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MODULES OVER DEDEKIND PRIME RINGS IV

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Throughout this paper, R will denote a Dedekind prime ring with the quotient ring Q. Let F be any non-trivial right additive topology. A short exact sequence $0 \to L \to M \to N \to 0$ is said to be F^∞-pure if the induced sequence $0 \to L_F \to M_F \to N_F \to 0$ is splitting exact, where M_F is the F-torsion submodule of M. A right R-module is said to be F^∞-pure injective if it has the injective property relative to the class of F^∞-pure exact sequences. Similarly we shall define the concept of F^∞-pure projective modules.

We have already determined the structures of F^∞-pure injective and F^∞-pure projective modules, under some conditions for F.

In this paper, we shall show how the results in [4] on these injectivity and projectivity can be carried over the case of modules over any topology, and discuss the relationships between F^∞-pure injective modules and F-injective modules. We shall show, in Theorem 2.2, that there is a duality between all F-reduced, F-torsion-free, F^∞-pure injective modules and all F-torsion, F-injective modules. This is a generalization of a theorem of Harrison [2]. By using the duality we shall give some properties of F-torsion-free and F^∞-pure injective modules.

1. F^∞-pure injective modules

Let R be a Dedekind prime ring with the two-sided quotient ring Q. We denote the (R, R)-bimodule Q/R by K. Let F be a non-trivial (right additive) topology. Then we denote the left additive topology corresponding to F by F_l (cf. [5]). For any module M, we denote the F-torsion submodule of M by M_F. Let $Q_F = \varinjlim I^{-1} (I \in F)$. Then $Q_F = \varinjlim J^{-1}$ $(J \in F_l)$ and $K_F = Q_F/R = K_{F_l}$ (cf. [5]). In this paper, F is a fixed non-trivial topology. Following [7], a module D is *F-injective* if $\mathrm{Ext}(R/I, D) = 0$ for every $I \in F$. For any module M, we denote the injective hull of M by $E(M)$ and denote the F-injective hull of it by $E_F(M)$. A module G is *F-cotorsion* if $\mathrm{Ext}(Q_F, G) = 0$. Let M be a module. The union of all F_l-divisible submodules of M is itself F_l-divisible and will be denoted by MF^∞: if $MF^\infty = 0$, then M will be said to be *F-reduced*. From the exact sequence

$0 \to R \xrightarrow{\alpha} Q_F \to K_F \to 0$ we derive the exact sequence: $\mathrm{Hom}(Q_F, M) \xrightarrow{\alpha^*} M \to \mathrm{Ext}(K_F, M)$. Then we have

Lemma 1.1. *Let M be a module. Then*
(i) *M/MF^∞ is F-reduced.*
(ii) *$Im\ \alpha^* = MF^\infty$.*

Proof. (i) If $(M/MF^\infty)F^\infty \neq 0$, then we have an exact sequence $0 \to MF^\infty \to N \to (M/MF^\infty)F^\infty \to 0$, where N is a submodule of M and $N \supsetneqq MF^\infty$. For any F-torsion module T, we have $0 = \mathrm{Ext}(T, MF^\infty) \to \mathrm{Ext}(T, N) \to \mathrm{Ext}(T, (M/MF^\infty)F^\infty) = 0$ by Lemma 2.5 of [5] and so $\mathrm{Ext}(T, N) = 0$ implies that N is F_l-divisible, a contradiction.

The proof of (ii) is similar to one of Proposition 2.3 of [4].

From Lemma 1.1 we know that a module is F-reduced and F-cotorsion if and only if it is F-cotorsion in the sense of [6].

Lemma 1.2. (i) *$Ext(K_F, M)$ is F-reduced and F-cotorsion for every module M.*

(ii) *Let G be F-reduced and F-cotorsion. Then $Ext(X, G) = 0$ for every F-torsion-free module X.*

Proof. (i) From the exact sequence $0 \to MF^\infty \to M \to M/MF^\infty \to 0$, we have the exact sequence $0 = \mathrm{Ext}(K_F, MF^\infty) \to \mathrm{Ext}(K_F, M) \to \mathrm{Ext}(K_F, M/MF^\infty) \to 0$. By Proposition 5.2 of [6] and Lemma 1.1, $\mathrm{Ext}(K_F, M/MF^\infty)$ is F-reduced and F-cotorsion. Hence $\mathrm{Ext}(K_F, M)$ is also F-reduced and F-cotorsion.

(ii) First assume that X is a Q_F-module. Then we have an exact sequence $0 \to \mathrm{Ker}\, f \to \sum \oplus Q_F \xrightarrow{f} X \to 0$, where f is a Q_F-homomorphism. Hence $\mathrm{Ker}\, f$ is a Q_F-submodule and so it is F_l-divisible. Applying $\mathrm{Hom}(\ , G)$ to this sequence we get the exact sequence: $0 = \mathrm{Hom}(\mathrm{Ker}\, f, G) \to \mathrm{Ext}(X, G) \to \mathrm{Ext}(Q_F, G) = 0$. Hence $\mathrm{Ext}(X, G) = 0$. Next assume that X is arbitrary. Since $X_F = 0$, we have the exact sequence $0 \to X \to X \otimes Q_F$ and so the sequence $0 = \mathrm{Ext}(X \otimes Q_F, G) \to \mathrm{Ext}(X, G) \to 0$ is exact. Hence $\mathrm{Ext}(X, G) = 0$ for every F-torsion-free module X.

Lemma 1.3. *Let $0 \to L \to M \to N \to 0$ be an F^∞-pure exact sequence. Then the sequence $0 \to Ext(K_F, L) \to Ext(K_F, M) \to Ext(K_F, N) \to 0$ is splitting exact.*

Proof. For any module X, we denote the module X/X_F by $\bar{X}$ and $\mathrm{Ext}(K_F, X)$ by X^*. It is easy to see that if X is F-torsion-free, then X^* is also F-torsion-free. Now we consider the following commutative diagrams:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L_F^* & \to & M_F^* & \to & N_F^* & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L^* & \to & M^* & \to & N^* & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \bar{L}^* & \to & \bar{M}^* & \to & \bar{N}^* & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since $\mathrm{Hom}(K_F, X)=\mathrm{Hom}(K_F, X_F)$ for every module X and R is hereditary, the rows and columns are all exact. Further, from the assumption and Lemma 1.2, we know that the columns, the top and bottom rows are splitting exact. Hence the middle row is also splitting exact.

Proposition 1.4. *Let G be an F-reduced module. Then G is F-cotorsion if and only if G is F^∞-pure injective.*

Proof. Assume that G is F-cotorsion. Then $G\cong\mathrm{Ext}(K_F, G)$ by Proposition 5.2 of [6]. Hence, by using Lemma 1.3, we can easily prove that G is F^∞-pure injective. Conversely assume that G is F^∞-pure injective. Since G is F-reduced and R is hereditary, the sequence $0\to G\to\mathrm{Ext}(K_F, G)\to\mathrm{Ext}(Q_F, G)\to 0$ is exact and F^∞-pure. So the sufficiency is clear.

From Proposition 5.2 of [6] and Proposition 1.4 we know that an F-reduced module G is F^∞-pure injective if and only if $G\cong\mathrm{Ext}(K_F, G)$. Let M be a submodule of G with $(G/M)_F=0$. Then G is an *F^∞-pure essential extension of M* if there are no nonzero submodules $H\subseteq G$ with $H\cap M=0$ and $[(G/(H\oplus M)]_F=0$. An extension G of M is an *F^∞-pure injective envelope* if G is F^∞-pure injective and the extension is F^∞-pure essential. By the same way as in §2 of [4], we easily obtain that F^∞-pure injective envelopes exist and are unique up to isomorphism. Let $M=D\oplus H$ be any module, where H is reduced and D is divisible. Then it is easy to see that $MF^\infty=D\oplus HF^\infty$ and so $E(MF^\infty)=D\oplus E(HF^\infty)$. Let $f_1\colon H\to E(HF^\infty)$ be an extension of the inclusion map $HF^\infty\to E(HF^\infty)$ and let $f\colon M\to E(MF^\infty)\colon f(d, x)=(d, f_1(x))$, where $d\in D$ and $x\in H$. Let the map $g\colon M\to \mathrm{Ext}(K_F, M)$ be the connecting homomorphism. Define $h\colon M\to E(MF^\infty)\oplus\mathrm{Ext}(K_F, M)\colon h(m)=(f(m), g(m))$. Then we have the following theorem by slight modifications to the proof of Theorem 2.9 in [4]:

Theorem 1.5. *Let M be a module. Then the sequence*

$$0\to M\to E(MF^\infty)\oplus Ext(K_F, M)\to Coker\ h\to 0$$

is the F^∞-pure injective resolution of M. $E(MF^\infty)\oplus Ext(K_F, M)$ is the F^∞-pure injective envelope of M, and $Coker\ h$ is injective and F-torsion-free.

REMARK. (i) An F^∞-pure injective module is F-cotorsion. The converse does not necessarily hold. For example, Q_F is F-cotorsion, because the sequence $0=\mathrm{Ext}(K_F, Q_F)\to\mathrm{Ext}(Q_F, Q_F)\to 0$ is exact. But it is not necessarily F^∞-pure injective.

(ii) By the similar way as in Theorem 2.1 of [4] we have a module is F^∞-pure projective if and only if it is a direct sum of a projective module and an F-torsion module. So the results in §3 of [4] hold for F^∞-pure exact sequences.

Let M be an F-reduced module. From the exact sequence $0\to M_F\to M\to M/M_F\to 0$ we get the exact sequence: $0\to\mathrm{Ext}(K_F, M_F)\to\mathrm{Ext}(K_F, M)\to\mathrm{Ext}(K_F, M/M_F)\to 0$. It is easy to see that $\mathrm{Ext}(K_F, M/M_F)$ is F-torsion-free. Thus we have

$$(1.6) \qquad \mathrm{Ext}(K_F, M)\cong\mathrm{Ext}(K_F, M_F)\oplus\mathrm{Ext}(K_F, M/M_F).$$

An F-reduced, F^∞-pure injective module is called *F-adjusted* if it has no nonzero F-torsion-free direct summands. For any F-torsion module T, $\mathrm{Ext}(K_F, T)$ is F-adjusted (cf. §55 of [1]). Thus if G is an F^∞-pure injective module, then, from Theorem 1.5 and (1.6) we get:

$$G = D\oplus A\oplus B\,,$$

where D is injective, A is F-adjusted and B is F-reduced, F-torsion-free, F^∞-pure injective. For F-adjusted modules we have

(1.7) The mapping $T\to\mathrm{Ext}(K_F, T)=G$ gives a one-to-one correspondence between all F-reduced, F-torsion modules T and all F-adjusted modules G (cf. Theorem 55.6 of [1]).

In the final section we shall study the structure of F^∞-reduced, F-torsion-free and F^∞-pure injective modules.

2. F-torsion, F-injective modules and F-reduced, F-torsion-free, F^∞-pure injective modules

In this section we consider the relations between F-torsion, F-injective modules and F-reduced, F-torsion-free, F^∞-pure injective modules.

Lemma 2.1. *Let D be an F-torsion, F-injective module. Then*
(i) *$Hom(K_F, D)$ is F-reduced, F-torsion-free and F^∞-pure injective.*
(ii) *$Hom(K_F, D)\otimes K_F\cong D$.*

Proof. (i) Since K_F is F-divisible as a left R-module, $\mathrm{Hom}(K_F, D)$ is F-torsion-free and so (i) follows from Proposition 5.1 of [6] and Proposition 1.4.

(ii) The mapping η: $\mathrm{Hom}(K_F, D)\otimes K_F\to D$ defined by $\eta(x\otimes\bar{q})=x(\bar{q})$, where $x\in\mathrm{Hom}(K_F, D)$, $\bar{q}\in K_F$, is an epimorphism, because D is F-torsion and

F-injective. Applying this map to $\mathrm{Hom}(K_F,\ )$ we have the exact sequence:

$0\to\mathrm{Hom}(K_F,\ \mathrm{Ker}\ \eta)\to\mathrm{Hom}(K_F,\ \mathrm{Hom}(K_F,\ D)\otimes K_F)\xrightarrow{\eta_*}\mathrm{Hom}(K_F,\ D)\to\mathrm{Ext}(K_F,\ \mathrm{Ker}\ \eta)\to 0$. Since $\mathrm{Hom}(K_F,\ D)\cong\mathrm{Ext}(K_F,\ \mathrm{Hom}(K_F,\ D))$ by Proposition 5.2 of [6] and (i), there is the isomorphism α: $\mathrm{Hom}(K_F,\ D)\cong\mathrm{Hom}(K_F,\ \mathrm{Hom}(K_F,\ D)\otimes K_F)$ by Lemma 2.7 of [5]. For any $\bar{q}\in K_F$, $x\in\mathrm{Hom}(K,\ D)$ we have $[\eta_*\alpha(x)](\bar{q})=\eta[\alpha(x)(\bar{q})]=\eta(x\otimes\bar{q})=x(\bar{q})$ and so $\eta_*\alpha=1$. Thus η_* is an isomorphism. Hence $\mathrm{Hom}(K_F,\ \mathrm{Ker}\ \eta)=0=\mathrm{Ext}(K_F,\ \mathrm{Ker}\ \eta)$ implies that $\mathrm{Ker}\ \eta=0$. Therefore $\mathrm{Hom}(K_F,\ D)\otimes K_F\cong D$.

Theorem 2.2. *The correspondence*

$$(*)\qquad D\to G=\mathrm{Hom}(K_F,\ D)$$

is one-to-one between all F-torsion, F-injective modules D and all F-reduced, F-torsion-free, F^∞-pure injective modules G. The inverse of $(*)$ is given by the correspondence $G\to G\otimes K_F$.

Proof. This is immediate from Proposition 5.2 of [6], Lemmas 2.5, 2.7, Corollary 2.6 of [5], and Proposition 1.4, Lemma 2.1.

This duality was exhibited by Harrison in [2] between all divisible, torsion groups and all reduced, torsion-free, cotorsion groups. The author generalized it to modules over bounded Dedekind prime rings (cf. [3]).

We define $\hat{R}_F=\varprojlim R/I$ $(I\in F)$ and $\hat{R}_{F_l}=\varprojlim R/J$ $(J\in F_l)$. Then $\hat{R}_F$ and $\hat{R}_{F_l}$ are both rings containing R. Let M be an F-torsion module. Then M is an $\hat{R}_F$-module as follows: For $m\in M$, $\hat{r}=([r_I+I])\in\hat{R}_F$, we define $m\hat{r}=mr_J$, where $J\subseteq O(m)=\{r\in R\mid mr=0\}$. Similarly, an F_l-torsion left module is an $\hat{R}_{F_l}$-module.

Proposition 2.3. *An F-reduced module is F-torsion-free, F^∞-pure injective if and only if it is a direct summand of a direct product of copies of $\hat{R}_{F_l}$.*

Proof. From Lemma 2.7 of [5] and Lemma 2.1, $\hat{R}_{F_l}$ is F-reduced, F-torsion-free and F^∞-pure injective and so the sufficiency is evident. Conversely let G be F-reduced, F-torsion-free and F^∞-pure injective. Then there exists an F-torsion, F-injective module D with $G=\mathrm{Hom}(K_F,\ D)$ by Theorem 2.2. It is easy to see that D can be embedded in an exact sequence $0\to D\xrightarrow{i}\prod K_F$ with sufficiently many copies of K_F. Hence we have the exact sequence $0\to\mathrm{Hom}(K_F,\ D)\xrightarrow{i_*}\prod\hat{R}_{F_l}\to\mathrm{Coker}\ i_*\to 0$. $\mathrm{Hom}(K_F,\ \prod K_F/D)$ is F-torsion-free and Coker i_* is so. Hence the above exact sequence splits. So G is a direct summand of $\prod\hat{R}_{F_l}$.

Lemma 2.4. (i) *An F-torsion, F-injective module is a direct sum of uniform,* *F-torsion and F-injective modules.*

(ii) *If M is F-torsion-free, then* $Tor(M, L)=0$ *for every* F_l*-torsion left module* L.

(iii) *If M is F-torsion-free with* $M\otimes K_F=0$, *then M is F-injective.*

Proof. (i) Let D be F-torsion and F-injective. Then $E(D)=\sum\oplus E_\alpha$, where E_α is uniform and injective. Hence $D=E(D)_F=\sum\oplus(E_\alpha)_F$ and $(E_\alpha)_F$ is uniform and F-injective.

(ii) Let L be an F_l-torsion left module. Then L can be embedded in an exact sequence $0\to L\to\sum\oplus K_F$ with sufficiently many copies of K_F. Hence we have the exact sequence $0\to \mathrm{Tor}(M, L)\to\sum\oplus\mathrm{Tor}(M, K_F)=0$.

(iii) Let J be any element in F_l. Then from an exact sequence $0\to R/J\to\sum\oplus K_F\to L\to 0$ we get the exact sequence $\mathrm{Tor}(M, L)\to M\otimes R/J\to M\otimes(\sum\oplus K_F)$. The first and last terms are zero by the assumption and (ii). Thus $M\otimes R/J=0$ implies that M is F-injective by Lemma 2.5 of [5].

A submodule B of an $\hat{R}_{F_l}$-module G is called an *F-basic submodule* if it satisfies the following conditions:

(a) B is a direct sum of indecomposable, cyclic $\hat{R}_{F_l}$-modules,

(b) $(G/B)_F=0$,

(c) G/B is F-injective.

Proposition 2.5. *Let G be an F-reduced, F-torsion-free and F^∞-pure injective module. Then*

(i) *G possesses an F-basic submodule.*

(ii) *Any two F-basic submodules of G are isomorphic.*

Proof. By Theorem 2.2 there exists an F-torsion, F-injective module D such that $G=\mathrm{Hom}(K_F, D)$. Write $D=\sum\oplus D_\alpha$, where D_α is uniform and F-injective. Further we let $B=\sum\oplus\mathrm{Hom}(K_F, D_\alpha)$ and $B_\alpha=\mathrm{Hom}(K_F, D_\alpha)$. Then we may assume that $G\supseteqq B$.

(i) We shall prove that B is an F-basic submodule of G. (a): Since D_α is isomorphic to a direct summand of K_F, B_α is a cyclic $\hat{R}_{F_l}$-module. From Theorem 2.2 it is evident that B_α is indecomposable. (b): Let $\bar{D}=(\prod D_\alpha)_F$. Then $D\subseteqq\bar{D}$ and $(\prod D_\alpha/\bar{D})_F=0$. So $0\to\bar{D}\to\prod D_\alpha\to\prod D_\alpha/\bar{D}\to 0$ is F^∞-pure. Hence $\bar{D}$ is F_l-divisible, because $\prod D_\alpha$ is F_l-divisible (cf. Lemma 2.4 of [4]). Applying $\mathrm{Hom}(K_F,\ \ )$ to the above sequence we get the exact sequence $0\to\mathrm{Hom}(K, \bar{D})\to\mathrm{Hom}(K_F, \prod D_\alpha)\to\mathrm{Hom}(K_F, \prod D_\alpha/\bar{D})=0$. Hence $\mathrm{Hom}(K_F, \bar{D})\cong\prod\mathrm{Hom}(K_F, D_\alpha)\cong\prod B_\alpha$. Since $\bar{D}/D$ is F-torsion, D is a direct summand of $\bar{D}$. So G is a direct summand of $\prod B_\alpha$. Let h be any element of $\prod B_\alpha$ with $hI\subseteqq B$ for some $I\in F$. Since hI is finitely generated, there are $\alpha_1, \cdots, \alpha_n$ such that $hI\subseteqq B_{\alpha_1}\oplus\cdots\oplus B_{\alpha_n}$. Hence $h\in B$, since $\prod B_\alpha$ is F-torsion-free and $B_{\alpha_1}\oplus\cdots\oplus B_{\alpha_n}$ is a direct summand of $\prod B_\alpha$. Therefore $(\prod B_\alpha/B)_F=0$ and so $(G/B)_F=0$. (c):

Applying $\otimes K_F$ to the exact sequence $0\to B\to G\to G/B\to 0$ we have the exact sequence $\mathrm{Tor}(G/B, K_F)\to B\otimes K_F\to G\otimes K_F\to (G/B)\otimes K_F\to 0$. By (b) and Lemma 2.4, the first term is zero. Further $B\otimes K_F\cong\sum\oplus D_\alpha=D\cong G\otimes K_F$ implies that $(G/B)\otimes K_F=0$. Hence G/B is F-injective by Lemma 2.4.

(ii) Let $B=\sum\oplus B_\alpha'$ be any F-basic different from B. Then by the same way as in the above (c), we have $D\cong\sum\oplus(B_\alpha'\otimes K_F)$. Hence we have $B\cong B'$ by Theorem 2.2 and Krull-Remak-Schmidt-Azumaya's theorem.

Let $S(K_F)$ be the right socle of K_F. Then $S(K_F)$ is a left module and is F_l-torsion. Hence it is a left $\hat{R}_{F_l}$-module. Let $G=\mathrm{Hom}(K_F, D)$, where D is an F-torsion and F-injective module. From the exact sequence $0\to S(K_F)\xrightarrow{i} K_F$, we have the exact sequence $0\to\mathrm{Ker}\, i^*\to G\xrightarrow{i^*}\mathrm{Hom}(S(K_F), D)\to 0$ as right $\hat{R}_{F_l}$-modules.

Lemma 2.6. *Ker $i^*=\cap GJ$, where J ranges over all maximal left ideals in F_l.*

Proof. It is evident that $S(K_F)=\sum J^{-1}/R$, where J ranges over all maximal left ideals in F_l. By Lemma 2.1, we have an isomorphism η: $G\otimes K_F\cong D$ and the exact sequence $0\to G\to G\otimes Q_F\to G\otimes K_F\to 0$. Let x be an element of G. Then we have

$$x(S(K_F))=0\Leftrightarrow x(J^{-1}/R)=0\Leftrightarrow \eta(x\otimes J^{-1}/R)=0\Leftrightarrow$$
$$x\otimes J^{-1}/R=0\Leftrightarrow x\otimes J^{-1}\subseteqq G \text{ in } G\otimes Q_F\Leftrightarrow x\in GJ$$

for every maximal left ideal J in F_l.

For any $\hat{R}_{F_l}$-module M, we put $J(M)=\cap MJ$ and $\bar{M}=M/J(M)$, where J ranges over all maximal left ideals in F_l. By Corollaries 2.8 and 2.9 of [5], $J(M)$ is an $\hat{R}_{F_l}$-module.

Corollary 2.7. (i) *The Jacobson radical of $\hat{R}_{F_l}$ coincides with $J(\hat{R}_{F_l})$.*

(ii) *Let $G=Hom(K_F, D)$, where D is F-torsion and F-injective. Then $\bar{G}\cong Hom(S(K_F), D)$.*

Proof. By Corollary 4.5 of [6], $\hat{R}_{F_l}\cong\mathrm{Hom}(K_F, K_F)$. Further, it is clear that K_F is quasi-injective and is an essential extension of $S(K_F)$. Hence (i) follows from Lemma 2.6. (ii) also follows from Lemma 2.6.

Let F_i be an atom contained in F (i.e., F_i is a minimal element in the lattice of all topologies) and let S_{F_i} be a simple, F_i-torsion module. Then $G_{F_i}=\mathrm{Hom}(K_F, E_F(S_{F_i}))$ is F-reduced, F-torsion-free, F^∞-pure injective and indecomposable. By Corollary 2.7, $\bar{G}_{F_i}\cong\mathrm{Hom}(S(K_F), E_F(S_{F_i}))=\mathrm{Hom}(S(K_F), S_{F_i})$ and so $\bar{G}_{F_i}$ is a simple $\hat{R}_{F_l}$-module. A simple $\hat{R}_{F_l}$-module which is isomorphic to the module

$\bar{G}_{F_i}$ is said to be *F_i-simple*. By the *F-rank M* of an $\hat{R}_{F_l}$-module M is meant the cardinal number of a maximal independent set of simple $\hat{R}_{F_l}$-modules of $\bar{M}$. Similarly the *F_i-rank M* of M is defined by using F_i-simple submodules of $\bar{M}$.

Theorem 2.8. *Let G and G_1 be F-reduced, F-torsion-free and F^∞-pure injective modules. Then $G \cong G_1$ if and only if F_i-rank $G = F_i$-rank G_1 for all atoms F_i contained in F.*

In particular, F-rank G coincides with the cardinal number of the numbers of cyclic, indecomposable direct summands of the F-basic submodule of G.

Proof. The necessity is clear. To prove the sufficiency we let $G=\mathrm{Hom}(K_F, D)$ and $G_1=\mathrm{Hom}(K_F, D_1)$, where D and D_1 are F-torsion, F-injective. Further let $D=\sum\oplus D_\alpha$, where D_α is uniform and F-injective and let $B=\sum\oplus B_\alpha$, where $B_\alpha=\mathrm{Hom}(K_F, D_\alpha)$. Then it is evident that $J(B)=\sum\oplus J(B_\alpha)$ and so $\bar{B}=\sum\oplus\bar{B}_\alpha$. For each α, D_α is isomorphic to $E_F(S_{F_i})$ for some atom F_i contained in F, where S_{F_i} is a simple and F_i-torsion module. Hence $\bar{B}_\alpha$ is a simple $\hat{R}_{F_l}$-module. The exact sequence $0\to B\to G\to G/B\to 0$ is F^∞-pure and so the sequence $0\to\bar{B}\to\bar{G}$ is exact (cf. Lemma 2.4 of [4]). Next we shall prove that $\bar{G}$ is an essential extension of $\bar{B}$ as an $\hat{R}_{F_l}$-module. The diagram is commutative with exact rows (cf. Corollary 2.7):

$$\begin{array}{ccccc} 0 & \longrightarrow & \bar{B} & \longrightarrow & \bar{G} \\ & & \wr\| & & \wr\| \\ 0 & \to & \sum\oplus\mathrm{Hom}(S(K_F), D_\alpha) & \to & \mathrm{Hom}(S(K_F), D)\,. \end{array}$$

So it is enough to show that $\mathrm{Hom}(S(K_F), D)$ is an essential extension of $\sum\oplus\mathrm{Hom}(S(K_F), D_\alpha)$. To prove this, let f be any nonzero element of $\mathrm{Hom}(S(K_F), D)$. Then there exists a simple direct summand S_α of $S(K_F)$ with $f(S_\alpha)\neq 0$. Write $S(K_F)=S_\alpha\oplus S'_\alpha$ and let e_α: $S(K_F)\to S_\alpha$ be the projection. Since $K_F=E_F(S_\alpha)\oplus E_F(S'_\alpha)$, there is $\hat{r}\in\hat{R}_{F_l}$ such that it is an extension of $S(K_F)\xrightarrow{e_\alpha}S_\alpha\hookrightarrow K_F$ and is a zero map on $E_F(S'_\alpha)$. Then $0\neq f\hat{r}$ and $f\hat{r}\in\sum\oplus\mathrm{Hom}(S(K_F), D_\alpha)$, as desired. Thus F_i-rank G coincides with the cardinal number of the numbers of F_i-simple direct summands of $\bar{B}$, and so it coincides with the cardinal number of the numbers of uniform, F-injective direct summands of D which contain an F_i-torsion and simple module for every atom F_i contained in F. Hence $D\cong D_1$ and so $G\cong G_1$ by Theorem 2.2. The last assertion is evident from the above discussion.

Theorem 2.9. *Let D be an F-torsion, F-injective module and let $G=\mathrm{Hom}(K_F, D)$. Then $\mathrm{Hom}(D, D)\cong\mathrm{Hom}_{\hat{R}_{F_l}}(G, G)$.*

Proof. The mapping Φ: $\mathrm{Hom}(D, D)\to\mathrm{Hom}_{\hat{R}_{F_l}}(G, G)$, defined by $\Phi(f)(g)=f\cdot g(f\in\mathrm{Hom}(D, D), g\in G)$ is clearly a ring homomorphism. Assume that $\Phi(f)=0$, where $f\in\mathrm{Hom}(D, D)$. Let $D=\sum\oplus D_\alpha$, where D_α is uniform and F-injective. For each α, D_α is isomorphic to a direct summand of K_α. Hence there is $g_\alpha \in G$ such that $g_\alpha(K_F)=D_\alpha$. Then we have $0=\Phi(f)(g_\alpha)(K_F)=f\cdot g_\alpha(K_F)=f(D_\alpha)$. Hence $f=0$, and thus Φ is a monomorphism. Finally we shall prove that Φ is an epimorphism. For any D_α, there exists a direct summand K_α of K_F such that $\theta_\alpha : K_\alpha \cong D_\alpha$. Let $e_\alpha : K_F \to K_\alpha$ be the projection and let $i_\alpha : K_\alpha \to K_F$ be the inclusion. Write $\phi_\alpha=\theta_\alpha e_\alpha$ and $\psi_\alpha=i_\alpha\theta_\alpha^{-1}$. Further let $B_\alpha=\mathrm{Hom}(K_F, D)$ and let $B=\sum\oplus B_\alpha$. Then $B_\alpha=\phi_\alpha\hat{R}_{F_l}$ and B is an F-basic submodule of G. Now let g be any element of $\mathrm{Hom}_{\hat{R}_{F_l}}(G, G)$. Then, as is easily verified, the mapping

$$f : D \ni d = \sum d_{\alpha_i} \to \sum g(\phi_{\alpha_i})\,[\psi_{\alpha_i}(d_{\alpha_i})] \in D\,,$$

where $d_{\alpha_i} \in D_{\alpha_i}$, is a homomorphism. Further, for any $k \in K_F$, we have

$$\begin{aligned}\Phi(f)\,(\phi_\alpha)\,(k) &= f\circ\phi_\alpha(k) = g(\phi_\alpha)\,[\psi_\alpha(\phi_\alpha(k))] \\ &= g(\phi_\alpha)e_\alpha(k) = g(\phi_\alpha e_\alpha)\,(k) = g(\phi_\alpha)\,(k)\,,\end{aligned}$$

because $e_\alpha \in \hat{R}_{F_l}$ and g is an $\hat{R}_{F_l}$-homomorphism. Hence $\Phi(f)\,(\phi_\alpha)=g(\phi_\alpha)$ for every ϕ_α. This implies that $\Phi(f)$ coincides with g on B. So $\Phi(f)-g$ induces a homomorphism from G/B into G. Since G/B is F_l-divisible and G is F-reduced, the map is zero and so $\Phi(f)=g$.

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