_\varepsilon u\|_{p-1}^2 \varepsilon^{-2s}(1+\delta^2/\varepsilon^2)^{-1}d\varepsilon/\varepsilon \leqq C'\|u\|_{(s+p-2,\delta)}^2$$

$$\int_0^1 \|[a, J_\varepsilon]\Lambda^p u\|_0^2 \varepsilon^{-2s}(1+\delta^2/\varepsilon^2)^{-1}d\varepsilon/\varepsilon \leqq C\|\Lambda^p u\|_{(s-2,\delta)}^2 \leqq C'\|u\|_{(s+p-2,\delta)}^2,$$

$$\int_0^1 \|J_\varepsilon[a, \Lambda^p]u\|_0^2 \varepsilon^{-2s}(1+\delta^2/\varepsilon^2)^{-1}d\varepsilon/\varepsilon \leqq C\|[a, \Lambda^p]u\|_{(s-1,\delta)}^2 \leqq C'\|u\|_{(p+s-2,\delta)}^2.$$

The proof is complete.

ii) By Proposition 2.7, $\|[a, J_\varepsilon']u\|_{m,p}^2$ is equivalent to

$$\sum_{j=0}^m (2\pi)^{1-n} \iint_0^\infty |D_n^j([a, J_\varepsilon']u)_n^\wedge(\xi', x_n)|^2(1+|\xi'|^2)^{m+p-j}d\xi'\,dx_n$$

and

$$D_n^j([a, J_\varepsilon']u) = \sum_{k=0}^j \binom{j}{k}([D_n^k a, J_\varepsilon']D_n^{j-k}u).$$

For fixed $x_n \geqq 0$, we have only to estimate the following

$$\int_0^1 \|[D_n^k a, J_\varepsilon']D_n^{j-k}u(\cdot, x_n)\|_{m+p-j}^2 \varepsilon^{-2s}(1+\delta^2/\varepsilon^2)^{-1}d\varepsilon/\varepsilon$$

$$\leqq C\|D_n^{j-k}u(\cdot, x_n)\|_{(m+p+s-j-2,\delta)}^2$$

$$\leqq C'\|D_n^{j-k}u(\cdot, x_n)\|_{(m+p+s-(j-k)-2,\delta)}^2,$$

since once this is established the desired inequality is obtained after integration with respect to x_n and using Fubini's theorem.

3. The proof of theorems

In order to prove Theorem 1 and 2, we make use of a special open covering $\{U_j\}_{j=0,\dots,N}$ of Ω, such that

i) $U_0 \subset \Omega$ and $U_j \cap \Gamma \neq \phi$ $(j=1, \cdots, N)$

ii) The matrix $\beta(x)$ is non-singular in $\bigcup_{j=1}^N U_j$,

iii) There exists a diffeomorphism α_j^{-1} from U_j $(j\neq 0)$ to some neighbourhood of the origin in R^n, such that $\alpha_j^{-1}|_{\Gamma\cap U_j}$ is also the diffeomorphism to some neighbourhood of the origin in R^{n-1}. Furthermore for all $x \in U_j$, y_n, which is n-th component of $\alpha_j^{-1}(x)$, is equal to $\pm$ distance (x, Γ) where if $x \in \Omega$ then $y_n = \mathrm{dist}(x, \Gamma)$ and if $x \in C\Omega$ then $y_n = -\mathrm{dist}(x, \Gamma)$.

Lemma 3.1. *Let the vector valued functions $u(x)$ and $Lu(x)$ belong to $L^2(\Omega)$.*

Then for an arbitrary $\varphi \in C_0^\infty(U_j)$ $(j \neq 0)$, $(\varphi u)\circ\alpha_j$ *belongs to* $H_{1,-1}(\bar{R}_+^n)$.

Proof. From the assumption $L(\varphi u)=f\in L^2(\Omega)$, and L is represented by the transformation α_j^{-1} in the following form

$$\beta(y)D_n+\sum_{k=1}^{n-1}E_k(y)D_k+E_0(y),$$

where $\beta(y)$ is non-singular in a neighbourhood of $\{y_n=0\}$ which is denoted by $\operatorname{Im}\alpha_j^{-1}$. Hence

$$\beta(y)D_n((\varphi u)\circ\alpha_j)=f\circ\alpha_j-\sum_{k=1}^{n-1}E_k(y)D_k((\varphi u)\circ\alpha_j)-E_0(y)((\varphi u)\circ\alpha_j).$$

From Proposition 2.7 and 2.11, the right hand side of the above equality belongs to $H_{0,-1}(\bar{R}_+^n)$. Thus we obtain the lemma by Proposition 2.5 with $\delta=1$.

Lemma 3.2. *Let the vector valued functions* $u(x)$ *and* $Lu(x)$ *be in* $L^2(\Omega)$ *and supp* $u\subset\bar{\Omega}\cap U_j$ $(j\neq 0)$. *Then*

$$(u, L^*v)=(Lu, v)+i\langle\beta u, v\rangle, \qquad v\in C^\infty(\bar{\Omega}).$$

By Lemma 3.1 *and Proposition* 2.9, *we may consider that* $u(x)$ *belongs to* $H_{-1/2}(\Gamma)$.

Proof. From Lemma 3.1, $u\circ\alpha_j$ is an element of $H_{1,-1}(\bar{R}_+^n)$. Hence, there exists a sequence $\{v_k\}\subset C_0^\infty(\bar{R}_+^n)$ such that $\{v_k\}$ converges to $u\circ\alpha_j$ in the topology of $H_{1,-1}(\bar{R}_+^n)$. Let a function φ be an element of $C_0^\infty(R^n)$, such that supp $\varphi\subset U_j$ and is equal to 1 in $\operatorname{supp}(u\circ\alpha_j)$. By Proposition 2.11,

$$\begin{aligned}\|\varphi v_k-u\circ\alpha_j\|_{1,-1}&=\|\varphi(v_k-u\circ\alpha_j)\|_{1,-1}\\&\leqq C\|v_k-u\circ\alpha_j\|_{1,-1}.\end{aligned}$$

Therefore the sequence $\{\varphi v_k\}$ also converges to $u\circ\alpha_j$ in the topology of $H_{1,-1}(\bar{R}_+^n)$. Clearly, it follows from Proposition 2.4 that $\{\varphi v_k\}$ converges to $u\circ\alpha_j$ in the topology of $L^2(R_+^n)$. Now, it we set $u_k=(\varphi v_k)\circ\alpha_j^{-1}$, then u_k is an element of $C^\infty(\bar{\Omega})$ and $\{u_k\}$ converges to u in the topology of $L^2(\Omega)$. On the other hand, by Proposition 2.9 and by the fact that $u_k\circ\alpha_j$ converges to $u\circ\alpha_j$ in the topology of $H_{1,-1}(\bar{R}_+^n)$, we conclude that $u_k\circ\alpha_j\to u\circ\alpha_j$ in the topology of $H_{-\frac{1}{2}}(R^{n-1})$. This shows that $u_k\to u$ in the topology of $H_{-\frac{1}{2}}(\Gamma)$ by deffinition. Now

$$(L(u_k-u), v)_\Omega=((L(u_k-u))\circ\alpha_j, J(v\circ\alpha_j))_{\bar{R}_+^n},$$

where J is the Jacobian of transformation α_j^{-1}. Hence

$$\begin{aligned}|(L(u_k-u), v)_\Omega|&\leqq\|(L(u_k-u))\circ\alpha_j\|_{0,-1}\|J(v\circ\alpha_j)\|_{0,1}\\&\leqq C\|u_k\circ\alpha_j-u\circ\alpha_j\|_{1,-1}.\end{aligned}$$

This shows that $(Lu_k\, v)_\Omega\to(Lu, v)_\Omega$. We obtain, by using Green's formula, the equality

$$(u_k, L^*v) = (Lu_k, v) + i\langle \beta u_k, v\rangle\,, \qquad v \in C^\infty(\bar{\Omega})\,.$$

When $k \to \infty$ in the above equality we obtain

$$(u, L^*v) = (Lu, v) + i\langle \beta u, v\rangle\,.$$

Lemma 3.3. *Let $\{\psi_j\}_{j=0,\cdots,N}$ be a partition of unity subordinated to an open covering $\{U_j\}_{j=0,\cdots,N}$, and let the vector-valued function $u(x)$ be a weak solution of the boundary value problem $Lu=f$, $Mu=g$. Then*

$$M(\psi_j u) = \psi_j g \qquad \text{in } \mathcal{D}'(\Gamma)\ (j \neq 0)\,.$$

Proof. If a vector valued function $v \in B^*$, then by commutativity of ψ_j and M, β, we obtain that $\psi_j v$ is also in B^*. Since $u(x)$ is a weak solution, we obtain that

$$(u, L^*((\psi_j v)) = (f, \psi_j v) + i\langle \beta g, \psi_j v\rangle\,, \qquad v \in B^*\,.$$

Now, let a function $\tilde{\psi}_j$ be in $C_0^\infty(R^n)$ such that $\operatorname{supp}\tilde{\psi}_j \subset U_j$ and $\tilde{\psi}=1$ in $\operatorname{supp}\psi_j$. Then

$$(\tilde{\psi}_j u, L^*(\psi_j v)) = (f, \psi_j v) + i\langle \beta g, \psi_j v\rangle\,, \qquad v \in B^*\,.$$

Since u is a weak solution, and $C_0^\infty(\Omega) \subset B^*$, we have $Lu=f$ in $\mathcal{D}'(\Omega)$. Hence Lu is equal to f in $L^2(\Omega)$. Therefore, $\tilde{\psi}_j u$ satisfies the conditions required in Lemma 3.2. So we apply Lemma 3.2 to $\tilde{\psi}_j u$ and obtain

$$\begin{aligned}(\tilde{\psi}_j u, L^*(\psi_j v)) &= (L(\tilde{\psi}_j u), \psi_j v) + i\langle \beta\tilde{\psi}_j u, \psi_j v\rangle \\ &= (f, \psi_j v) + i\langle \beta u, \psi_j v\rangle\,.\end{aligned}$$

A vector-valued function u is approximated by the elements of $C^\infty(\bar{\Omega})$. Thus

$$\langle u, \beta^*\psi_j v\rangle = \langle Mu, \beta^*\psi_j v\rangle\,.$$

Therefore, we obtain

$$\langle M(\psi_j u), \beta^* v\rangle = \langle \psi_j g, \beta^* v\rangle\,, \qquad v \in B^*\,.$$

For an arbitrary $w \in C^\infty(\bar{\Omega})$, we have a decomposition

$$\beta^* w = M^*\beta^* w + (I - M^*)\beta^* w\,.$$

Since M is a idempotent matrix,

$$\langle M\psi_j u, (I-M^*)\beta^* w\rangle = \langle \psi_j g, (I-M^*)\beta^* w\rangle = 0\,.$$

This and $\beta^{*-1}M^*\beta^* w \in B^*$ show that

$$\langle M(\psi_j u), \beta^* w\rangle = \langle \psi_j g, \beta^* w\rangle\,, \qquad w \in C^\infty(\bar{\Omega})\,.$$

Since the matrix β is non-singular, and the restriction mapping: $C^\infty(\bar{\Omega})\to C^\infty(\Gamma)$ is surjective, we obtain the required lemma.

Lemma 3.4. *Let us assume the following estimate is valid*

$$C_1\|u\|_{0,0}^2\leqq C_2\|\tilde{L}u\|_{0,0}^2+C_3\|u\|_{-1,0}^2,$$
$$u\in C_0^\infty(\bar{R}_+^n\cap\omega) \text{ and } \tilde{M}u=0 \text{ in } \{x_n=0\},$$

where $\tilde{L}$ is a first order partial differential operator, $\tilde{M}$ is a smooth idempotent matrix and ω is an open set in R^n. Then we obtain the following inequality

$$\text{(P. 1)}'\qquad C_1\|w\|_{0,0}^2\leqq C_2\|\tilde{L}w\|_{0,0}^2+C_3'\|w\|_{1,-2}^2+C_4\langle\tilde{M}u\rangle_{\frac{1}{2}}^2,$$

where w is an arbitrary element of $C_0^\infty(\bar{R}_+^n\cap\omega)$ and the constant C_4 is independent of C_1 and C_3.

Proof. In Proposition 2.10, we take $\tilde{M}w$ in place of f_0, and 0 for f_k ($k=1,\cdots,m-1$). Then there exists $v\in C_0^\infty(\Omega)$ such that

$$\tilde{M}w|_{x_n=0}=v|_{x_n=0}$$

and

$$\|v\|_{m,s}\leqq C_{m,s}\langle\tilde{M}w\rangle_{m+s-\frac{1}{2}}.$$

Since $\tilde{M}(w-v)|_{x_n=0}=\tilde{M}(I-\tilde{M})w|_{x_n=0}=0$, inserting $w-v$ in the inequality of the assumption, we see the following

$$C_1\|w-v\|_{0,0}^2\leqq C_2\|\tilde{L}(w-v)\|_{0,0}^2+C_3\|w-v\|_{-1,0}^2.$$

On the other hand by the trace theorem we see that

$$\|\tilde{L}v\|_{0,0}^2\leqq C\|v\|_{1,0}^2\leqq C'\langle\tilde{M}w\rangle_{\frac{1}{2}}^2,$$
$$\begin{aligned}\|v\|_{0,0}^2&=\iint_0^\infty|\hat{v}_n(\xi',x_n)|^2dx_nd\xi'\\&\leqq C_1^{-1}\iint_0^\infty|\hat{v}_n(\xi,x_n)|^2(1+|\xi'|^2)dx_nd\xi'\\&\quad+C_1\iint_0^\infty|\hat{v}_n(\xi',x_n)|^2(1+|\xi'|^2)^{-1}dx_nd\xi'\\&\leqq C_1^{-1}C\|v\|_{0,1}^2+C_1C'\|v\|_{0,-1}^2\\&\leqq C_1^{-1}C''\langle\tilde{M}w\rangle_{\frac{1}{2}}^2+C'''\|v\|_{0,-1}^2,\end{aligned}$$

and

$$\|v\|_{-1,0}\leqq\|v\|_{0,-1}\leqq\|v\|_{1,-2}\leqq C\langle w\rangle_{-\frac{3}{2}}\leqq C'\|w\|_{1,-2}.$$

Thus applying the three inequalities above, we obtain the lemma.

Proof of Theorem 1. We may assume without loss of generality that the non-singular matrix $\beta(x)$ is the identity matrix. For, if u is a weak solution of boundary value problem $Lu=f$, $Mu=g$, then the $L^2(\Omega)$-vector u is also a weak solution of $\beta^{-1}Lu=\beta^{-1}f$, $Mu=g$, where β is extended over Ω. Inductively we assume $u\in H_{r-1}(\Omega)$, since by hypothesis $u\in L^2(\Omega)$. It suffices, by Leibnitz' formula, to show each $\psi_j u$ belongs to $H_r(\Omega)$. For $j=0$, $\psi_0 u\in H_r(\Omega)$ if and only if $\psi_0 u\in H_r(R^n)$. For $j>0$, it suffices to show $(\psi_j u)\circ\alpha_j\in H_r(\bar{R}^n_+)$. Since Γ is non-characteristic for L and $Lu=f$ in $L^2(\Omega)$, the normal derivative $D_n(\psi_j u)\circ\alpha_j$ is expressed by f and tangential derivatives of $(\psi_j u)\circ\alpha_j$. Therefore, if we assume $(\psi_j u)\circ\alpha_j\in H_{0,r}(\bar{R}^n_+)$ then by Proposition 2.5, $(\psi_j u)\circ\alpha_j$ belongs to $H_{1,r-1}(\bar{R}^n_+)$. Using above fact and Proposition 2.5 inductively, we can show that $(\psi_j u)\circ\alpha_j$ belongs to $H_{k,r-k}(\bar{R}^n_+)$ $(0\leqq k\leqq r)$. Thus, it will suffice to show $\psi_0 u\in H_r(R^n)$ and $(\psi_j u)\circ\alpha_j\in H_{0,r}(\bar{R}^n_+)$. That is, in view of Proposition 2.1 and Proposition 2.8, it suffices to prove

$$\text{(3.1)} \qquad ||\psi_0 u||^2_{r-1,\delta}+\sum_1^N||(\psi_j u)\circ\alpha_j||^2_{0,(r-1,\delta)}\leqq C$$

for all δ with $0<\delta\leq 1$. Here we have to remark that by the assumption of induction, Proposition 2.5 and $Lu=f$, $(\psi_j u)\circ\alpha_j$ belongs to $H_{r,-1}(\bar{R}^n_+)$. We begin with the estimation of

$$\text{i)} \qquad ||(\psi_j u)\circ\alpha_j||_{0,(r-1,\delta)}\,.$$

From the inequality (P. 1)

$$q||u||^2_0\leqq||Lu||^2_0+C_q||u||^2_{-1},\ u\in B,\ \text{supp}\ u\subset U_j\,.$$

By Proposition 2.1, when u is an element of $H_s(\Omega)$ with supp $u\subset U_j$, then $||u||_{s,\Omega}$ and $||u\circ\alpha_j||_{s,0}$ are equivalent norms. Therefore we obtain

$$q||u\circ\alpha_j||^2_{0,0}\leqq C||\tilde{L}(u\circ\alpha_j)||^2_{0,0}+C_q'||u\circ\alpha_j||^2_{-1,0}$$

where $\tilde{L}(\cdot)=(L(\cdot\circ\alpha_j^{-1}))\circ\alpha_j$. Thus from Lemma 3.4, we see that

$$q||w||^2_{0,0}\leqq C||\tilde{L}w||^2_{0,0}+C'\langle\tilde{M}w\rangle^2_{\frac{1}{2}}+C_q'||w||^2_{1,-2}\,,$$

where $w\in C_0^\infty(\bar{R}^n_+\cap\text{Im}\,\alpha_j^{-1})$, $\tilde{M}=M\circ\alpha_j$ and C' is independent of q. From the Lemma 3.1, when u is a weak solution, then $J_\varepsilon'(\psi_j u)\circ\alpha_j$ is an element of $H_1(R^n_+)$. Here $\{J_\varepsilon'\}$ is a tangential mollifier of type $r+1$ with kernel $\chi'(x')$, whose support is small, containing the origin of R^n, such that for all $\varepsilon\in(0, 1]$, supp $J_\varepsilon'(\psi_j u)\circ\alpha_j$ is contained in $\text{Im}\,\alpha_j^{-1}$. Since $J_\varepsilon'(\psi_j u)\circ\alpha_j$ is approximated by an element of $C_0^\infty(\bar{R}^n_+\cap\text{Im}\,\alpha_j^{-1})$ in the topology of $H_1(\bar{R}^n_+)$, we have

$$\text{(3.2)} \qquad q||J_\varepsilon'(\psi_j u)\circ\alpha_j||^2_{0,0}\leqq C||\tilde{L}J_\varepsilon'(\psi_j u)\circ\alpha_j||^2_{0,0}+C'\langle\tilde{M}J_\varepsilon'(\psi_j u)\circ\alpha_j\rangle^2_{\frac{1}{2}} +C_q'||J_\varepsilon'(\psi_j u)\circ\alpha_j||^2_{1,-2}\,.$$

Here, the first term on the right of (3.2)

$$\tilde{L}J_\varepsilon'(\psi_j u)\circ\alpha_j = [\tilde{L}, J_\varepsilon'](\psi_j u)\circ\alpha_j + J_\varepsilon'[\tilde{L}, \psi_j\circ\alpha_j](u\circ\alpha_j) + J_\varepsilon'(\psi_j\circ\alpha_j)\tilde{L}(u\circ\alpha_j).$$

Furthermore we observe that as a distribution in R_+^n

$$J_\varepsilon'(\psi_j\circ\alpha_j)\tilde{L}(u\circ\alpha_j) = J_\varepsilon'(\psi_j f)\circ\alpha_j.$$

For

$$\begin{aligned}(J_\varepsilon'(\psi_j\circ\alpha_j)\tilde{L}(u\circ\alpha_j), \phi) &= (\tilde{L}(u\circ\alpha_j), (\psi_j\circ\alpha_j)J_\varepsilon'^*\phi)\\ &= (f\circ\alpha_j, (\psi_j\circ\alpha_j)J_\varepsilon'^*\phi)\\ &= (J_\varepsilon'(\psi_j f)\circ\alpha_j, \phi), \qquad \phi\in C_0^\infty(R_+^n)\end{aligned}$$

where we use the fact $(\psi_j\circ\alpha_j)J_\varepsilon'^*\phi\in C_0^\infty(R_+^n)$. By using Lemma 3.3,

$$\begin{aligned}\tilde{M}J_\varepsilon'(\psi_j u)\circ\alpha_j &= [\tilde{M}, J_\varepsilon'](\psi_j u)\circ\alpha_j + J_\varepsilon'\tilde{M}(\psi_j u)\circ\alpha_j\\ &= [\tilde{M}, J_\varepsilon'](\psi_j u)\circ\alpha_j + J_\varepsilon'(\psi_j g)\circ\alpha_j.\end{aligned}$$

Therefore we obtain the inequality

$$\begin{aligned}(3.3)\qquad & q\|J_\varepsilon'(\psi_j u)\circ\alpha_j\|_{0,0}^2\\ &\leqq C\{\|[\tilde{L}, J_\varepsilon'](\psi_j u)\circ\alpha_j\|_{0,0}^2\\ &\quad + \|J_\varepsilon'[\tilde{L}, \psi_j\circ\alpha_j](u\circ\alpha_j)\|_{0,0}^2 + \|J_\varepsilon'(\psi_j f)\circ\alpha_j\|_{0,0}^2\\ &\quad + \langle[\tilde{M}, J_\varepsilon'](\psi_j u)\circ\alpha_j\rangle_{\frac{1}{2}}^2 + \langle J_\varepsilon'(\psi_j g)\circ\alpha_j\rangle_{\frac{1}{2}}^2\}\\ &\quad + C_q'\|J_\varepsilon'(\psi_j u)\circ\alpha_j\|_{1,-2}^2.\end{aligned}$$

Since the coefficient of D_n is the identity matrix and $[\tilde{L}, \psi_j\circ\alpha_j]$ is the smooth function, the first and second terms in the right hand side (3.3) are estimated by the following form

$$\begin{aligned}\sum_{k=1}^{n-1}\|[E_k(y), J_\varepsilon']D_{y_k}(\psi_j u)\circ\alpha_j\|_{0,0}^2 + \|[E_0(y), J_\varepsilon'](\psi_j u)\circ\alpha_j\|_{0,0}^2\\ + \|J_\varepsilon' F(y)\tilde{\psi}_j(u\circ\alpha_j)\|_{0,0}^2,\end{aligned}$$

where $E_k\in C_0^\infty(\operatorname{Im}\alpha_j^{-1})$, $k=0, \cdots, n-1$, $F(y)\in C_0^\infty(\operatorname{Im}\alpha_j^{-1})$ and $\tilde{\psi}_j\in C_0^\infty(\operatorname{Im}\alpha_j^{-1})$ is equal 1 in $\operatorname{supp}(\psi_j\circ\alpha_j)$. Let us insert this in (3.3) and multiply the inequality thus obtained by $\varepsilon^{-2r-1}(1+\delta^2/\varepsilon^2)^{-1}$, and then integrate with respect to ε in the interval (0, 1). Then applying Proposition 2.12 and Proposition 2.13, we see

$$\begin{aligned}&\int_0^1\|J_\varepsilon' F(y)\tilde{\psi}_j(u\circ\alpha_j)\|_{0,0}^2\varepsilon^{-2r}(1+\delta^2/\varepsilon^2)^{-1}d\varepsilon/\varepsilon\\ &\quad\leqq C\|\tilde{\psi}_j(u\circ\alpha_j)\|_{0,(r-1,\delta)}^2\\ &\quad\leqq C'\sum_{k=0}^{N}\|(\psi_k(\tilde{\psi}_j\circ\alpha_j^{-1})u)\circ\alpha_j\|_{0,(r-1,\delta)}^2\\ &\quad\leqq C''\{\|\psi_0(\tilde{\psi}_j\circ\alpha_j^{-1})u\|_{0,(r-1,\delta)}^2 + \sum_{k=1}^{N}\|(\psi_k(\tilde{\psi}_j\circ\alpha_j^{-1})u)\circ\alpha_k\|_{0,(r-1,\delta)}^2\\ &\quad\leqq C'''\{\|\psi_0 u\|_{(r-1,\delta)}^2 + \sum_{j=1}^{N}\|(\psi_j u)\circ\alpha_j\|_{0,(r-1,\delta)}^2\}\end{aligned}$$

where we used Proposition 2.1 and the fact that $(1+|\xi'|^2)^r(1+|\delta\xi'|^2)^{-1}\leqq(1+|\xi|^2)^r(1+|\delta\xi|^2)^{-1}$ $(r\geqq 1)$

$$\begin{aligned}&\int_0^1\langle[\tilde{M}, J_\varepsilon'](\psi_j u)\circ\alpha_j\rangle_{\frac{1}{2}}^2\varepsilon^{-2r}(1+\delta^2/\varepsilon^2)^{-1}d\varepsilon/\varepsilon\\&\leqq C\langle(\psi_j u)\circ\alpha_j\rangle^2_{(r-\frac{3}{2},\delta)}\leqq C'\|(\psi_j u)\circ\alpha_j\|^2_{1,(r-2,\delta)}\\&\leqq C''\{\|D_n(\psi_j u)\circ\alpha_j\|^2_{0,(r-2,\delta)}+\|(\psi_j u)\circ\alpha_j\|^2_{0,(r-1,\delta)}\}\\&\leqq C'''\{\|f\|^2_{r-1}+\|u\|^2_{r-1}+\|(\psi_j u)\circ\alpha\|^2_{0,(r-1,\delta)}\},\end{aligned}$$

where we used that Γ is non-characteristic for L and Proposition 2.5. Therefore the required inequality becomes

$$\begin{aligned}(3.4)\qquad&(qC_1-C_2)\sum_1^N\|(\psi_j u)\circ\alpha_j\|^2_{0,(r-1,\delta)}\\&\leqq C_3\|\psi_0 u\|^2_{(r-1,\delta)}+C\{\|f\|_r^2+\langle g\rangle^2_{r+\frac{1}{2}}\\&\quad+\|u\|^2_{r-1}+\sum_1^N\|(\psi_j u)\circ\alpha_j\|^2_{r,-1}\}.\end{aligned}$$

ii) Next we shall estimate $\|\psi_0 u\|_{(r-1,\delta)}$.

Let $\{J_\varepsilon\}$ be a full mollifier of type $r+1$ with kernel $\chi(x)$, whose support is small such that supp $\chi+$supp $\psi_0\subset\Omega$. Applying $(P.\ 1)$ to $J_\varepsilon(\psi_0 u)$, we have

$$q\|J_\varepsilon(\psi_0 u)\|^2\leqq\|LJ_\varepsilon(\psi_0 u)\|_0^2+C_q\|J_\varepsilon(\psi_0 u)\|^2_{-1}.$$

By the analogous calculation for $\|(\psi_j u)\circ\alpha_j\|_{0,(r-1,\delta)}$, we obtain

$$\begin{aligned}(3.5)\qquad&(qC_4-C_5)\|\psi_0 u\|^2_{(r-1,\delta)}\\&\leqq C(\|f\|_r^2+\|u\|^2_{r-1})+C_6\sum_1^N\|(\tilde{\psi}_0\psi_j u)\circ\alpha_j\|^2_{(r-1,\delta)},\end{aligned}$$

where $\tilde{\psi}_0\in C_0^\infty(\Omega)$ is equal to 1 on supp ψ_0. By using inducitvely that Γ is non-characteristic for L and from $Lu=f$ in $L^2(\Omega)$, we obtain

$$\begin{aligned}(3.6)\qquad&\sum_1^N\|(\tilde{\psi}_0\psi_j u)\circ\alpha_j\|^2_{(r-1,\delta)}\leqq\sum_1^N\|(\tilde{\psi}_0\psi_j u)\circ\alpha_j\|^2_{r,(-1,\delta)}\\&\leqq C\sum_1^N\|(\psi_j u)\circ\alpha_j\|^2_{r,(-1,\delta)}\\&\leqq C_7\sum_1^N\|(\psi_j u)\circ\alpha_j\|^2_{0,(r-1,\delta)}+C(\|f\|_r^2+\|u\|^2_{r-1}).\end{aligned}$$

Combining (3.4), (3.5) and (3.6), we have that for certain C_8, C_9 and C

$$\begin{aligned}&(qC_8-C_9)(\|\psi_0 u\|^2_{(r-1,\delta)}+\sum_1^N\|(\psi_j u)\circ\alpha_j\|^2_{0,(r-1,\delta)})\\&\leqq C(\|f\|^2_{r,\Omega}+\langle g\rangle^2_{r+\frac{1}{2},\Gamma}+\|u\|^2_{r-1,\Omega}+\sum_1^N\|(\psi_j u)\circ\alpha_j\|^2_{r,-1}).\end{aligned}$$

If q is larger than C_9/C_8, then the inequality (3.1) is completed.

The proof of Theorem 2 is performed by the same method as Theorem 1.

Instead of the above inequality in this case we obtain the following one. Let us replace (P. 1)′ by (P. 2), then by $\langle\cdot\rangle_r$ we can do $\langle\cdot\rangle_{r+\frac{1}{2}}$ in the above proof in Theorem 1. Therefore we see

$$\begin{aligned}(qC_{10}-C_{11})(\|\psi_0 u\|^2_{(r-1,\delta)}+\sum_1^N\|(\psi_j u)\circ\alpha_j\|^2_{0,(r-1,\delta)})\\ \leqq C(\|f\|^2_{r,\Omega}+\langle g\rangle^2_{r,\Gamma}+\|u\|^2_{r-1,\Omega}+\sum_1^N\|(\psi_j u)\circ\alpha_j\|^2_{r,-1})\,.\end{aligned}$$

Thus taking a sufficient large q such that $q>C_{11}/C_{10}$, the proof is completed.

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