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MODULES OVER DEDEKIND PRIME RINGS III

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Let R be a Dedekind prime ring with the quotient ring Q. Let F be any right additive topology (cf. [11]). Then R is a topological ring with elements of F as the neighborhoods of zero. Let M be a topological right R-module with submodule neighborhoods of zero. M is called F-linearly compact if

(a) it is Hausdorff,

(b) if every finite subset of the set of congruences $x \equiv m_\alpha \pmod{N_\alpha}$, where N_α are closed submodules of M, has a solution in M, then the entire set of the congruences has a solution in M.

The purpose of this paper is to study the algebraic and topological properties of F-linearly compact modules.

After discussing some properties on R which need in this paper, we show, in Section 2, that the Kaplansky's duality theorem holds for F-linearly compact modules (Theorem 2.12). By using the duality theorem we determine, in Section 3, the algebraic and topological structures of F-linearly compact modules when F is bounded. Moreover we define the concepts of F^ω-pure injective and F^∞-pure injective modules, and investigate the relations of between these concepts and F-linearly compact modules.

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1. Topologies on Dedekind prime rings

Throughout this paper, R will denote a Dedekind prime ring which is not artinian, and Q will denote the quotient ring of R. We will denote the (R, R)-bimodule Q/R by K. A subring of Q containing R is called an *overring* of R. For any essential right ideal I, the *left order* of I is defined by $0_l(I)=\{q\in Q \mid qI\subseteq I\}$. We define the *inverse* of I to be $I^{-1}=\{q\in Q \mid IqI\subseteq I\}$. Then we obtain $II^{-1}=0_l(I)$ and $I^{-1}I=R$. Let I be a right ideal of R. By Theorem 1.3 of [1], R/I is an artinian R-module if and only if I is an essential right ideal of R. For any right ideal I and any element a of R, we define $a^{-1}I=\{r\in R \mid ar\in I\}$. Let M be a (right R-) module. M is said to be *torsion* if, for every $m\in M$, $mI=0$ for some essential right ideal I. We say that M is *divisible* if $MJ=M$ for every essential left ideal J of R. Let F be any (right additive) topology (cf. [11]). We say that $m \in M$ is an *F-torsion element* if $O(m)=\{r \in R \mid mr=0\} \in F$, and denote the submodule of F-torsion elements by M_F. If $M_F=0$, then we say that M is *F-torsion-free*. A topology F is *trivial* if all modules are F-torsion or F-torsion-free. If $F=\{R\}$, then it is clear that all modules are F-torsion-free. Assume that F contains a non essential right ideal I of R, then F-torsion module R/I is a direct sum of a torsion module and a non-zero projective module C by Theorem 2.1 cf [1]. By Theorem 2.4 of [1], a finite copies of C contains R as right modules and so R is F-torsion. Hence all modules are F-torsion. So if F is a non-trivial topology, then F consists of essential right ideals. Conversely a topology F consists of essential right ideals, then it is non-trivial, because R is F-torsion-free and R/I is F-torsion ($I \in F$).

From now on, F will denote a non-trivial topology. We define $Q_F=\varinjlim I^{-1}$, where I ranges over all elements of F. Clearly Q_F is an overring of R.

Proposition 1.1 (i) *The mapping $F \to Q_F$ is one-to-one correspondence between all non-trivial topologies and all overrings of R properly containing R.*

(ii) *A module M is F-torsion if and only if $M \otimes Q_F=0$.*

(iii) *For any module M, $M_F=\mathrm{Tor}(M, Q_F/R)$.*

Proof. By Corollary 13.4 of [11], F is perfect. Hence (ii) and (iii) follow from Exercise 2 of [11, p. 81].

(i) Let Q_0 be an overring of R properly containing R. Then it is well known that Q_0 is R-flat and that the inclusion map: $R \to Q_0$ is an epimorphism (cf. [11, p. 75]). Hence, by Theorem 13.10 of [11], $F_0=\{I \mid IQ_0=Q_0, I \text{ is a right ideal}\}$ is a topology. Since $Q_0 \otimes Q_0 \cong Q_0$ and Q_0 is R-flat, we have $Q_0/R \otimes Q_0=0$. Hence $0 \neq Q_0/R$ is F_0-torsion. It is evident that R is not F_0-torsion. Hence F_0 is non-trivial. Thus (i) follows from Theorem 13.10 of [11].

Let $\{S_\alpha \mid \alpha \in \Lambda\}$ be the representative class of simple modules which are non-isomorphic mutually. For any subset Γ of Λ, we denote the set of R and of essential right ideals I such that any composition factor of the module R/I is isomorphic to S_γ for some $\gamma \in \Gamma$ by $F(\Gamma)$.

Proposition 1.2. *A non-empty family of right ideals of R is a non-trivial topology if and only if it is of the form $F(\Gamma)$ for some subset Γ of Λ.*

Proof. First we shall prove that $F(\Gamma)$ is a non-trivial topology. (i) If $I \in F(\Gamma)$ and $a \in R$, then $a^{-1}I \in F(\Gamma)$, because $R/a^{-1}I \cong (aR+I)/I$. (ii) Let I be a right ideal of R. Assume that there exists $J \in F(\Gamma)$ such that $a^{-1}I \in F(\Gamma)$ for every $a \in J$. Again, since $R/a^{-1}I \cong (aR+I)/I$ for every $a \in J$, we obtain that $(I+J)/I$ is a torsion module. Hence R/I is also torsion and so I is an essential right ideal. By Theorem 3.3 of [1], $I+J=aR+I$ for some $a \in J$, and thus $R/a^{-1}I \cong (I+J)/I$. Therefore $I \in F(\Gamma)$. Thus $F(\Gamma)$ is a topology. Since $F(\Gamma)$

consists of essential right ideals, it is non-trivial. Conversely let F be any topology and let $\Gamma=\{\gamma\in\Lambda \mid S_\gamma\simeq R/I$ for some $I\in F\}$. From Lemma 3.1 of [11], we have $\Gamma\neq\phi$. We shall prove that $F=F(\Gamma)$. For an essential right ideal I of R, $I\in F$ if and only if $R/I\otimes Q_F=0$ by Proposition 1.1 and so $F\supseteq F(\Gamma)$. Assume that $F\supsetneqq F(\Gamma)$. Then there is $I\in F$ such that some composition factor of R/I is isomorphic to S_α for some $\alpha\in\Lambda-\Gamma$. So there are right ideals $J_1\supsetneqq J_2\supseteq I$ such that $J_1/J_2\simeq S_\alpha$. Take $a\in J_1$ with $a\notin J_2$. Then we get: $R/a^{-1}J_2\simeq J_1/J_2\simeq S_\alpha$. Hence, since $a^{-1}J_2\in F$, we have $\alpha\in\Gamma$, which is a contradiction.

Corollary 1.3. *The lattice of all overrings of R is a Boolean lattice.*

The family F_l of left ideals J of R such that $Q_FJ=Q_F$ is a left additive topology. We call it the *left additive topology corresponding* to F. F_l is also non-trivial by Proposition 1.1. Thus F_l consists of essential left ideals of R. We put $Q_{F_l}=\varinjlim J^{-1}(J\in F_l)$. A module M is said to be *F_l-divisible* if $MJ=M$ for every $J\in F_l$. In a similar way, we define the concepts of *F_l-torsion* and *F-divisible* for any left module.

Proposition 1.4. (i) *$Q_F=Q_{F_l}$ and so Q_F is (F, F_l)-divisible.*

(ii) *$K_F=K_{F_l}=Q_F/R$, where $K=Q/R$. Thus K_F is also (F, F_l)-divisible.*

(iii) *Let I be an essential right ideal of R. Then $I\in F$ if and only if I^{-1}/R is F_l-torsion.*

Proof. (i) follows from Proposition 1.1 of [10] and the definitions. (ii) is clear.

(iii) Since Q_F is flat as R-modules, the sequence $0\to Q_F\to Q_F\otimes I^{-1}\to Q_F\otimes I^{-1}/R\to 0$ is exact. Further, since $Q_F\otimes Q_F\simeq Q_F$, we obtain that $I\in F$ if and only if $Q_F\otimes I^{-1}/R=0$. So $I\in F$ if and only if I^{-1}/R is F_l-torsion.

2. Duality theorem for F-linearly compact modules

Let F be any non-trivial topology. We define $\hat{R}_F=\varprojlim R/I(I\in F)$ and $\hat{R}_{F_l}=\varprojlim R/J(J\in F_l)$. It is easy to see that both $\hat{R}_F$ and $\hat{R}_{F_l}$ are rings containing R (cf. §4 of [10]). Let M be an F-torsion module. Then M is an $\hat{R}_F$-module as follows: For $m\in M$, $\hat{r}=([r_I+I])\in\hat{R}_F$, we define $m\hat{r}=mr_J$, where $J\subseteq O(m)$. Similarly, an F_l-torsion left module is an $\hat{R}_{F_l}$-module.

Lemma 2.1. *A module is F-linearly compact in the discrete topology if and only if it is F-torsion and artinian.*

Proof. The sufficiency follows from Proposition 5 of [13]. Conversely assume that M is F-linearly compact in the discrete topology. Take $m\in M$. Then, by the continuity of multiplication, there exists $I\in F$ such that $mI=0$.

Thus M is F-torsion. By Lemma 2.3 of [9], M is finite dimensional in the sense of Goldie. So the socle $S(M)$ of M is finitely generated and M is an essential extension of $S(M)$. Let N be any submodule of M. Then, since N is an open and closed submodule, $\bar{M}=M/N$ is also F-linearly compact in the discrete topology by Proposition 2 of [13]. Thus the socle $S(\bar{M})$ of $\bar{M}$ is also finitely generated and $\bar{M}$ is an essential extension of $S(\bar{M})$. This implies that M is an artinian module by Proposition 2* of [12].

Corollary 2.2. *Let M be F-linearly compact and let N be a submodule. Then N is a neighborhood of zero if and only if M/N is F-torsion and artinian.*

Proof. If N is a neighborhood of zero, then M/N is F-linearly compact in the discrete topology. So the necessity follows from Lemma 2.1. Conversely, assume that M/N is F-torsion and artinian. Let $\{M_\alpha\}$ be the set of submodule neighborhoods of zero. Since the topology is Hausdorff, $\cap M_\alpha=0$, and so $\cap \bar{M}_\alpha=\bar{0}$ in $\bar{M}=M/N$. Therefore there are finite submodules $M_{\alpha_1}, \cdots, M_{\alpha_n}$ such that $\cap_{i=1}^n \bar{M}_{\alpha_i}=\bar{0}$, i.e., $\cap_{i=1}^n M_{\alpha_i}\subseteq N$. Thus N is open.

Corollary 2.3. *If a module is F-linearly compact in two topologies, then these topologies coincide.*

Lemma 2.4. *A module is F-linearly compact if and only if it is an inverse limit of F-torsion and artinian modules.*

Proof. The sufficiency follows from Proposition 4 of [13] and Lemma 2.1. To prove the necessity let $\{N_\alpha\}$ be the set of submodule neighborhoods of zero. Then the modules M/N_α with the natural maps: $[m+N_\alpha]\to[m+N_\beta]$, where $N_\alpha\subseteq N_\beta$, form an inverse system. Write $\hat{M}=\varprojlim M/N_\alpha$. Then it is a topological module; each M/N_α has the discrete topology and the product topology on $\prod M/N_\alpha$ induces a subspace topology on $\hat{M}$. Since $\cap N_\alpha=0$, the canonical map $f\colon M\to\hat{M}$ is a monomorphism. It is easy to see that f is a topological isomorphism from M onto $f(M)$ and that $f(M)$ is dense in $\hat{M}$. On the other hand, M is complete by Proposition 8 of [13] and so $f(M)=\hat{M}$. Further M/N_α is F-torsion and artinian by Corollary 2.2.

Following [11], a module D is *F-injective* if $\mathrm{Ext}(R/I, D)=0$ for every $I\in F$. By Proposition 6.2 of [11], D is F-injective if and only if $\mathrm{Ext}(T, D)=0$ for every F-torsion T. Further, since every F-torsion module T can be embedded in an exact sequence $0\to T\to\sum\oplus K_F$ with sufficiently many copies of K_F, D is F-injective if and only if $\mathrm{Ext}(K_F, D)=0$. For any module M, we denote the injective hull of M by $E(M)$ and denote the F-injective hull of it by $E_F(M)$ (cf. [11]).

Lemma 2.5. (i) *A module is F-injective if and only if it is F_I-divisible.*

(ii) *Let M be a module with $M_F=0$. Then $E_F(M)=M\otimes Q_F$.*

Proof. (i) Assume that D is F-injective. Let $J\in F_l$. Then J^{-1}/R is F-torsion by Proposition 1.4 and so the necessity follows from Proposition 3.2 of [10]. Conversely assume that D is F_l-divisible. Let I be any element of F. Then $I^{-1}/R=\sum_{i=1}^n \oplus R/J_i$ for $J_i\in F_l$. By Proposition 3.3 of [10], we have

$R/I\cong \mathrm{Hom}(I^{-1}/R, K_F)\cong \sum_{i=1}^n \oplus \mathrm{Hom}(R/J_i, K_F)\cong \sum_{i=1}^n \oplus J_i^{-1}/R$, and so Ext $(R/I, D)\cong \sum_{i=1}^n \oplus \mathrm{Ext}(J_i^{-1}/R, D)\cong \sum_{i=1}^n \oplus D/DJ_i=0$. Therefore D is F-injective.

(ii) By Proposition 1.1, $M_F=\mathrm{Tor}(M, K_F)$. Hence from the exact sequence $0\to R\to Q_F\to K_F\to 0$ we get an exact sequence $0\to M\to M\otimes Q_F\to M\otimes K_F\to 0$.

By Proposition 1.4 and (i), $M\otimes Q_F$ is F-injective and so $M\otimes Q_F=E_F(M)$.

Corollary 2.6. *Let M be a module. Then $M\otimes Q_F$ and $M\otimes K_F$ are both F-injective.*

For a module M, we define $\hat{M}_{F_l}=\varprojlim M/MJ (J\in F_l)$. $\hat{M}_{F_l}$ is an $\hat{R}_{F_l}$-module (cf. §4 of [10]). Similarly, for a left module N, we can define a left $\hat{R}_F$-module $\hat{N}_F$.

Lemma 2.7. *Let M be a module with $M_F=0$. Then there are commutative diagrams:*

$$\begin{array}{ccccc} \hat{M}_{F_l} & \cong & \mathrm{Hom}(K_F, M\otimes K_F) & \cong & \mathrm{Ext}(K_F, M) \\ \uparrow & & \uparrow \alpha & & \uparrow \beta \\ M & = & M & = & M \end{array},$$

where $\alpha(m)(\bar{q})=m\otimes\bar{q}$ ($m\in M$, $\bar{q}\in K_F$) and β is the connecting homomorphism.

Proof. From the exact sequence $0\to R\to Q_F\to K_F\to 0$, we get an exact sequence:

$$(1) \qquad 0=\mathrm{Tor}(M, K_F)\to M\to M\otimes Q_F\to M\otimes K_F\to 0.$$

Hence the assertion of the first diagram follows from the similar way as in Theorem 4.4 of [10]. Applying Hom(K_F,) to the sequence (1), we obtain the exact sequence:

$\mathrm{Hom}(K_F, M\otimes Q_F)\to \mathrm{Hom}(K_F, M\otimes K_F)\to \mathrm{Ext}(K_F, M)\to \mathrm{Ext}(K_F, M\otimes Q_F)$. The first and last terms are zero, because $M\otimes Q_F$ is F-torsion-free and F-injective. Hence $\mathrm{Hom}(K_F, M\otimes K_F)\cong \mathrm{Ext}(K_F, M)$. We consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & M\otimes Q_F & \to & M\otimes K_F & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & E(M) & \to & E(M)/M & \to & 0 . \end{array}$$

If $[x+M]I=0$, where $x\in E(M)$ and $I\in F$, then $xI\subseteqq M$ and so $x\in M\otimes Q_F$ by Proposition 6.3 of [11] and Lemma 2.5. Hence $[x+M]\in M\otimes K_F$. This implies that $(E(M)/M)_F=M\otimes K_F$. It is evident that $E(M)_F=0$. Thus we have Ext $(K_F, M)=\mathrm{Hom}(K_F, E(M)/M)=\mathrm{Hom}(K_F, M\otimes K_F)$. Now it is easy to see that $\alpha=\beta$.

Corollary 2.8. (i) *$\hat{R}_F/R$ is F-divisible.*
(ii) *$R/I\cong\hat{R}_F/I\hat{R}_F$ for every $I\in F$.*

Proof. (i) Applying Lemma 2.7 to the left module R, we get an isomorphism: $\hat{R}_F/R\cong\mathrm{Ext}(Q_F, R)$. Since $\mathrm{Ext}(Q_F, R)$ is a left Q_F-module, it is F-divisible and so $\hat{R}_F/R$ is also F-divisible.

(ii) It is evident that $I\hat{R}_F\cap R=I$. Hence (ii) follows from (i).

By Lemma 2.4, $\hat{R}_F$ is an F-linearly compact module in the topology which is defined by taking as a subbase of neighborhoods of zero the set $\{\pi_I^{-1}(0)\cap\hat{R}_F\mid I\in F\}$, where π_I: $\prod R/I\to R/I$ is the projection. Further we have

Corollary 2.9. (i) *$\pi_I^{-1}(0)\cap\hat{R}_F=I\hat{R}_F$ for every $I\in F$.*

(ii) *$\hat{R}_F$ is a complete topological ring in the topology which has the set $\{I\hat{R}_F\mid I\in F\}$ as neighborhoods of zero.*

Proof. (i) Clearly $\pi^{-1}(0)\cap\hat{R}_F\supseteqq I\hat{R}_F$. By Corollary 2.8, there exists a right ideal $J\supseteqq I$ such that $J/I\cong[\pi_I^{-1}(0)\cap\hat{R}_F]/I\hat{R}_F$, i.e., $\pi_I^{-1}(0)\cap\hat{R}_F=I\hat{R}+J=J\hat{R}_F$, because $\pi_I^{-1}(0)\cap\hat{R}_F$ is an $\hat{R}_F$-module. From this fact we easily obtain that $J=I$ and so $\pi_I^{-1}(0)\cap\hat{R}_F=I\hat{R}_F$.

(ii) For any $\hat{x}\in\hat{R}_F$, we define $\hat{x}^{-1}(I\hat{R}_F)=\{\hat{r}\in\hat{R}_F\mid\hat{x}\hat{r}\in I\hat{R}_F\}$, where $I\in F$. Then we have the natural isomorphisms $\hat{R}_F/\hat{x}^{-1}(I\hat{R}_F)\overset{\theta}{\cong}(\hat{x}\hat{R}_F+I\hat{R}_F)/I\hat{R}_F\overset{\varphi}{\cong}J/I$ for some $J\supseteqq I$. Define $\varphi\theta([1+\hat{x}^{-1}(I\hat{R}_F)]=[a+I](a\in J)$. Then $J=aR+I$ and so $J/I\overset{\eta}{\cong}R/a^{-1}I$, where $\eta([a+I])=[1+a^{-1}I]$. Therefore we get the natural isomorphisms $\hat{R}_F/\hat{x}^{-1}(I\hat{R}_F)\cong R/a^{-1}I\cong\hat{R}_F/(a^{-1}I)\hat{R}_F$. Thus we have $(a^{-1}I)\hat{R}_F=\hat{x}^{-1}(I\hat{R}_F)$. This implies that $\hat{R}_F$ is a topological ring. The completeness of $\hat{R}_F$ follows from Proposition 8 of [13].

Let $\hat{F}=\{I\hat{R}_F\mid I\in F\}$. For any $\hat{R}_F$-module, we can define the concept of $\hat{F}$-linearly compact modules.

Proposition 2.10. *A module is F-linearly compact if and only if it is an $\hat{R}_F$-module and is $\hat{F}$-linearly compact.*

Proof. Assume that M is F-linearly compact. By Lemma 2.4, M is an $\hat{R}_F$-module. Let N be a closed submodule of M. Then N is F-linearly compact by Proposition 3 of [13], and so it is an $\hat{R}_F$-submodule. Hence it is enough to prove that M is a topological $\hat{R}_F$-module. Take $m\in M$, $\hat{r}\in\hat{R}_F$. Then we define $m^{-1}N=\{\hat{s}\in\hat{R}_F\mid m\hat{s}\in N\}$ for any submodule neighborhood N of zero. Since M/N is F-torsion, we have $m^{-1}N\in\hat{F}$. Further we have $(m+N)(\hat{r}+m^{-1}N)\subseteqq m\hat{r}+N$ and so M is a topological $\hat{R}_F$-module. Conversely assume that M is $\hat{F}$-linearly compact as an $\hat{R}_F$-module. Let $\{N_\alpha\}$ be the set of submodule neighborhoods of zero. Then, by a similar way as in Lemmas 2.1, 2.4 and Corollaries 2.2, 2.8, we have $M=\varprojlim M/N_\alpha$ and M/N_α is F-torsion and artinian. Thus M is F-linearly compact.

Let M be F-linearly compact. M^* will mean the module of all continuous homomorphisms from M into K_F, where K_F has been awarded the discrete topology. It is evident that an element $f\in\mathrm{Hom}(M,K_F)$ is continuous if and only if $\mathrm{Ker}\,f$ is open.

Lemma 2.11. *Let M be F-linearly compact. Then*

(i) *M^* is an $\hat{R}_{F_l}$-module.*

(ii) *Let N^* be a finitely generated left $\hat{R}_{F_l}$-submodule of M^* and let $g\in \mathrm{Hom}_{\hat{R}_{F_l}}(M^*, K_F)$. Then there exists an element $m\in M$ such that $(f)g=f(m)$ for every $f\in N^*$.*

Proof. (i) For $f\in M^*$ and $\hat{r}\in\hat{R}_{F_l}$, we have $\mathrm{Ker}(\hat{r}f)\supseteqq\mathrm{Ker}\,f$ and so $\hat{r}f\in M^*$.

We shall prove (ii) by Müller's method (cf. Lemma 1 of [8]). Write $N^*=\hat{R}_{F_l}f_1+\cdots+\hat{R}_{F_l}f_n$, where $f_i\in N^*$, and let $W=\{(f_1(m),\cdots,f_n(m))\mid m\in M\}\subseteqq\sum^n\oplus K_F$. Assume that $x=((f_1)g,\cdots,(f_n)g)\notin W$. Then $O(\bar{x})=\{r\in R\mid \bar{x}r=0\}\in F$, where $\bar{x}=[x+W]$ in $\sum\oplus K_F/W$. Hence there exists a map θ: $\bar{x}R\to K_F$ with $\theta(\bar{x})\neq 0$. Since K_F is F-injective, the map θ is extended to the map $\tilde{\theta}$: $\sum\oplus K_F/W\to K_F$. Hence there exists a map φ: $\sum^n\oplus K_F\to K_F$ with $\varphi(x)\neq 0$. From Lemma 2.7 we have $\mathrm{Hom}(\sum\oplus K_F, K_F)=\sum\oplus\hat{R}_{F_l}$ and so $\varphi=(\hat{r}_1,\cdots,\hat{r}_n)$ for some $\hat{r}_i\in\hat{R}_{F_l}$. Thus we get: $0\neq\varphi(x)=\sum_{i=1}^n\hat{r}_i[(f_i)g]=\sum_{i=1}^n(\hat{r}_if_i)g$ and so $\sum_{i=1}^n\hat{r}_if_i\neq 0$. On the other hand, $0=\varphi(w)=\sum_{i=1}^n\hat{r}_if_i(m)$ for every $w=(f_1(m),\cdots,f_n(m))$, where $m\in M$. Hence $\sum_{i=1}^n\hat{r}_if_i=0$, a contradiction.

Let G be a left $\hat{R}_{F_l}$-module. We denote the right module $\mathrm{Hom}_{\hat{R}_{F_l}}(G, K_F)$ by $G^\sharp$, and define its finite topology by taking the submodules $\mathrm{Ann}_{G^\sharp}(N)=\{f\in G^\sharp\mid (N)f=0\}$ as a fundamental system of neighborhoods of zero, where N ranges over all finitely generated $\hat{R}_{F_l}$-submodules of G. The following theorem was proved by I. Kaplansky [4] for modules over commutative, complete discrete valuation rings.

Theorem 2.12. *Let M be an F-linearly compact module. Then M is isomorphic to $M^{*\sharp}$ as topological modules.*

Proof. Let α be the canonical homomorphism from M into $M^{*\sharp}$ which is defined by $\alpha(m)(f)=f(m)$, where $m\in M$ and $f\in M^*$.

(i) First we shall prove that α is a monomorphism. To prove this, we assume that $\alpha(m)=0$ and $0\neq m\in M$. Then there exists an open submodule N with $N\not\ni m$. Let $\bar{m}=[m+N]$ in M/N. Then $O(\bar{m})\in F$ by Lemma 2.1. So we can define a homomorphism f: $\bar{m}R\to K_F$ with $f(\bar{m})\neq 0$. This map can be extended to a homomorphism g from M/N into K_F. Let h: $M\to M/N$ be the natural homomorphism. Then $g\cdot h\in M^*$ and $(g\cdot h)(m)\neq 0$. This implies that $\alpha(m)\neq 0$, a contradiction, and so α is a monomorphism.

(ii) Secondly, we shall prove that α is an epimorphism. Let x be any element of $M^{*\#}$. Then, for every $f\in M^*$, there exists an element $m_f\in M$ such that $(f)x=f(m_f)$ by Lemma 2.11. We consider the congruences

$$(1) \qquad x\equiv m_f(\operatorname{Ker} f)\,.$$

Again, by Lemma 2.11, any finite number of congruences (1) have a solution. Further Ker f is open and so it is closed. By F-linearly compactness of M, there exists a solution $m\in M$. Hence $(f)x=f(m_f)=f(m)$ for every $f\in M^*$ and so $x=\alpha(m)$.

(iii) Finally we shall prove that α is a topological isomorphism. Let S be any submodule neighborhood of zero in the finite topology. Then $S=\operatorname{Ann}_{M^{*\#}}(f_1)\cap\cdots\cap\operatorname{Ann}_{M^{*\#}}(f_n)$, where $f_i\in M^*$. It is evident that $S=\operatorname{Ker} f_1\cap\cdots\cap\operatorname{Ker} f_n$ in M and so it is open in the orginal topology. Conversely, let N be any open submodule in the orginal topology. Then M/N is F-torsion and artinian. So M/N can be embedded in an exact sequence $0\to M/N\xrightarrow{\theta}\sum^n\oplus K_F$ with finite copies of K_F. Let π_i: $\sum^n\oplus K_F\to K_F$ be the projection $(1\leqq i\leqq n)$ and let η: $M\to M/N$ be the natural map. Then we have $N=\cap_{i=1}^n\operatorname{Ker} g_i$, where $g_i=\pi_i\cdot\theta\cdot\eta\in M^*$ and so N is open in the finite topology.

3. In case F is bounded.

A topology F is said to be *bounded* if, for every $I\in F$, there is an nonzero ideal A such that $I\supseteqq A$. When F is bounded, we shall determine, in this section, the algebraic and topological structures of F-linearly compact modules. Let P be a prime ideal of R and let $F_P=\{I\mid I\supseteqq P^n$ for some n, I is a right ideal of $R\}$. Then F_P is a bounded atom in the lattice of all topologies. F_P-linearly compact modules is called *P-linearly compact*. Write $\hat{R}_P=\varprojlim R/P^n$. Then it is evident that $\hat{R}_{F_P}=\hat{R}_P=\hat{R}_{(F_P)_l}$ as rings. It is well-known that $\hat{R}_P$ is a prime, principal ideal ring and that $\hat{P}=P\hat{R}=\hat{R}_PP$, where $\hat{P}$ is the unique maximal ideal of $\hat{R}_P$. In this section, we shall use the following notations: $Q_P=Q_{F_P}$; $K_P=K_{F_P}$; $R(P^n)=e\hat{R}_P/e\hat{P}^n$; $R(P^n)_l=\hat{R}_Pe/\hat{P}^ne$; $R(P^\infty)=\varinjlim e\hat{R}_P/e\hat{P}^n$; $R(P^\infty)_l=\varinjlim \hat{R}_Pe/\hat{P}^ne$, where e is a uniform idempotent in $\hat{R}_P$. First we shall study P-linearly compact modules.

Lemma 3.1. *$Q\otimes\hat{R}_P$ is the quotient ring of $\hat{R}_P$.*

Proof. From the exact sequence $0\to R\to\hat{R}_P\to\hat{R}_P/R\to 0$, we get the exact sequence: $0=\mathrm{Tor}(K_P, \hat{R}_P/R)\to K_P\to K_P\otimes\hat{R}_P\to K_P\otimes\hat{R}_P/R=0$, since $\hat{R}_P/R$ is P-divisible and has no P-primary submodules, and so $K_P\simeq K_P\otimes\hat{R}_P$. Hence we have the exact sequence $0\to\hat{R}_P\to Q\otimes\hat{R}_P\to K_P\to 0$. Thus $Q\otimes\hat{R}_P$ is an essential extension of $\hat{R}_P$ as a right $\hat{R}_P$-module. Since $\hat{P}^n=P^n\hat{R}_P=\hat{R}_PP^n$ and $\hat{R}_P$ is bounded, local, we obtain that $Q\otimes\hat{R}_P$ is divisible as an $\hat{R}_P$-module. Hence $Q\otimes\hat{R}_P$ is an $\hat{R}_P$-injective hull of $\hat{R}_P$. By Theorem of [2, p 69], it is the maximal quotient ring of $\hat{R}_P$ in the sense of [2] and so it is the quotient ring of $\hat{R}_P$.

For an $\hat{R}_P$-module M, we let $M^{\#}=\mathrm{Hom}_{\hat{R}_P}(M, K_P)$.

Lemma 3.2. (i) *$R(P^n)^{\#}\simeq R(P^n)_l$.*
(ii) *$R(P^\infty)^{\#}\simeq\hat{R}_Pe$.*
(iii) *$(e\hat{R}_P)^{\#}\simeq R(P^\infty)_l$.*
(iv) *$[e(Q\otimes\hat{R}_P)]^{\#}\simeq(Q\otimes\hat{R}_P)e$.*
These modules are all P-linearly compact.

Proof. (i) is evident. (ii) $R(P^\infty)^{\#}=[\varinjlim R(P^n)]^{\#}\simeq\varprojlim R(P^n)_l\simeq\hat{R}_Pe$.

(iii) $R(P^\infty)_l$ is F_P-torsion and artinian. Hence it is P-linearly compact and so $R(P^\infty)_l\simeq[R(P^\infty)_l]^{*\#}=(\varinjlim R(P^n)_l)^{*\#}\simeq(\varprojlim R(P^n))^{\#}\simeq(e\hat{R}_P)^{\#}$.

(iv) From the exact sequence $0\to e\hat{R}_P\to e(Q\otimes\hat{R}_P)\to R(P^\infty)\to 0$, we get the exact sequence $0\to\hat{R}_Pe\to[e(Q\otimes R_P)]^{\#}\to R(P^\infty)_l\to 0$ as left $\hat{R}_P$-modules. Let f be any element of $[e(Q\otimes\hat{R}_P)]^{\#}$. Assume that $P^nf=0$ for some n. Then $P^nf(e(Q\otimes\hat{R}_P))=0$ implies that $0=f(e(Q\otimes\hat{R}_P))P^n=f(e(Q\otimes\hat{R}_P))$ and so $f=0$. Hence $[e(Q\otimes\hat{R}_P))]^{\#}$ is torsion-free as a left $\hat{R}_P$-module. Thus $[e(Q\otimes\hat{R}_P)]^{\#}$ is an essential extension of $\hat{R}_Pe$. Hence we may assume that $\hat{R}_Pe\subseteq[e(Q\otimes\hat{R}_P)]^{\#}\subseteq(Q\otimes\hat{R}_P)e$. From Lemma 3.2 of [6], we easily obtain that $[e(Q\otimes\hat{R}_P)]^{\#}=(Q\otimes\hat{R}_P)e$.

By Lemma 2.1, $R(P^n)$ and $R(P^\infty)$ are P-linearly compact in the discrete topology. By Lemma 2.4 and Corollary 2.9, $e\hat{R}_P$ is P-linearly compact in the P-adic topology. $e(Q\otimes\hat{R}_P)$ is a topological module by taking as neighborhoods of zero the submodules $\{e\hat{P}^n \mid n=0, \pm 1, \pm 2, \cdots\}$. Further the exact sequence $0\to e\hat{R}_P\to e(Q\otimes\hat{R}_P)\to R(P^\infty)\to 0$ satisfies the assumption of Proposition 9 of [13] and so $e(Q\otimes\hat{R}_P)$ is P-linearly compact in the above topology.

Lemma 3.3. *Let $0\to L\to M\to N\to 0$ be an exact sequence of $\hat{R}_P$-modules. If the sequence is P^ω-pure in the sense of* [7], *then the exact sequence $0\to N^{\#}\to M^{\#}\to L^{\#}\to 0$ is also P^ω-pure.*

Proof. Since $\hat{R}_P$ is a principal ideal ring, the proof of the lemma is similar to the one of Proposition 44.7 of [3] (see, also Lemma 1.1 of [7]).

Theorem 3.4. (i) *A module is P-linearly compact if and only if it is isomorphic, as topological modules to a direct product of modules of the following types*: $R(P^n)$, $R(P^\infty)$, $e\hat{R}_P$, $e(Q\otimes\hat{R}_P)$, *where* e *is a uniform idempotents in* $\hat{R}_P$ *and the topologies of these modules are defined in the proof of Lemma* 3.2.

(ii) *A module* M *is P-linearly compact, then* M^* *is isomorphic to a direct sum of modules of the following types*: $R(P^n)_l$, $R(P^\infty)_l$, $\hat{R}_Pe$, $(Q\otimes\hat{R}_P)e$, *where* e *is a uniform idempotent in* $\hat{R}_P$.

Proof. (i) Since each of these modules does admit a P-linearly compact topology, the sufficiency is evident from Proposition 1 of [13]. Conversely, let M be P-linearly compact. Then M^* is a left $\hat{R}_P$-module and $\hat{R}_P$ is a complete g-discrete valuation ring in the sense of [6] (cf. p. 432 of [6]). So M^* possesses a basic submodule B by Theorem 3.6 of [6]. Further any finitely generated module and any injective module over a Dedekind prime ring are both a direct sum of indecomposable modules. Hence, from the definition of basic submodules, Corollary 4.4 of [6] and Lemma 3.1 we have $B=\sum_n\oplus\sum\oplus R(P^n)_l\oplus\sum\oplus\hat{R}_Pe$ and $M^*/B=\sum\oplus R(P^\infty)_l\oplus\sum\oplus(Q\otimes\hat{R}_P)e$. By Theorem 1.5 of [7] and Lemmas 3.2, 3.3, the exact sequence $0\rightarrow(M^*/B)^\#\rightarrow M^{*\#}\rightarrow B^\#\rightarrow 0$ splits and so, from Theorem 2.12 and Lemma 3.2, we get:

$$(1) \qquad M\overset{\varphi}{\cong}\prod_n\prod R(P^n)\oplus\prod R(P^\infty)\oplus\prod e\hat{R}_P\oplus\prod e(Q\otimes\hat{R}_P)\,.$$

The right sided module is P-linearly compact and so, by Corollary 2.3, φ is an isomorphism as topological modules.

Since the topology of the left sided of (1) is the product topology, (ii) follows easily from Lemma 3.2.

From Theorem 1.5 of [7], Theorem 3.4 and definitions, we have the following chain of implications;

(P^n-pure injective) $\searrow$
$\qquad\qquad\qquad\qquad$ (P^ω-pure injective) $\Rightarrow$ (P^∞-pure injective).
(P-linearly compact) $\nearrow$

Let F be a bounded topology and let M be F-linearly compact. Then we know from Lemma 2.4 that $M=\varprojlim M_i$, where M_i is F-torsion and artinian. By the same way as in Theorem 3.2 of [5], we have $M_i=\sum\oplus M_{iP}$, where $M_{iP}=\{x\in M_i \mid xP^n=0 \text{ for some } n\}$ and P ranges over all prime ideals contained in F. Write $M_P=\varprojlim M_{iP}$. Then M_P is P-linearly compact and M is isomorphic naturally to $\prod M_P$ as topological modules, where $\prod M_P$ will carry the product topology. It is evident that $K_F=\sum\oplus K_P$, where P ranges over all prime ideals in F. Further we can easily prove that $M^*=\sum\oplus M_P^*$ and that $M^{*\#}=\prod M_P^{*\#}$, where M_P^* consists of all continuous maps of M_P into K_P. Thus, from Theorem 3.4, we have

Theorem 3.5. *Let F be a bounded topology. Then*

(i) *A module is F-linearly compact if and only if it is isomorphic as topological modules to a direct product of modules of the following types*: $R(P^n)$, $R(P^\infty)$, $e_P\hat{R}_P$, $e_P(Q\otimes\hat{R}_P)$, *where P ranges over all prime ideals in F and e_P is a uniform idempotent in $\hat{R}_P$.*

(ii) *If M is F-linearly compact, then M^* is isomorphic to a direct sum of modules of the following types*: $R(P^n)_l$, $R(P^\infty)_l$, $\hat{R}_Pe_P$, $(Q\otimes\hat{R}_P)e_P$.

Let F be any topology. A short exact sequence

$$(E):\ 0\to L\to M\to N\to 0$$

is said to be *F^ω-pure* if $MJ\cap L=LJ$ for every $J\in F_l$. (E) is said to be *F^∞-pure* if the induced sequence $0\to L_F\to M_F\to N_F\to 0$ is splitting exact. A module is called *$F^\omega(F^\infty)$-pure injective* if it has the injective property relative to the class of $F^\omega(F^\infty)$-pure exact sequences. The structure of F^∞-pure injective modules is investigated in the forthcoming paper.

Lemma 3.6. *Let F be a bounded topology. Then (E) is F^ω-pure if and only if (E) is P^ω-pure for every prime ideal $P\in F$.*

Proof. For any prime ideal P, it is clear that $P\in F$ if and only if $P\in F_l$. So the necessity is evident. Conversely assume that (E) is P^ω-pure for $P\in F$. Let J be any element of F_l. Then there is a nonzero ideal A with $J\supseteq A$. Write $A=P_1^{\alpha_1}\cdots P_n^{\alpha_n}$, where P_i are prime ideals. Then $P_i\in F$ and $X/XA\simeq X/XP_1^{\alpha_1}\oplus\cdots\oplus X/XP_n^{\alpha_n}$ for every module X. Hence by Lemma 1.1 of [7] the sequence $0\to L/LA\to M/MA\to N/NA\to 0$ is splitting exact. Hence $MJ\cap L=LJ$ and so (E) is F^ω-pure.

From the same ways as (1.2), (1.4), (1.5) of [7] and Lemma 3.6 we have

Proposition 3.7. *Let F be a bounded topology. Then a module G is F^ω-pure injective if and only if it is isomorphic to the module $E(GF^\omega)\oplus\prod_P\hat{G}_P$, where P ranges over all prime ideals in F, $GF^\omega=\cap GJ(J\in F_l)$ and $\hat{G}_P=\varprojlim G/GP^n$.*

Let F be a bounded topology. Then from Theorem 3.5, Proposition 3.7 and definitions, we get the following chain of implications;

(F-linarly compact) $\Rightarrow$ (F^ω-pure injective) $\Rightarrow$ (F^∞-pure injective).

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