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MODULES OVER DEDEKIND PRIME RINGS II

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This paper is mainly concerned with the investigation of modules over Dedekind prime rings. Throughout this paper R will denote a Dedekind prime ring and P will denote a nonzero prime ideal of R. For an exact sequence (E): $0 \to L \to M \to N \to 0$ of right R-modules, we shall define the concepts of P^n-purity ($n \leqq \omega$), P^∞-purity and T^∞-purity as follows:

(i) (E) is P^n-pure if and only if $MP^m \cap L = LP^m$ for every natural number $m \leqq n$.

(ii) (E) is P^∞-pure if and only if the sequence $0 \to L_P \to M_P \to N_P \to 0$ is splitting exact.

(iii) (E) is T^∞-pure if and only if the sequence $0 \to L_T \to M_T \to N_T \to 0$ is splitting exact,

where M_P is the P-primary submodule of M and M_T is the torsion submodule of M. In case of abelian groups, these purities were discussed by [14] and [6] from the point of view of relative homological algebra. An essential right ideal I of R is said to be completely faithful if R/I is completely faithful (cf. [2]). A torsion R-module M is said to be C-primary if, for every $m \in M$, $mI=0$ for some completely faithful right ideal I of R. By the same way as in (ii) above, we can define the concept of C^∞-purity. The concepts of P^n ($n \leqq \omega$)-pure, P^∞-pure, C^∞-pure and T^∞-pure injective envelopes of R-modules will be introduced by an analogy of pure injective abelian groups (cf. [4]). One of our purposes of this paper is to generalize some results in [6] on these purites in abelian groups to the case of modules over Dedekind prime rings and to determine the structures of these four kinds of pure injective envelopes (Sections 1 and 2). As an application of Sections 1 and 2, we study, in Section 3, relationships between short exact sequences and long exact sequences on the relative homological algebra. Some of results in this section are extensions of those of abelian groups to modules over Dedekind prime rings, and new are the other results. When R is a commutative Dedekind domain, it is well known that $\mathrm{Ext}(M, R)=0$ and R is not cotorsion, then every submodule of M with countable rank is projective. Further if $\mathrm{Ext}(M, R)=0=\mathrm{Hom}(M, R)$, then M is divisible, torsion-free or $M=0$ (cf. [13]). In Section 4, we shall generalize these results to a Dedekind prime ring which is not simple. In Section 5, the concept of a

P-basic submodule of an R-module will be introduced by an analogy of that of an abelian group (cf. [4]). Under the assumption $\dim R = \dim R/P$, we show that any R-module possesses a P-basic submodule and that the dimension of any two P-basic submodules of a module is an invariant for the module. If R is a commutative Dedekind domain, then $\dim R = \dim R/P$.

In an appendix we shall present some elementary facts on cotorsion R-modules which are obtained by modifying the methods used in the corresponding ones on abelian groups. Some of these results are used in this paper.

1. P^n-pure projective and P^n-pure injective modules ($n \leqq \omega$)

Throughout this paper, R will denote a Dedekind prime ring with the two-sided quotient ring Q and $K = Q/R$. By a module we shall understand a unitary right R-module. In place of $\otimes_R$, Hom_R, Ext_R and Tor^R, we shall just write $\otimes$, Hom, Ext, and Tor, respectively. Since R is hereditary, $\mathrm{Tor}_n = 0 = \mathrm{Ext}^n$ for all $n > 1$, and so we shall use Ext for Ext^1 and Tor for Tor_1. Let P be a prime ideal of R and let $\hat{R}_P$ be the completion of R at P in the sense of Goldie [5]. Then $\hat{R}_P = (\hat{D})_k$, where $\hat{D}$ is a complete, discrete valuation ring with a unique maximal ideal $\hat{P}_0$ (cf. Theorem 1.1 of [7]). In particular, $\hat{R}_P$ is a bounded Dedekind prime ring. If M is P-primary, then M is in a natural way an $\hat{R}_P$-module and is torsion as an $\hat{R}_P$-module. So if M is indecomposable, P-primary with $O(M) = P^n$, then M is isomorphic to $e\hat{R}_P/e\hat{P}^n$, where e is a uniform idempotent in $\hat{R}_P$, and we denote it by $R(P^n)$. For any module M and a subset A of R, we define $M[A] = \{m \mid m \in M,\ mA = 0\}$.

A short exact sequence

$$(E)\ 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

of modules is said to be *P^n-pure* if $MP^m \cap f(L) = f(L)P^m$ for every $m \leqq n$, where m and n are natural numbers. (E) is said to be *P^ω-pure* if it is P^n-pure for every natural number n. A module G is said to be *P^n-pure projective* if it has the projective property relative to the class of P^n-pure exact sequences. Similarly, a module I is said to be *P^n-pure injective* if it has the injective property relative to the class of P^n-pure exact sequences. P^ω-pure projective and P^ω-pure injective modules are defined in an obvious way.

Lemma 1.1. *For an extension (E), the following three conditions are equivalent:*

(i) *(E) is P^n-pure.*

(ii) *The sequence $0 \to L[P^n] \xrightarrow{f} M[P^n] \xrightarrow{g} N[P^n] \to 0$ is splitting exact.*

(iii) *The sequence $0 \to L/LP^m \xrightarrow{\bar{f}} M/MP^m \xrightarrow{\bar{g}} N/NP^m \to 0$ is splitting exact for every $m \leqq n$.*

Proof. The equivalence of (i) and (ii) follows from the same argument as in Theorem 5.1 of [13].

(iii) $\Rightarrow$ (i): This is trival.

(i) $\Rightarrow$ (iii): Let $\bar{M}=M/MP^m$ and let $\bar{L}=L/LP^m$. Since $\bar{M}P^m=0$, $\bar{M}$ is an $\hat{R}_P$-module, where $\hat{R}_P=(\hat{D})_k$ and $\hat{D}$ is a complete, discrete valuation ring. Let e_{11} be the matrix unit with 1 in the (1, 1) position and zeros elsewhere. It is evident that $\bar{L}e_{11}$ is pure in $\bar{M}e_{11}$ as a $\hat{D}$-module, and so $\bar{L}e_{11}$ is a direct summand of $\bar{M}e_{11}$ by Theorem 3.12 of [9]. Thus $\bar{L}$ is a direct summand of $\bar{M}$.

Lemma 1.2. *Every module can be embedded as a P^ω-pure submodule in a direct sum of a divisible module and a direct product of the modules $R(P^n)$ $(n=1, 2, \cdots)$.*

Proof. Let $M\subseteqq D$ be modules, where D is divisible. Define $k; M\rightarrow D\oplus\prod_n M/MP^n$: $k(x)=(x, \prod(x+MP^n))$, where $x\in M$. Then it is evident that k is a monomorphism and that M is P^ω-pure in $D\oplus\prod_n M/MP^n$. Since a direct sum of modules is embedded in the direct product of the modules as a P^ω-pure submodule, and M/MP^n is a direct sum of the modules $R(P^m)$ $(1\leqq m\leqq n)$, we obtain that M is embedded as a P^ω-pure submodule in a direct sum of a divisible module and a direct product of the modules $R(P^n)$ $(n=1, 2, \cdots)$.

From Lemmas 1.1 and 1.2, we have

(1.3) A module G is P^n-pure injective if and only if $G=D\oplus T$, D is a divisible module and T is a module with $TP^n=0$ (cf. Theorem 2 of [6]).

(1.4) A module G is P^ω-pure injective if and only if it is a direct summand of a direct sum of a divisible module and a direct product of the modules $R(P^n)$ $(n=1, 2, \cdots)$.

Let M be a P^n-pure submodule of a module G. We call the module G a *P^n-pure essential extension* of M if there are no nonzero submodules $S\subseteqq G$ with $S\cap M=0$ and the image of M is P^n-pure in G/S. By the similar arguments as in §41 of [4], we obtain that maximal P^n-pure essential extensions of M exist and are unique up to isomorphism over M. Further G is a maximal P^n-pure essential extension of M if and only if it is a minimal P^n-pure injective module containing M as a P^n-pure submodule. We may call a minimal P^n-pure injective module containing M as a P^n-pure submodule the *P^n-pure injective envelope* of M. Similarly, we can define the P^n-pure injective envelope of the module.

A module M is said to be *P-divisible* if $MP=M$. The union of all the P-divisible submodules of M is itself P-divisible and will be denoted by MP^∞: if $MP^\infty=0$, then M will be said to be *P-reduced*. We write $MP^\omega=\cap_n MP^n$. In general, for any ordinal α we define $MP^{\alpha+1}=(MP^\alpha)P$, and if α is a limit ordinal, then we define $MP^\alpha=\cap_\beta MP^\beta$ for $\beta<\alpha$. There exists an ordinal τ such that $MP^\tau=MP^{\tau+1}$. It is clear that $MP^\infty=MP^\tau$, Now let $E(MP^{\omega})$ be the injective envelope of MP^{ω} and let $h: M\rightarrow E(MP^{\omega})$ be an extension of the inclusion map $MP^{\omega}\rightarrow E(MP^{\omega})$. Define $k: M\rightarrow E(MP^{n})\oplus M/MP^{n}$: $k(m)=(h(m), m+MP^{n})$, where $m\in M$, and define $g: M\rightarrow E(MP^{\omega})\oplus\hat{M}_{P}$: $g(m)=(h(m), f(m))$, where $\hat{M}_{P}=\varprojlim M/MP^{n}$ and $f: M\rightarrow\hat{M}_{P}$ is the canonical map. Then we have

Theorem 1.5. (i) *The sequence*

$$(1)\qquad 0\rightarrow M\xrightarrow{k}E(MP^{n})\oplus M/MP^{n}\rightarrow \mathit{Coker}\,k\rightarrow 0$$

is a P^{n}-pure injective resolution of M. $E(MP^{n})\oplus M/MP^{n}$ is a P^{n}-pure injective envelope of M and Coker k is divisible.

(ii) *The sequence*

$$(2)\qquad 0\rightarrow M\xrightarrow{g}E(MP^{\omega})\oplus\hat{M}_{P}\rightarrow \mathit{Coker}\,g\rightarrow 0$$

is a P^{ω}-pure injective resolution of M. $E(MP^{\omega})\oplus\hat{M}$ is a P^{ω}-pure injective envelope of M and Coker g is divisible.

Proof. (i) It is clear that k is a monomorphism. First we shall prove that Coker k is divisible. Let $(d, m+MP^{n})$ be any element in $E(MP^{n})\oplus M/MP^{n}$ and let c be any regular element of R. We put $y=m-mc$. Then $(d, m+MP^{n})-(d-h(y), mc+MP^{n})=(h(y), y+MP^{n})\in k(M)$. Let d' be an element of $E(MP^{n})$ with $d-h(y)=d'c$. Then we obtain $(d, m+MP^{n})+k(M)=[(d', m+MP^{n})+k(M)]c$. Hence Coker k is divisible. Using (iii) of Lemma 1.1, we can easily show that the sequence (1) is P^{n}-pure. It remains to show that $E(MP^{n})\oplus M/MP^{n}$ is the P^{n}-pure injective envelope of M. Let $G=D\oplus C$ be the P^{n}-pure injective envelope of M, where D is divisible and $CP^{n}=0$. We may assume that $M\subseteqq G\subseteqq E(MP^{n})\oplus M/MP^{n}$ by the same way as in Lemma 41.3 of [4]. Since $D=GP^{n}$, we have $D\supseteqq D\cap M=GP^{n}\cap M=MP^{n}$ and so $D\supseteqq E(MP^{n})$. Thus $D=E(MP^{n})$, because $E(MP^{n})$ is the maximal divisible submodule of $E(MP^{n})\oplus M/MP^{n}$. Thus we may assume that $C\subseteqq M/MP^{n}$. On the other hand, since M/MP^{n} can be embedded, in a natural way, into $G/D\,(\simeq C)$, we have $C\supseteqq M/MP^{n}$ and thus $C=M/MP^{n}$. Therefore $E(MP^{n})\oplus M/MP^{n}$ is a P^{n}-pure injective envelope of M.

Since $\hat{M}_{P}/f(M)$ is divisible, (ii) follows from the same argument as in (i).

REMARK. The results on P^{n}-pure projective and P^{ω}-pure projective modules are obtained by modifying the methods used in the corresponding ones on abelian groups (cf. Theorems 2, 3, 28 and 31 of [6]). So we shall give these results without the proofs.

(1.6) A module G is P^{n}-pure projective if and only if $G=F\oplus T$, where F is a projective module and T is a module such that $TP^{n}=0$.

(1.7) A module G is P^{ω}-pure projective if and only if it is a direct sum of a projective module and the modules $R(P^n)$ $(n=1, 2, \cdots)$.

Let $F \xrightarrow{f} M \to 0$ be exact with F projective. Define g: $F \oplus M[P^n] \to M$: $g(x, y) = f(x) + y$ $(x \in F$ and $y \in M[P^n])$, and define h: $F \oplus \sum_n \oplus M[P^n] \to M$: $h(x, y) = f(x) + y_1 + \cdots + y_k$, where $x \in F$ and $y = y_1 + \cdots + y_k \in \sum_n \oplus M[P^n]$. Then we have

(1.8) The sequence

$$0 \to \operatorname{Ker} g \to F \oplus M[P^n] \xrightarrow{g} M \to 0$$

is a P^n-pure projective resolution of M.

(1.9) The sequence

$$0 \to \operatorname{Ker} h \to F \oplus \textstyle\sum_n \oplus M[P^n] \xrightarrow{h} M \to 0$$

is a P^{ω}-pure projective resolution of M.

2. S^{∞}-pure projective and S^{∞}-pure injective modules ($S = P$, C, or T)

Let M be a module. M is said to be *completely faithful* if every submodule of every factor module of M is faithful (cf. [2]). An essential right ideal I of R is *completely faithful* if R/I is completely faithful. Let I and J are completely faithful right ideals of R. Then $I \cap J$ and $r^{-1}I = \{x \mid x \in R, rx \in I\}$ are both completely faithful, where $r \in R$. Thus $M_C = \{m \mid m \in M, mI = 0$ for some completely faithful right ideal I of $R\}$ is a submodule of M and it is said to be a *C-primary submodule* of M. We will denote the torsion submodule of M by M_T and will denote the P-primary submodule of M by M_P. By Theorem 1.4 of [8] and Theorem 3.2 of [9], $M_T = M_C \oplus \sum_P \oplus M_P$. Let I be an essential right ideal of R. Define $I^{-1} = \{q \mid q \in Q, qI \subseteqq R\}$. We put $Q_C = \cup I^{-1}$, where I ranges over all completely faithful right ideals of R. By Proposition 5.1 of [8], $Q_C = \cup J^{-1}$, where J ranges over all completely faithful left ideals of R. The union of the submodules P^{-n} of Q for all $n \geqq 0$ will be denoted by Q_P. We will denote the (R, R)-bimodule Q/R by K. It is evident that $K = Q_C/R \oplus \sum_P \oplus Q_P/R$ and that $K_C = Q_C/R$, $K_P = Q_P/R$. We put $\hat{M} = \varprojlim M/MI$, where I ranges over all essential left ideals of R. Then $\hat{R}$ is a ring and $\hat{M}$ is an $\hat{R}$-module (cf. §4 of [15]). Further we can easily see that $\hat{R} = \hat{R}_C \oplus \prod_P \hat{R}_P$ as a ring, where $\hat{R}_C = \varprojlim R/I$, where I ranges over all completely faithful left ideals of R. If M is a C-primary left R-module, then M is a left $\hat{R}_C$-module. A module M is said to be *C-divisible* if $MI = M$ for every completely faithful left ideal I of R. We will denote the maximal C-divisible submodule of M by MC^{∞}; if $MC^{\infty} = 0$, then M is said to be *C-reduced*. We write $MC^1 = \cap MI$, where I ranges over all completely faithful left ideals of R. By induction, we can define the submodule MC^{α} for every ordinal α. There exists an ordinal τ such that $MC^{\tau}=MC^{\tau+1}$. It is evident that $MC^{\infty}=MC^{\tau}$. We put $MT^{1}=\cap MI$, where I ranges over all completely faithful left ideals and all nonzero ideals of R. Similarly, we can define the submodule MT^{α} for every ordinal α. There exists an ordinal σ such that $MT^{\sigma}=MT^{\sigma+1}$. It is evident that MT^{σ} is the maximal divisible submodule of M. We will denote the maximal divisible submodule of M by MT^{∞}.

Let S be any one of the set $\{P, C, T\}$. A short exact sequence $0\to L\to \to M\to N\to 0$ is said to be *S^{∞}-pure* if the sequence $0\to L_S\to M_S\to N_S\to 0$ is splitting exact. In this section, we shall determine the structure of S^{∞}-pure projective and S^{∞}-pure injective modules. For a convenience, we call the torsion submodule M_T of the module M the *T-primary submodule* of M. Let $F\xrightarrow{f} M\to 0$ be exact, where F is projective. Define $g\colon F\oplus M_S\to M\colon g(x, y)=f(x)+y$, where $x\in F$ and $y\in M_S$. Then we have

Theorem 2.1. *Let S be any one of the set $\{P, C, T\}$. Then*

(i) *A module G is S^{∞}-pure projective if and only if $G=H\oplus L$, where H is projective and L is S-primary.*

(ii) *The sequence*

$$0\to \operatorname{Ker} g\to F\oplus M_S\xrightarrow{g} M\to 0$$

is an S^{∞}-pure projective resolution of M (cf. Theorems 8, 10, 11 and 12 of [6]).

Proof. (i) The sufficiency is clear. Conversely suppose that G is S^{∞}-pure projective. The S^{∞}-pure exact sequence

$$0\to G_S\xrightarrow{\alpha} G\xrightarrow{\beta} G/G_S\to 0$$

yields the exact sequence

$$0=\operatorname{Hom}(G_S, X)\to \operatorname{Ext}(G/G_S, X)\xrightarrow{\beta^*}\operatorname{Ext}(G, X),$$

where X is any projective module. Now let $(F)\colon 0\to X\to Z\to G/G_S\to 0$ be any extension of X by G/G_S. Then we consider the following commutative diagram with exact rows:

$$\begin{array}{lccccccccc}
(F\beta)\colon & 0 & \to & X & \to & Y & \to & G & \to & 0\\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle\beta} & & \\
(F)\colon & 0 & \to & X & \to & Z & \to & G/G_S & \to & 0\,.
\end{array}$$

Since (F) is S^{∞}-pure and G is S^{∞}-pure projective, it is evident that $(F\beta)$ splits. Hence $\operatorname{Im}\beta^*=0$ and thus $\operatorname{Ext}(G/G_S, X)=0$ for every projective module X.

Since R is hereditary, this implies that G/G_S is projective. Therefore we have $G=G/G_S\oplus G_S$, as desired.

(ii) This is trivial.

A module G is called *cotorsion* if $\mathrm{Ext}(N, G)=0$ for every torsion-free module N. Since any torsion-free module can be embedded into a direct sum of copies of Q, G is cotorsion if and only if $\mathrm{Ext}(Q, G)=0$. The properties of cotorsion modules are investigated in the appendix and some of these results are needed in this section.

Theorem 2.2. *Let S be a prime ideal P of R or C. Then*

(i) *A module G is S^∞-pure injective if and only if $G=D\oplus H$, where D is divisible, and H is reduced, cotorsion and is an $\hat{R}_S$-module.*

(ii) *A module G is T^∞-pure injective if and only if $G=D\oplus H$, where D is divisible and H is reduced, cotorsion* (cf. Theorems 9 and 10 of [6]).

Proof. (i) Since the proof for the case $S=C$ is the similar to the proof for the case $S=P$, we shall only give the proof for the case $S=P$. First assume that $G=D\oplus H$, where D is divisible, and H is reduced, cotorsion and is an $\hat{R}_P$-module. We shall prove that $\mathrm{Ext}(X, H)=0$ for every module X with $X_P=0$. Since H is reduced and cotorsion, we have $H=\mathrm{Ext}(K_C, H)\oplus \prod_{P_i}\mathrm{Ext}(K_{P_i}, H)$ by (A.4) in the appendix, where P_i ranges over all nonzero prime ideals of R. Since H is an $\hat{R}_P$-module, $HI=H$ for every prime ideal I $(\neq P)$ and for every completely faithful left ideal I of R. Hence $\mathrm{Ext}(K_{P_i}, H)=0=\mathrm{Ext}(K_C, H)$ for every $P_i\neq P$. It is clear that X is embedded in a direct sum of minimal right ideals of Q, copies of K_{P_i} $(P\neq P_i)$ and K_C. So we have $\mathrm{Ext}(X, H)=0$ from the above discussion. Let $0\to A\to Y\to B\to 0$ be any P^∞-pure exact sequence. Then we obtain the following commutative diagram with exact rows and column:

$$\begin{tikzcd}
 & & \mathrm{Ext}(B/B_P, H) \arrow[d] \\
\mathrm{Hom}(Y, H) \arrow[r] \arrow[d] & \mathrm{Hom}(A, H) \arrow[r, "\delta_1"] \arrow[d] & \mathrm{Ext}(B, H) \arrow[d] \\
\mathrm{Hom}(Y_P, H) \arrow[r] & \mathrm{Hom}(A_P, H) \arrow[r, "\delta_2"] & \mathrm{Ext}(B_P, H).
\end{tikzcd}$$

Since $\mathrm{Im}\,\delta_2=0=\mathrm{Ext}(B/B_P, H)$, we have $\mathrm{Im}\,\delta_1=0$. Therefore H is P^∞-pure injective, as desired. Conversely, suppose that $G=D\oplus H$ is P^∞-pure injective, where D is divisible and H is reduced. It is clear that $\mathrm{Ext}(Y, G)=0$ for every torsion-free module Y, and so G is cotorsion. Therefore $H\cong\mathrm{Ext}(K_C, H)\oplus \prod \mathrm{Ext}(K_{P_i}, H)$. Since all extensions of H by K_{P_i} $(P_i\neq P)$ and of H by K_C are P^∞-pure, we obtain $\mathrm{Ext}(K_{P_i}, H)=0=\mathrm{Ext}(K_C, H)$. Hence we have $H\cong\mathrm{Ext}(K_P, H)$.

(ii) follows from the similar arguments as in (i)

Let S be any one of the set $\{P, C, T\}$. A submodule M of a module G is said to be *FS^∞-pure* if G/M has no S-primary submodules. Let M be an FS^∞-pure submodule of G. Then G is called an *FS^∞-pure essential extension of* M if there are no nonzero submodules $L\subseteq G$ with $L\cap M=0$ and the image of M is FS^∞-pure in G/L. Let M be an S^∞-pure submodule of a module G. We call the module G an *S^∞-pure essential extension* of M if there are no nonzero submodules $L\subseteq G$ with $L\cap M=0$ and the image of M is S^∞-pure in G/L.

In the remainder of this section, we shall define an S^∞-pure injective envelope of a module and determine the structure of the S^∞-pure injective envelope of a module. For this purpose we need to extend one of the result of Nunke [13], which is also useful in §3. The exact sequence $0\to R\to Q_S\to K_S\to 0$ yields the exact sequences

$$0 \to \mathrm{Tor}(M, K_S) \to M \xrightarrow{f} M\otimes Q_S\,,$$

$$\mathrm{Hom}(K_S, M) \to \mathrm{Hom}(Q_S, M) \xrightarrow{g} M \to \mathrm{Ext}(K_S, M)\,,$$

where $f(m)=m\otimes 1$ and $g(\alpha)=\alpha(1)$. In particular, if $S=T$, then $Q_T=Q$ and $K=K_T$. A module M is said to be *T-reduced* if it is reduced.

Proposition 2.3. *Let S be any one of the set $\{P, C, T\}$. Then* (i) *$Ker f=M_S$.* (ii) *$Im\, g=MS^\infty$. If $M_S=0$, then g induces an isomorphism*

$$Hom(Q_S, M)\cong MS^\infty.$$

The module M is S-reduced if and only if $Hom(Q_S, M)=0$.

Proof. (i) follows from the similar way as in (a) of Theorem 3.2 of [13].

In order to prove (ii) we need two lemmas.

Lemma 2.4. *Let $0\to L\to M\to N\to 0$ be a C^∞-pure exact sequence, then $MJ\cap L=LJ$ for every completely faithful left ideal J of R.*

Proof. This is clear from the definition of C^∞-purity and the similar arguments as in Lemma 5.2 of [13].

Lemma 2.5. *Let S be a prime ideal P of R or C. If M is S-divisible, then $Ext(K_S, M)=0=Ext(Q_S, M)$ and $Im\, g=M$.*

Proof. Since the proof for the case $S=C$ is the similar to the one for the case $S=P$, we shall only give the proof for the case $S=P$. From the exact sequence $0\to R\to Q_P\to K_P\to 0$, we obtain the exact sequence $\mathrm{Ext}(K_P, M)\to \mathrm{Ext}(Q_P, M)\to 0$. Hence it suffice to prove that $\mathrm{Ext}(K_P, M)=0$. First, if $M=\sum\oplus Q_P$, then the exact sequence $0\to\sum\oplus Q_P\to\sum\oplus Q\to\sum\oplus Q/Q_P\to 0$ yields the exact sequence

$$0 = \mathrm{Hom}(K_P, \sum\oplus Q/Q_P) \to \mathrm{Ext}(K_P, Q_P) \to \mathrm{Ext}(K_P, Q) = 0$$

(the first term is zero, since $\sum\oplus Q/Q_P$ has no P-primary submodules). Hence $\mathrm{Ext}(K_P, M)=0$. Next, if M is torsion-free, then $0\to M\to M\otimes Q$ is exact. Since M is P-divisible, $MP^{-n}=M$ in $M\otimes Q$ for every n. Hence M is a Q_P-module, because $Q_P=\cup_n P^{-n}$. Thus we obtain an exact sequence $\sum\oplus Q_P\to M\to 0$, and this sequence induces the exact sequence

$$0 = \mathrm{Ext}(K_P, \sum\oplus Q_P) \to \mathrm{Ext}(K_P, M) \to 0\,.$$

Finally if M is arbitrary, then we may assume that M is reduced. It is evident that M_T has no P-primary submodules. Hence $\mathrm{Ext}(K_P, M_T)=0$, since the injective hull of M_T has also no P-primary submodules. Applying $\mathrm{Ext}(K_P,\ )$ to the exact sequence $0\to M_T\to M\to M/M_T\to 0$, we have

$$0 = \mathrm{Ext}(K_P, M_T) \to \mathrm{Ext}(K_P, M) \to \mathrm{Ext}(K_P, M/M_T) = 0\,.$$

Thus we have $\mathrm{Ext}(K_P, M)=0$. The second assertion follows from the following exact sequence:

$$0 \to \mathrm{Hom}(K_P, M) \to \mathrm{Hom}(Q_P, M) \to M \to \mathrm{Ext}(K_P, M) = 0\,.$$

Returning to the proof of the proposition, the exact sequences

$$0 \to MP^\infty \to M \to M/MP^\infty \to 0 \quad \text{and} \quad 0 \to R \to Q_P \to K_P \to 0$$

yield the commutative diagram with exact rows and columns:

$$\begin{array}{ccccc} & \mathrm{Hom}(Q_P, MP^\infty) & \to & MP^\infty & \to \mathrm{Ext}(K_P, MP^\infty) = 0 \\ & \downarrow & & \downarrow & \downarrow \\ & \mathrm{Hom}(Q_P, M) & \to & M & \to \mathrm{Ext}(K_P, M) \\ & \downarrow & & \downarrow & \downarrow \\ 0 = & \mathrm{Hom}(Q_P, M/MP^\infty) & \to & M/MP^\infty & \to \mathrm{Ext}(K_P, M/MP^\infty)\,. \end{array}$$

Since M/MP^∞ is P-reduced, we have $\mathrm{Hom}(Q_P, M/MP^\infty)=0$. From this diagram we easily obtain that $\mathrm{Im}\, g=MP^\infty$.

The proofs for the cases $S=C$ or $S=T$ are similar to the one for the case $S=P$.

Lemma 2.6. *Let $0\to L\to M\to N\to 0$ be an exact sequence such that $MJ\cap L=LJ$ for every completely faithful left ideal J of R. Then the sequence $0\to L_C\to M_C\to N_C\to 0$ is exact.*

Proof. Let I be any completely faithful right ideal of R. Since I^{-1}/R is finitely generated C-primary, there are completely faithful left ideals J_i $(1\leqq i\leqq n)$ such that $I^{-1}/R\cong\sum_{i=1}^n\oplus R/J_i$ by Theorem 3.11 of [2]. On the other hand, by the assumption, we obtain the sequence $0\to L/LJ\to M/MJ\to N/NJ\to 0$ is exact for every completely faithful left ideal J of R, and so the sequence $0 \to L\otimes I^{-1}/R \to M\otimes I^{-1}/R \to N\otimes I^{-1}/R \to 0$ is exact, because $M\otimes R/J \cong M/MJ$ for every left ideal J of R. Thus the sequence $0\to L\otimes K_C \to M\otimes K_C \to N\otimes K_C \to 0$ is exact, since $K_C = \varinjlim I^{-1}/R$, where I ranges over all completely faithful right ideals of R. Hence, by Proposition 2.3, the sequence $0\to L_C \to M_C \to N_C \to 0$ is exact.

Let S be a prime ideal P of R or C and let $M=D\oplus H$ be any module, where D is divisible and H is reduced. We have $MS^\infty = D\oplus HS^\infty$, and so $E(MS^\infty)=D\oplus E(HS^\infty)$. Let f_1: $H\to E(HS^\infty)$ be an extension of the inclusion map $HS^\infty \to E(HS^\infty)$ and let f: $M\to E(MS^\infty)$: $f(d, x)=(d, f_1(x))$, where $d\in D$ and $x\in H$. From the exact sequence $0\to R\to Q_S\to K_S\to 0$, we obtain the map g: $M\to \mathrm{Ext}(K_S, M)$. Define h: $M\to E(MS^\infty)\oplus\mathrm{Ext}(K_S, M)$: $h(m)=(f(m), g(m))$. Then we have

Lemma 2.7. *Let S be a prime ideal P of R or C. Then the exact sequence*

$$(1) \qquad 0\to M \xrightarrow{h} E(MS^\infty)\oplus\mathrm{Ext}(K_S, M)\to \mathrm{Coker}\, h\to 0$$

is FS^∞-pure, and Coker h is divisible.

Proof. Since the proof for the case $S=C$ is similar to the proof for the case $S=P$, we shall only give the proof for the case $S=P$. First we shall prove that Coker h is divisible. From the following commutative diagram, it is evident that $\mathrm{Ext}(K_P, M)/g(M)$ is divisible and has no P-primary submodules.

$$\begin{tikzcd}
M \arrow[r] \arrow[d, equal] & \mathrm{Ext}(K, M) \arrow[r] \arrow[d] & \mathrm{Ext}(Q, M) \arrow[r] \arrow[d] & 0 \\
M \arrow[r, "g"] & \mathrm{Ext}(K_P, M) \arrow[r] & \mathrm{Ext}(Q_P, M) \arrow[r] \arrow[d] & 0 \\
 & & 0. &
\end{tikzcd}$$

Let (d, x) be any element of $E(MP^\infty)\oplus\mathrm{Ext}(K_P, M)$ and let c be any regular element of R. Since $\mathrm{Ext}(K_P, M)/g(M)$ is divisible, there exist $y\in\mathrm{Ext}(K_P, M)$ and $m\in M$ such that $x-yc=g(m)$. Then $(d, x)-(d-f(m), yc)=(f(m), g(m))=h(m)$. Let d_1 be any element of $E(MP^\infty)$ with $d-f(m)=d_1c$. Then we obtain $(d, x)+h(M)=[(d_1, y)+h(M)]c$, as desired. Next we shall prove that the sequence (1) is P^ω-pure. To prove this, let $P^n=Rp_1+\cdots+Rp_l$ and let $h(m)=\sum_{i=1}^l (d_i, x_i)p_i$ be any element of $h(M)\cap[E(MP^\infty)\oplus\mathrm{Ext}(K_P, M)]P^n$. Then $f(m)=\sum_{i=1}^l d_ip_i$ and $g(m)=\sum_{i=1}^l x_ip_i$. Since $\mathrm{Ext}(K_P, M)/g(M)$ has no P-primary submodules, $g(M)$ is a P^∞-pure submodule of $\mathrm{Ext}(K_P, M)$. Therefore $g(m)=\sum_{i=1}^l g(m_i)p_i$ for some $m_i\in M$. By Proposition 2.3, $m-\sum_{i=1}^l m_ip_i\in MP^\infty \subseteq MP^n$. Hence there are elements $m_i'\in M$ such that $m=\sum_{i=1}^l m_i'p_i$. Thus we obtain $h(m)=\sum_{i=1}^{l}h(m_i')p_i\in h(M)P^n$, and so (1) is P^ω-pure. This implies that the sequence

$$0 \to h(M)_P \to [E(MP^\infty)\oplus \mathrm{Ext}(K_P, M)]_P \to (\mathrm{Coker}\ h)_P \to 0$$

is exact. Finally we shall prove that $(\mathrm{Coker}\ h)_P=0$. To prove this, we put $M=D\oplus H$, where D is divisible and H is reduced. Then $MP^\infty=D\oplus HP^\infty$.

It follows immediately that $(HP^\infty)_P=0$, and so $[E(MP^\infty)]_P=D_P$. It is also evident that $g(M)_P=\mathrm{Ext}(K_P, M)_P$. From the exact sequence $0\to HP^\infty\to H\to H/HP^\infty\to 0$, we obtain the exact sequence:

$$0 \to \mathrm{Tor}(HP^\infty, K_P) \to \mathrm{Tor}(H, K_P) \to \mathrm{Tor}(H/HP^\infty, K_P) \to HP^\infty\otimes K_P = 0$$

(The last term is zero, since HP^∞ is P-divisible and K_P is P-primary). Hence, by Proposition 2.3, we have the exact sequence:

$$(2) \qquad 0=(HP^\infty)_P \to H_P \to (H/HP^\infty)_P \to 0\,.$$

Now let (d, x) be any element of $[E(MP^\infty)\oplus\mathrm{Ext}(K_P, M)]_P$. Then we may assume that $x=g(y)$ with $y\in H$. Further we consider the following commutative diagram with exact rows and column:

$$(3) \qquad \begin{array}{ccccc} & & & & 0 \\ & & & & \downarrow \\ \mathrm{Hom}(Q_P, H) & \longrightarrow & H & \longrightarrow & \mathrm{Ext}(K_P, H)\ (=\mathrm{Ext}(K_P, M)) \\ \downarrow & & \downarrow & & \downarrow \\ 0=\mathrm{Hom}(Q_P, H/HP^\infty) & \to & H/HP^\infty & \to & \mathrm{Ext}(K_P, H/HP)^\infty\,. \end{array}$$

By (2) and (3), we may assume that $y\in H_P$. Hence $f(y)=0$, because $E(HP^\infty)_P=0$. It is clear that $g(d)=0$. Therefore we obtain $(d, x)=(f(d), g(d))+(f(y), g(y))\in(h(M))_P$, and so $(\mathrm{Coker}\ h)_P=0$. Thus (1) is FP^∞-pure.

By Lemma 2.7, every module can be embedded as an FS^∞-pure submodule in an S^∞-pure injective module and so we can adapt the Maranda's method (cf. §41 of [4]) to the FS^∞-pure extensions of the module. Thus we obtain the following two results:

(i) Maximal FS^∞-pure essential extensions of the module exist and are unique up to isomorphism.

(ii) Any maximal FS^∞-pure essential extension is S^∞-pure injective. Further we have

Lemma 2.8. *Let S be a prime ideal P of R or C. For any modules $M\subseteq G$, the following three conditions are equivalent*:

(i) *G is a maximal FS^∞-pure essential extension of M.*

(ii) *G is a maximal S^∞-pure essential extension of M.*

(iii) *G is a minimal S^∞-pure injective module containing M as an S^∞-pure submodule.*

Proof. Let N be an S^∞-pure injective module containing M as an FS^∞-pure submodule, and let B be an S^∞ (or an FS^∞)-pure essential extension of M. Then it is evident that the identity map of M can be extended to a monomorphism of B into N. Let G be a maximal FS^∞-pure essential extension of M. If there exists a submodule $0 \neq L \subset G$ with $L \cap M=0$ and the image of M is S^∞-pure in G/L, then we obtain a monomorphism: $G \xrightarrow{f} G/L \to G$, where f is a natural homomorphism and so $L=0$, which is a contradiction. Therefore G is an S^∞-pure essential extension of M. Now from the above discussions, the equivalency of (i), (ii) and (iii) are evident.

We may call a minimal S^∞-pure injective module containing a given module M as an S^∞-pure submodule the *S^∞-pure injective envelope of M.*

Now it is easy to characterize the S^∞-pure injective envelope of a module M.

Theorem 2.9. *Let S be a prime ideal P of R or C. Then the sequence*

$$0 \to M \xrightarrow{h} E(MS^\infty) \oplus Ext(K_S, M) \to Coker\ h \to 0$$

is the S^∞-pure injective resolution of M. $E(MS^\infty) \oplus Ext(K_S, M)$ is the S^∞-pure injective envelope of M, and Coker h is divisible, where h is as Lemma 2.7.

Proof. By Lemma 2.7, we may only prove that $E(MS^\infty) \oplus \mathrm{Ext}(K_S, M)$ is the S^∞-pure injective envelope of M. To prove this, suppose that $M \subseteqq B \subseteqq G$, where B is a maximal FS^∞-pure essential extension of M and $G=E(MS^\infty) \oplus \mathrm{Ext}(K_S, M)$. Let $B=D \oplus H$, where D is divisible and H is reduced. Since $HS^\infty=0$ (see Theorem 2.2), it is evident that $D=E(MS^\infty)$. Thus we have $B=E(MS^\infty) \oplus (\mathrm{Ext}(K_S, M) \cap B)$. Since $MS^\infty = BS^\infty \cap M$ and $BS^\infty = GS^\infty = E(MS^\infty)$, we may assume that $\bar{M}=M/MS^\infty \subseteqq \bar{B}=B/BS^\infty \subseteqq \bar{G}=G/GS^\infty$ in a natural way. These inclusions yield the following commutative diagram (see (A.4) in the appendix):

$$\begin{array}{ccccc} \bar{M} & \xrightarrow{\alpha} & \bar{B} & \xrightarrow{\beta} & \bar{G} \\ \downarrow & & \wr\| & & \wr\| \\ \mathrm{Ext}(K_S, \bar{M}) & \xrightarrow{\alpha_*} & \mathrm{Ext}(K_S, \bar{B}) & \xrightarrow{\beta_*} & \mathrm{Ext}(K_S, \bar{G}), \end{array}$$

because $\bar{B}$ and $\bar{G}$ are S-reduced and S^∞-pure injective. On the other hand, since $\bar{G}/\bar{M}$ ($\cong \mathrm{Ext}(K_S, M)/g(M)$) is divisible and has no S-primary submodules, we have an exact sequence $0=\mathrm{Hom}(K_S, \bar{G}/\bar{M}) \to \mathrm{Ext}(K_S, \bar{M}) \xrightarrow{(\beta\alpha)_*} \mathrm{Ext}(K_S, \bar{G}) \to \mathrm{Ext}(K_S, \bar{G}/\bar{M})=0$. Thus we obtain that $\beta_*\alpha_*$ is an isomorphism. Hence β is an isomorphism and so $B=G$.

Let $M=D \oplus H$ be any module, where D is divisible and H is reduced, and let $f: M \to D$ be the extension of the identity map: $D \to D$ such that $f(H)=0$. From the exact sequence $0 \to R \to Q \to K \to 0$, we obtain the map $h: M \to \mathrm{Ext}(K, M)$. Define $g: M \to D \oplus \mathrm{Ext}(K, M)$: $g(m)=(f(m), h(m))$. Then, by the similar argument as in Theorem 2.9, we have

Theorem 2.10. *The sequence*

$$0 \to M \xrightarrow{g} D \oplus Ext(K, M) \to Coker\, g \to 0$$

is a T^{∞}-pure injective resolution of M. $D \oplus Ext(K, M)$ is a T^{∞}-pure injective envelope of M, and Coker g is divisible and torsion-free.

3. Long exact sequences

Let (E): $0 \to L \to M \to N \to 0$ be an extension of L by N. Then the mapping $(E) \to \chi(E)$ establishes a one-to-one correspondence between the equivalence classes of extensions of L by N and the elements of $\mathrm{Ext}(N, L)$, where $\chi(E)=\delta(i)$, δ: $\mathrm{Hom}(L, L) \to \mathrm{Ext}(N, L)$ is the connecting homomorphism defined by (E) and i is the identity endomorphism. Further, Baer multiplication in the equivalence classes of extensions of L by N is carried into the addition in $\mathrm{Ext}(N, L)$ (cf. Theorem 1.1 of [1, ch. XIV]). Let S be any one of the set $\{P^n\ (n \leqq \omega), P^{\infty}, C^{\infty}, T^{\infty}\}$. If a short exact sequence (E): $0 \to L \to M \to N \to 0$ is S-pure, then (E) is said to be an *S-pure extension* of L by N. It is evident that the set of equivalence classes of S-pure extensions of L by N is a subgroup of the equivalent classes of extensions of L by N. We will denote the corresponding subgroup of $\mathrm{Ext}(N, L)$ by $\mathrm{Sext}(N, L)$. First we shall give some elementary facts about $\mathrm{Sext}(N, L)$.

Proposition 3.1. (i) *Let S be any element of the set $\{P^n\ (n \leqq \omega), P^{\infty}, C^{\infty}, T^{\infty}\}$ and let f: $M \to N$ be a homomorphism. Then $f_*\ (Sext(X, M)) \subseteqq Sext(X, N)$ and $f^*\ (Sext(N, X)) \subseteqq Sext(M, X)$ for every module X.*

(ii) *Let S be any element of the set $\{P, C, T\}$. If $N_S=0$ or L is S-divisible, then $Ext(N, L)=S^{\infty}\ ext(N, L)$. In case $S=P$, the converse also holds.*

Proof. (i) follows from the definition and Lemmas 1.1, 2.4.

(ii) If $N_S=0$, then it is clear that $\mathrm{Ext}(N, L)=S^{\infty}\,\mathrm{ext}(N, L)$. If L is S-divisible, then $L_S=0$; because we may assume that L is reduced. Now let $0 \to L \to X \to N \to 0$ be any extension of L by N. From this exact sequence, we obtain the exact sequence:

$$0=\mathrm{Tor}(L, K_S) \to \mathrm{Tor}(X, K_S) \to \mathrm{Tor}(N, K_S) \to L \otimes K_S = 0\,.$$

(The last term is zero, because L is S-divisible). Thus, by Proposition 2.3, the exact sequence $0 \to L \to X \to N \to 0$ is S^{∞}-pure. Therefore $\mathrm{Ext}(N, L)= S^{\infty}\,\mathrm{ext}(N, L)$. Finally, in case $S=P$, assume that $\mathrm{Ext}(N, L)=P^{\infty}\,\mathrm{ext}(N, L)$

and that $N_P \neq 0$. Then N contains a simple, P-primary module. So we may assume that $\sum^n \oplus N \supseteqq P^{-1}/R$. This inclusion map yields the exact sequence $\mathrm{Ext}(\sum \oplus N, L) = P^\infty \mathrm{ext}(\sum \oplus N, L) \to \mathrm{Ext}(P^{-1}/R, L) \to 0$. Hence we have $\mathrm{Ext}(P^{-1}/R, L) = P^\infty \mathrm{ext}(P^{-1}/R, L)$ by (i). Since $P^\infty \mathrm{ext}(P^{-1}/R, L) = 0$, we have $L/LP = 0$ by Proposition 3,2 of [15], and so L is P-divisible.

Now we can proceed as in [4] and [6] to obtain the following fundamental results (cf. Theorem 53.7 of [4] and Theorems 5, 13 of [6]).

Theorem 3.2. *Let S be any one of the set $\{P^n\ (n \leqq \omega), P^\infty, C^\infty, T^\infty\}$. If a short exact sequence*

$$(1) \qquad 0 \to L \to M \to N \to 0$$

is S-pure, then for every module X, the following sequences are exact:

$$Hom(X, N) \xrightarrow{\delta_1} S\,ext(X, L) \to S\,ext(X, M) \to S\,ext(X, N) \to 0\,,$$

$$Hom(L, X) \xrightarrow{\delta_2} S\,ext(N, X) \to S\,ext(M, X) \to S\,ext(L, X) \to 0\,,$$

where δ_i are the connecting homomorphisms induced by (1).

Lemma 3.3. *Let S be any one of the set $\{P, C, T\}$. If $0 \to L \to M \to N \to 0$ is S^∞-pure exact, then $MS^\alpha \cap L = LS^\alpha$ for all ordinals α.*

Proof. First we shall prove that $XS^\alpha \cap X_S = X_S S^\alpha$ for any module X and any ordinal α. The exact sequence $0 \to X_S \to X \to X/X_S \to 0$ is S^∞-pure and so $XS^1 \cap X_S = X_S S^1$ by Lemmas 1.1 and 2.4. Hence, for any ordinal $\beta < \alpha$, we may assume that $XS^\beta \cap X_S = X_S S^\beta$. If α is a limit ordinal, then the assertion is clear from the definition. If α is not a limit ordinal, then $XS^{\alpha-1} \cap X_S = X_S S^{\alpha-1}$. Thus $XS^{\alpha-1}/X_S S^{\alpha-1}$ is a submodule of X/X_S by a natural way. This implies that the exact sequence $0 \to X_S S^{\alpha-1} \to XS^{\alpha-1} \to XS^{\alpha-1}/X_S S^{\alpha-1} \to 0$ is S^∞-pure. Hence we obtain

$$XS^\alpha \cap X_S = XS^\alpha \cap (XS^{\alpha-1} \cap X_S) = (XS^{\alpha-1})S^1 \cap X_S S^{\alpha-1} = X_S S^\alpha\,.$$

Now we shall prove the assertion by induction on α. Assume that $MS^\beta \cap L = LS^\beta$ for every $\beta < \alpha$. If α is a limit ordinal, then it is evident that $MS^\alpha \cap L = LS^\alpha$. If α is not a limit ordinal, then $MS^{\alpha-1} \cap L = LS^{\alpha-1}$ and so $MS^{\alpha-1}/LS^{\alpha-1} \subseteqq M/L$. Thus we have $(MS^{\alpha-1}/LS^{\alpha-1})_S \subseteqq (M/L)_S \cap (M/L)S^{\alpha-1} = [(M/L)_S]S^{\alpha-1}$. So, from the spliteness of the sequence $0 \to L_S \to M_S \to N_S \to 0$, we obtain the spliteness of the sequence $0 \to (LS^{\alpha-1})_S \to (MS^{\alpha-1})_S \to (MS^{\alpha-1}/LS^{\alpha-1})_S \to 0$. Hence we get $MS^\alpha \cap L = MS^\alpha \cap (MS^{\alpha-1} \cap L) = (MS^{\alpha-1})S^1 \cap LS^{\alpha-1} = LS^\alpha$.

Theorem 3.4. *Let S be any element of the set $\{P^n\ (n \leqq \omega), P^\infty, C^\infty, T^\infty\}$*

and let $0 \to M \xrightarrow{f} G \xrightarrow{g} G/M \to 0$ *be an exact sequence. Then* $f(M) \subseteqq GS$ *if and only if the sequence*

$$Ext(X, M) \xrightarrow{f_*} Sext(X, G) \xrightarrow{g_*} Sext(X, G/M) \to 0$$

is exact for every module X. *In particular, if* $S = T^\infty$, *then* $Im\, f_* = 0$.

Proof. If R is the ring of integers and the sequence $0 \to L \to M \to N \to 0$ is pure, then this result was proved by Irwin, Walker and Walker (cf. Theorem 22 of [6]). If $S = P^n$, then the theorem follows from the similar way as in Theorem 22 of [6] replacing integers by the generators of P^n as a left R-module. If $S = P^\infty$, C^∞ or T^∞, then, by the validity of Lemma 3.3, the proof of the sufficiency proceeds just like that of Theorem 22 of [6] did. To prove the necessity we consider the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & GS \arrow[r] \arrow[d] & GS/M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \arrow[r, "f"] & G \arrow[r, "g"] & G/M \arrow[r] & 0 . \end{tikzcd}$$

This diagram yields the commutative diagram:

$$\begin{tikzcd} \mathrm{Ext}(X, M) \arrow[r] \arrow[d, equal] & \mathrm{Ext}(X, GS) \arrow[d] \\ \mathrm{Ext}(X, M) \arrow[r, "f_*"] & \mathrm{Ext}(X, G) . \end{tikzcd} \tag{1}$$

By Proposition 3.1, $\mathrm{Ext}(X, GS) = \mathrm{Sext}(X, GS)$. Hence $\mathrm{Im}\, f_* \subseteqq \mathrm{Sext}(X, G)$. It is clear that $g_*(\mathrm{Sext}(X, G)) \subseteqq \mathrm{Sext}(X, G/M)$. To prove that g_* is an epimorphism, let $0 \to H \to F \to X \to 0$ be an S-pure projective resolution of X. We may assume that H is projective from the construction of H (see Theorem 2.1). From the above exact sequence we obtain the following commutative diagram with the exact first row and columns:

$$\begin{tikzcd} \mathrm{Hom}(H, M) \arrow[r] \arrow[d] & \mathrm{Hom}(H, G) \arrow[r] \arrow[d] & \mathrm{Hom}(H, G/M) \arrow[r] \arrow[d] & 0 \\ \mathrm{Sext}(X, M) \arrow[r] \arrow[d] & \mathrm{Sext}(X, G) \arrow[r] \arrow[d] & \mathrm{Sext}(X, G/M) \arrow[d] & \\ 0 & 0 & 0 & \end{tikzcd}$$

From this diagram, we can easily show that g_* is an epimorphism.

If $S = T^\infty$, then, from the diagram (1), we have $\mathrm{Im}\, f_* = 0$.

Theorem 3.5. *Let* S *be any one of the set* $\{P, C, T\}$ *and let* $0 \to L \xrightarrow{f} M \xrightarrow{g} M/L \to 0$ *be an exact sequence. Then* $M_S \subseteqq f(L)$ *if and only if the sequence*

$$Ext(M/L, X) \to S^{\infty}\,ext(M, X) \to S^{\infty}\,ext(L, X) \to 0$$

is exact for every module X.

Proof. First suppose that $M_S \subseteq f(L)$. Then from the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & M_S & \to & M & \to & M/M_S & \to & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & L & \to & M & \to & M/L & \to & 0\,,
\end{array}$$

we obtain the commutative diagram:

$$\begin{array}{ccc}
\mathrm{Ext}(M/L, X) & \xrightarrow{g^*} & \mathrm{Ext}(M, X) \\
\downarrow & & \| \\
\mathrm{Ext}(M/M_S, X) & \longrightarrow & \mathrm{Ext}(M, X)\,.
\end{array}$$

By Proposition 3.1, $\mathrm{Ext}(M/M_S, X) = S^{\infty}\,\mathrm{ext}(M/M_S, X)$ and so $\mathrm{Im}\, g^* \subseteq S^{\infty}\,\mathrm{ext}(M, X)$. It is clear that $f^*(S^{\infty}\,\mathrm{ext}(M, X)) \subseteq S^{\infty}\,\mathrm{ext}(L, X)$. Finally we shall prove that f^* is an epimorphism. Let $0 \to X \to Y \to Z \to 0$ be an S^{∞}-pure injective resolution of X, where Z is divisible. From this exact sequence, we obtain the following commutative diagram with exact columns:

$$\begin{array}{ccccccc}
\mathrm{Hom}(M/L, Z) & \longrightarrow & \mathrm{Hom}(M, Z) & \longrightarrow & \mathrm{Hom}(L, Z) & \to 0 & \text{(exact)} \\
\downarrow & & \downarrow & & \downarrow & & \\
S^{\infty}\,\mathrm{ext}(M/L, X) & \to & S^{\infty}\,\mathrm{ext}(M, X) & \xrightarrow{f^*} & S^{\infty}\,\mathrm{ext}(L, X) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

From this diagram, we can easily show that f^* is an epimorphism. To prove the sufficiency, let (E): $0 \to H \to F \to M/L \to 0$ be a projective resolution of M/L. By assumption, we obtain the exact sequence:

$$\mathrm{Ext}(M/L, H) \to S^{\infty}\,\mathrm{ext}(M, H) \to S^{\infty}\,\mathrm{ext}(L, H) \to 0\,.$$

Now we consider the following commutative diagram with exact rows:

$$\begin{array}{lccccccccc}
(Eg): & 0 & \to & H & \to & Y & \xrightarrow{\alpha} & M & \to & 0 \\
 & & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle g} & & \\
(E)\;: & 0 & \to & H & \to & F & \xrightarrow{\gamma} & M/L & \to & 0\,,
\end{array}$$

where $M \oplus F \supseteq Y = \{(m, x) \mid \gamma(x) = m + L\}$. Since $H_S = 0$ and $0 \to H_S \to Y_S \to M_S \to 0$ is splitting exact, we obtain the isomorphism α: $Y_S \cong M_S$. Let m be any element of M_S. Then there exists an element $y=(m', x)\in Y_S$ such that $\alpha(y)=m$. Since $\alpha(y)=m'$, we have $m'=m$. Further F is torsion-free, and so we have $x=0$. Hence $m+L=g\alpha(y)=\gamma\beta(y)=0$, and thus $M_S\subseteq f(L)$.

4. On the properties of modules M and N which follow from the relation Ext$(M, N)=0$

For an abelian group M, it is well-known that if $\mathrm{Ext}(M, Z)=0$, then every submodule of M with countable rank is free, and that if $\mathrm{Ext}(M, Z)=0=\mathrm{Hom}(M, Z)$, then $M=0$, where Z is the ring of the rational integers. These results was proved for modules over commutative Dedekind domains by Nunke [13]. In this section we shall extend these results to modules over Dedekind prime rings which are not simple.

Lemma 4.1. *Let M be a torsion-free module. Then every submodule of M with countable dimension is projective if and only if every submodule of M with finite dimension is projective.*

Proof. The necessity is evident. To prove the sufficiency, we can assume that M itself has a countable dimension. We now show that M is projective. Since M has a countable dimension, there are countable infinite uniform submodules $\{U_i\}$ of M such that $M'\supset\sum\oplus U_i$. We put $M_i=(U_1\oplus\cdots\oplus U_i)Q\cap M$. Then it is clear that $M=\cup_i M_i$, $\dim M_i=i$, $\dim M_{i+1}/M_i=1$ and M_{i+1}/M_i is torsion-free. Since M_{i+1} is projective, it is finitely generated. Hence M_{i+1}/M_i is projective by Theorem 3.1 of [9], and thus M is projective.

Lemma 4.2. *Let R be not cotorsion as a right R-module and let R be not simple. If M is a finitely generated, projective left R-module and if M is a left $\hat{R}$-module, then $M=0$.*

Proof. Assume that $M\neq 0$. Then there are finitely generated left free $\hat{R}$-module $F=\hat{R}\oplus\cdots\oplus\hat{R}$ and a left R-module N such that $F\cong M\oplus N$ as a left R-module. Let P be a nonzero prime ideal of R. Then we have ${}_P\hat{F}\cong{}_P\hat{M}\oplus{}_P\hat{N}$. Since $\hat{R}=\hat{R}_C\oplus\prod_P\hat{R}_P$, $P\hat{R}_C=\hat{R}_C$ and $P\hat{R}_{P_i}=\hat{R}_{P_i}$ $(P\neq P_i)$, we obtain that ${}_P(\hat{\hat{R}})\cong\varprojlim\hat{R}_P/P^n\hat{R}_P\cong\hat{R}_P$ by the natural correspondence. So the canonical map $f: F\to{}_P\hat{F}$ is an epimorphism. Hence the restriction map $g=f/M$: $M\to{}_P\hat{M}$ is also epimorphism. On the other hand, since M is projective, g is a monomorphism and thus $M\cong{}_P\hat{M}$. Hence there is a left ideal I of R such that the sequence ${}_P\hat{R}\to I\to 0$ is exact. Hence ${}_P\hat{R}\cong I\oplus L$, where L is a left R-module. By Theorem 2.4 of [2], we have $I\oplus\cdots\oplus I\cong R\oplus J$ for some left ideal J of R. Since ${}_P\hat{I}\cong I$, we have ${}_P\hat{R}\cong R$ as a left R-module. Hence R is complete in the P-adic topology and so R is P^ω-pure injective as a right R-module by Theorem 1.5. Therefore R is cotorsion as a right R-module, which is a contradiction. So we have $M=0$.

Lemma 4.3. *If $Ext(M, N)=0$ and if $NI \neq N$ for every maximal left ideal I of R, then M is torsion-free.*

Proof. Assume that M is not torsion-free. Then M contains a simple module S. It is clear that $S \cong I^{-1}/R$, where I is a maximal left ideal of R. Hence we obtain the exact sequence $0=\mathrm{Ext}(M, N) \to \mathrm{Ext}(I^{-1}/R, N) \to 0$ and so $\mathrm{Ext}(I^{-1}/R, N)=0$. On the other hand, by Proposition 3.2 of [15] $\mathrm{Ext}(I^{-1}/R, N) \cong N/NI$. This is a contradiction from the assumption and so M is torsion-free.

Theorem 4.4. *Let R be not cotorsion as a right R-module and let R be not simple. If $Ext(M, R)=0$, then M is torsion-free and every submodule of M with countable dimension is projective* (cf. Theorem 8.4 of [13]).

Proof. By Lemma 4.3, M is torsion-free. Further, by Lemma 4.1, we need only show that every submodule of M with finite dimension is projective. $\mathrm{Ext}(M, R)=0$ implies that $\mathrm{Ext}(L, R)=0$ for very submodule L of M. So we may assume that M itself has a finite dimension and show that it is projective. If $\dim M=n<\infty$, then there are a finitely generated projective submodule U and a torsion module T such that

$$(1) \qquad 0 \to U \to M \to T \to 0$$

is exact, where $\dim U=\dim M$. The sequence (1) yields the exact sequence as a left R-module:

$$(2) \qquad \mathrm{Hom}(U, R) \to \mathrm{Ext}(T, R) \to \mathrm{Ext}(M, R) = 0\,.$$

Thus $\mathrm{Ext}(T, R)$ is a finitely generated left R-module. Applying $\mathrm{Hom}(T, \ )$ to $0 \to R \to Q \to K \to 0$, we obtain

$$(3) \qquad 0 = \mathrm{Hom}(T, Q) \to \mathrm{Hom}(T, K) \to \mathrm{Ext}(T, R) \to \mathrm{Ext}(T, Q) = 0\,.$$

Hence $\mathrm{Hom}(T, K)$ is finitely generated as a left R-module. Since K is an $\hat{R}$-module, $\mathrm{Hom}(T, K)$ is a left $\hat{R}$-module. If $\mathrm{Hom}(T, K)$ is torsion-free, then $\mathrm{Hom}(T, K)=0$ by Lemma 4.2. If $\mathrm{Hom}(T, K)_T \neq 0$, then $\mathrm{Hom}(T, K)/\mathrm{Hom}(T, K)_T$ is torsion-free and is an $\hat{R}$-module. So it is zero and thus $\mathrm{Hom}(T, K)$ is torsion. By Theorem 3.11 of [2], $\mathrm{Hom}(T, K) \cong R/I_1 \oplus \cdots \oplus R/I_n$, where I_i is an essential left ideal of R. Since $\mathrm{Hom}(R/I, K) \cong I^{-1}/R$ as a right R-module for every essential left ideal I of R, we have $\mathrm{Hom}(\mathrm{Hom}(T, K), K) \cong I_1^{-1}/R \oplus \cdots \oplus I_n^{-1}/R$. Now the map α: $T \to \mathrm{Hom}(\mathrm{Hom}(T, K), K)$ defined by $\alpha(t)(f)=f(t)$, where $t \in T, f \in \mathrm{Hom}(T, K)$, is a homomorphism. It is evident that α is a monomorphism, and so T is finitely generated. Hence M is also finitely generated and thus M is projective.

Theorem 4.5. *Let R be a Dedekind prime ring, let R be not simple and let*

M be a module with $Hom(M, R)=0=Ext(M, R)$. Then

(i) *If R is cotorsion as a right R-module, then M is divisible.*

(ii) *If R is not cotorsion as a right R-module, then* $M=0$ (cf. Theorem 8.5 of [13]).

Proof. By Lemma 4.3, M is torsion-free.

(i) Since M is flat, we get an exact sequence $0\to M\to M\otimes Q\to M\otimes K\to 0$ and so $M\otimes K$ is divisible, torsion. Assume that M is not divisible. Then $M\otimes K\neq 0$ and so there are homomorphisms $f\in \mathrm{Hom}(K, M\otimes K)$ and $g\in \mathrm{Hom}(M\otimes K, K)$ such that $g\circ f\neq 0$. Hence the map α: $\mathrm{Hom}(K, M\otimes K)\to \mathrm{Hom}(K, K)$ defined by $\alpha(h)=gh$ for $h\in \mathrm{Hom}(K, M\otimes K)$, is a nonzero homomorphism. Thus, by (A.6) in the appendix, there exists a homomorphism $0\neq\beta$: $\mathrm{Ext}(K, M)\to \mathrm{Ext}(K, R)=R$. From the exact sequence $0\to R\to Q\to K\to 0$, we obtain the map $\delta: M\to \mathrm{Ext}(K, M)$. Then $\beta\delta\in \mathrm{Hom}(M, R)=0$ and so β induces the map $0\neq\bar{\beta}$: $\mathrm{Ext}(K, M)/\delta(M)\to R$. Since $\mathrm{Ext}(K, M)/\delta(M)$ is divisible, $\bar{\beta}=0$. This is a contradiction and so M is divisible.

(ii) If M is not reduced, then M contains a minimal right ideal eQ of Q as a direct summand. Hence $\mathrm{Ext}(M, R)=0$ implies that $\mathrm{Ext}(Q, R)=0$. This is a contradiction and so M is reduced. Assume that $M\neq 0$. Then by Theorem 4.4 we may assume that $\dim M>\aleph_0$. There is a submodule N of M such that $\dim M/N=1$ and M/N is torsion-free.

If $\mathrm{Hom}(N, R)=0$, then we have the following exact sequence

$$0=\mathrm{Hom}(N, R)\to \mathrm{Ext}(M/N, R)\to \mathrm{Ext}(M, R)=0\,.$$

Hence by Theorem 4.4, M/N is projective and thus $M=N\oplus M/N$. So $\mathrm{Hom}(M, R)\neq 0$, which is a contradiction.

If $\mathrm{Hom}(N, R)\neq 0$, then there is a nonzero homomorphism f: $N\to R$. Since $\hat{R}_P\cong \mathrm{Ext}(K_P, R)$ and the sequence $0\to N\to M\to M/N\to 0$ is P^∞-pure, f can be extended to a homomorphism f: $M\to \hat{R}_P$. So there is a nonzero map g: $M\to R/P^n$ for some n. Applying $\mathrm{Hom}(M, \;)$ to the exact sequence $0\to P^n\to R\to R/P^n\to 0$, we get the exact sequence

$$0=\mathrm{Hom}(M, R)\to \mathrm{Hom}(M, R/P^n)\to \mathrm{Ext}(M, P^n)=0\,.$$

(The last term is zero, since P^n is finitely generated, projective and $\mathrm{Ext}(M, R)=0$). Hence $\mathrm{Hom}(M, R/P^n)=0$, which is a contradiction. Thus we have $M=0$.

Finally we shall study the module M which have the following property: $\mathrm{Ext}(M, T)=0$ for every torsion module T. Modules with this property are dual of cotorsion modules. In case of modules over commutative Dedekind domains, these modules was investigated by Nunke [13]. If R is bounded, then we have

Theorem 4.6. *Let R be a bounded Dedekind prime ring. If $Ext(M, T)=0$ for every torsion module T, then M is torsion-free and every submodule of M with countable dimeniosn is projective* (cf. Theorem 8.4 of [13]).

Proof. By the same way as in Theorem 4.4, we may assume that M itself has a finite dimension, and show that it is projective. Assume that $\dim M=n$. Then there are a finitely generated projective module U and a torsion module T such that

$$(1) \qquad 0 \to U \to M \to T \to 0$$

is exact. Now let $N=\sum\oplus A^{-1}/R$, where A ranges over nonzero ideals of R. From the sequence (1) we get the exact sequence (as a left R-module)

$$(2) \qquad \mathrm{Hom}(U, N) \to \mathrm{Ext}(T, N) \to \mathrm{Ext}(M, N) = 0\,.$$

Since $\mathrm{Hom}(U, N)$ is torsion, we obtain that $\mathrm{Ext}(T, N)$ is also torsion. First we shall prove that $\mathrm{Ext}(T, N)$ is of bounded order as a left R-module. The exact sequence

$$(3) \qquad 0 \to N \to \sum\oplus Q/R \xrightarrow{f} \sum\oplus Q/A^{-1} \to 0$$

is an injective resolution of N. Applying $\mathrm{Hom}(T, \ )$ to the sequence (3), we get an exact sequence:
$\mathrm{Hom}(T, \sum\oplus Q/R) \to \mathrm{Hom}(T, \sum\oplus Q/A^{-1}) \to \mathrm{Ext}(T, N) \to 0$. $\mathrm{Hom}(T, \sum\oplus Q/A)$ and $\mathrm{Hom}(T, Q/A^{-1})$ are both reduced, algebraically compact by the similar way as in Theorem 46.1 of [4]. Thus $\mathrm{Ext}(T, N)$ is of bounded order by (A.1), (A.3), (A.8) and (A.9) in the appendix. Next we shall prove that T is of bounded order. Assume that T is not bounded order. If T is reduced, then by Theorem 3.2 of [9] and Lemma 1.3 of [11], there are submodules $\{T_n,\}$ $\{K_n\}$ of T with the following properties:

(i) $T_1 \subset T_2 \subset \cdots$, and $K_1 \supset K_2 \supset \cdots$,

(ii) $T=T_n \oplus K_n$,

(iii) $O(T_1) \supset O(T_2) \supset \cdots$, and $O(T_n)$ is of bounded order.

Let $O(T_n)=A_n$. Then there is a map $\varphi_n\colon T_n \to Q/A_n^{-1}$ such that the submodule $\varphi_n(T_n)$ has an order A_n (cf. Theorems 3.7 and 3.38 of [9]). It is easily seen that $A\varphi_n \neq \{0\}$ for every ideal A containing A_n and that $0 \neq r\varphi_n$ $(r \in R)$ is not factored $T_n \to \sum\oplus Q/R \xrightarrow{f} \sum\oplus Q/A^{-1}$. Thus we obtain that $\mathrm{Ext}(T, N)$ $(\cong \mathrm{Hom}(T, \sum\oplus Q/A^{-1})/\mathrm{Im} f_*)$ is of unbounded order, which is a contradiction. Thus T is of bounded order. If T is not reduced, then T contains a module of type P^∞ as a direct summand. Now we consider the exact sequence $0 \to P^{-n}/R \to Q_P/R \to Q_P/P^{-n} \to 0$, which is an injective resolution of P^{-n}/R. Let q be a nonzero element of P^{-n}. We define a mapping $q_l\colon K_P$ $(=Q_P/R) \to Q_P/P^{-n}$ by $(x+R) \to (qx+P^{-n})$. Then it is easily verified that q_l is factored $K_P \to Q_P/R$

$\to Q_P/P^{-n}$ if and only if $q \in R$. Thus $\mathrm{Ext}(K_P, P^{-n}/R)$ contains an element of order P^n. Since K_P is a direct sum of a finite copies of the module of type P^∞, $\mathrm{Ext}(T, N)$ is of unbounded order, which is a contradiction. Thus T is of bounded order. Finally, we shall prove that M is projective. Since M is finite dimensional and torsion-free, there is a positive integer m such that $U \subseteq M \subseteq \sum^m \oplus Q$. Since T is of bounded order, there exists a nonzero ideal A of R such that $MA \subseteq U$. Thus we obtain $M \subseteq UA^{-1}$ in $\sum^m \oplus Q$. It is clear that UA^{-1} is finitely generated, and thus M is also finitely generated. So M is projective by Theorem 3.1 of [9].

REMARK. From the proof of Theorem 4.6, we know that if $\mathrm{Ext}(M, N)=0$, where $N=\sum \oplus A^{-1}/R$, A ranges over all nonzero ideals of R and R is bounded, then M is torsion-free and every submodule of M with countable dimension is projective. In case of modules over commutative Dedekind domains, this result was also proved by Nunke [13]. But if R is not bounded, then the above result does not hold. For example, let I be a completely faithful right ideal of R. Then it is evident that $\mathrm{Ext}(R/I, N)=0$ and R/I is not projective.

5. P-basic submodules

Let P be a prime ideal of R. A submodule B of a module M is called a *P-basic submodule* if it satisfies the following conditions:

(i) B is a direct sum of uniform right ideals and modules of type $R(P^n)$ $(n=1, 2, \cdots)$,

(ii) B is P^ω-pure in M,

(iii) M/B is P-divisible.

In this section, we shall show, under the assumption $\dim R = \dim R/P$, that a P-basic submodule of a module exists and that the dimension of any two P-basic submodules of the module is an invariant for the module.

We now give some examples of R satisfying the condition $\dim R = \dim R/P$.

(i) A commutative Dedekind domain R and the total matrix ring over R satisfy the condition.

(ii) If R is a g-discrete valuation ring in the sense of [10], then $\dim R = \dim R/P$.

Let R_P be the local ring of R with respect to P and let $C(P)=\{r \mid rR+P=R, r \in R\}=\{r \mid Rr+P=R\}$. Then R satisfies the Ore condition with respect to $C(P)$ and $R_P=\{ac^{-1} \mid a \in R, c \in C(P)\}$. Further $R^n/P \cong R_P/P'^n$ for every n, where $P'=P_PR=R_PP$ (cf. [8]).

Throughout this section, we assume that $\dim R = \dim R/P$. Then, since $\dim R = \dim R_P$, we have $\dim R_P = \dim R_P/P'$. Thus, by Hilfssatz 3.7 of [12], idempotents in R_P/P' can be lifted to R_P and so R_P is a g-discrete valuation ring in the sense of [10].

Lemma 5.1. *Assume that* $\dim R=\dim R/P$. *If* M *is uniform, P-reduced and torsion-free, then it contains a P-basic submodule.*

Proof. It is evident that a torsion-free module X is P-reduced if and only if $X\otimes R_P$ is reduced as an R_P-module. Thus if M is uniform, P-reduced and torsion-free, then $M\otimes R_P$ is uniform and reduced as an R_P-module. By Lemma 3.3 of [10], $M\otimes R_P\cong eR_P$, where e is a uniform idempotent in R_P. It is evident that eR is a P-basic submodule of eR_P and that $(eR_P/eR)_P=0$. Thus $M\otimes R_P$ has a P-basic submodule N'. Now we let $N=M\cap N'$. Then, since $(M\otimes R_P)/N'\supseteqq M/N$ and $(M\otimes R_P/N')_P=0$, we have the exact sequence $0\to N\to M\to M/N\to 0$ is P^{ω}-pure. Since R is hereditary, N is projective. From the exact sequence $0\to(MP+N)\to M\to M/(MP+N)\to 0$, we have the exact sequence $0\to(MP+N)\otimes R_P\to M\otimes R_P\to M/(MP+N)\otimes R_P\to 0$. Since $(MP+N)\otimes R_P=(M\otimes R_P)P+N\otimes R_P=M\otimes R_P$, we have $M/(MP+N)\otimes R_P=0$ and so $M/(MP+N)=0$, because $M/(MP+N)$ is P-primary. Thus M/N is P-divisible.

Lemma 5.2. *Assume that* $\dim R=\dim R/P$. *If* M *is not a P-divisible module, then there exists a* P^{ω}*-pure, P-reduced and uniform submodule* U *of* M. *Further, if* U *is torsion, then it is a module of type* $R(P^n)$, *and if* U *is torsion-free, then it is projective.*

Proof. We may assume that M is reduced.

(a) If M is torsion-free and P-reduced, then for any uniform submodule V of M, we put $V^*=\{x\in M\mid xP^{\omega}\subseteqq V$ for some $n\}$. It is clear that V^* is P^{ω}-pure in M. By Lemma 5.1, V^* contains a P-basic submodule U. It is evident that U is projective, uniform and P^{ω}-pure in M.

(b) If M is torsion-free and not P-reduced, then $MP^{\infty}\neq 0$. It is evident that M/MP^{∞} is P-reduced. From the proof of Lemma 2.5, MP^{∞} is a Q_P-module and thus $\bar{M}=M/MP^{\infty}$ has no P-primary submodules. Therefore $\bar{M}$ is torsion-free, because $\bar{M}$ is P-reduced. By (a), there exists a P^{ω}-pure, uniform and projective submodule $\bar{U}$ of $\bar{M}$. Let N be the inverse image in M of $\bar{U}$. Then $N=MP^{\infty}\oplus U$ with $U\cong\bar{U}$. It can be easily checked that N is P^{ω}-pure in M, and so U is also P^{ω}-pure in M.

(c) If M is not torsion-free, then $M_T\neq 0$. Suppose that M_P is not divisible. Then, by Theorem 3.24 of [9], M_P contains a module U of type $R(P^n)$ as a direct summand. It is clear that U is P^{ω}-pure in M. Next suppose that M_P is divisible. Then M_T is P-divisible. Applying $\mathrm{Hom}(K_P,\ )$ to the P^{∞}-pure exact sequence $0\to M_T\to M\to M/M_T\to 0$, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0=\mathrm{Ext}(K_P, M_T) & \longrightarrow & \mathrm{Ext}(K_P, M) & \longrightarrow & \mathrm{Ext}(K_P, M/M_T) & \to & 0\\
\| & & \cup\!\| & & \cup\!\| & & \\
0=P^{\infty}\,\mathrm{ext}(K_P, M_T) & \to & P^{\infty}\,\mathrm{ext}(K_P, M) & \to & P^{\infty}\,\mathrm{ext}(K_P, M/M_T) & \to & 0\,.
\end{array}$$

If M/M_T is P-divisible, then, from the above diagram and Proposition 3.1, it follows that M is also P-divisible, which is a contradiction. Hence $\tilde{M}=M/M_T$ is not P-divisible. By (a) or (b), there exists a P^ω-pure, uniform and projective submodule $\tilde{U}$ of $\tilde{M}$. Let N be the inverse image in M of $\tilde{U}$. Then $N=M_T\oplus U$ and $U\cong\tilde{U}$. We can easily prove that N is P^ω-pure in M, and so U is P^ω-pure in M.

Lemma 5.3. *Assume that $\dim R=\dim R/P$. Let S be a P^ω-pure submodule of a module M such that M/S is not P-divisible. Then there exists a uniform submodule U such that $S\cap U=0$ and $S\oplus U$ is again P^ω-pure. If U is torsion-free, then it is projective. If U is torsion, then it is a module of type $R(P^n)$.*

Proof. Let $\bar{M}=M/S$. Then, by Lemma 5.2, there exists a P^ω-pure and uniform submodule $\bar{U}$ of $\bar{M}$. Let N be the inverse image in M of $\bar{U}$. Then the exact sequence $0\to S\to N\to \bar{U}\to 0$ is P^ω-pure. If $\bar{U}$ is projective, then $N=S\oplus U$ and $U\cong\bar{U}$. If $\bar{U}$ is a module of type $R(P^n)$, then, by Lemma 1.1, the sequence splits and so $N=S\oplus U$ and $U\cong\bar{U}$. From the P^ω-purity of $\bar{U}$ and S, we obtain at once that N is again P^ω-pure.

From Lemma 5.3 and Zorn's lemma we have

Theorem 5.4. *Assume that $\dim R=\dim R/P$. Then every module contains a P-basic submodule.*

Let B be a P-basic submodule of a module M. We collect the uniform direct summands of the same order in a decomposition of B, and form their direct sums to obtain

(1) $B=B_0\oplus B_1\oplus B_2\oplus\cdots\oplus B_n\oplus\cdots$, where

(2) B_0 is a direct sum of uniform right ideals and $B_n=\sum\oplus R(P^n)$.

Now the proof of the following theorem proceeds as that of Theorem 32.4 of [4] replacing the prime integer p^n by the generators of P^n as a left R-module.

Theorem 5.5. *Assume that $\dim R=\dim R/P$. Let B be a submodule of a module M, and assume that B satisfies the conditions* (1) *and* (2). *Then B is a P-basic submodule of M if and only if it satisfies the following two conditions*:

(i) *B_0 is P^ω-pure in M.*

(ii) *$M=B_1\oplus\cdots\oplus B_n\oplus(B_n^*+MP^n)$ for every n,*

where $B_n^*=B_0\oplus B_{n+1}\oplus B_{n+2}\oplus\cdots$ (cf. Theorem 32.4 of [4]).

Lemma 5.6. *Assume that $\dim R=\dim R/P$. Let $B=B_0\oplus B_1\oplus\cdots\oplus B_n\oplus\cdots$ be a P-basic submodule of M. Then*

(i) $M_P\cap(B_0+MP^n)=M_PP^n$.

(ii) $B_0\cap(M_P+MP^n)=B_0P^n$.

Proof. Since $0\to B_0\to M\to M/B_0\to 0$ is P^ω-pure and $(B_0)_P=0$, we have

$M_P \cong (M/B_0)_P = (M_P + B_0)/B_0$. Hence $[M/(M_P+B_0)]_P = 0$ and so $0 \to (M_P+B_0) \to M \to M/(M_P+B_0) \to 0$ is P^ω-pure. Thus we have $MP^n \cap (M_P+B_0) = (M_P+B_0)P^n$. From this equality, the lemma follows immediately.

Lemma 5.7. *Assume that $\dim R = \dim R/P$. If U is a uniform right ideal of R, then U/UP is a simple R-module.*

Proof. From the exact sequence $0 \to UP \to U \to U/UP \to 0$, we obtain the exact sequence $0 \to UP \otimes R_P \to U \otimes R_P \to (U/UP) \otimes R_P \to 0$. It is clear that $U \otimes R_P$ is reduced and uniform as an R_P-module, and so $U \otimes R_P \cong eR_P$ by Lemma 3.3 of [10]. By Lemma 3.1 of [10], $(U/UP) \otimes R_P$ $(\cong U/UP)$ is a simple R_P-module. Thus U/UP is a simple R-module.

Theorem 5.8. *Assume that $\dim R = \dim R/P$. Let B be a P-basic submodule of a module M, and let $B = B_0 \oplus B_1 \oplus \cdots \oplus B_n \oplus \cdots$ be as in Theorem 5.5. Then*

(i) *$B_P = B_1 \oplus \cdots \oplus B_n \oplus \cdots$ is a basic submodule of M_P and so B_P is unique up to isomorphism.*

(ii) *The dimension of B_0 is an invariant for M.*

Proof. (i) Since M_P is a fully invariant submodule of M, we have $M_P = (B_1 \oplus \cdots \oplus B_n) \oplus [M_P \cap (B_n^* + MP^n)]$, where $B_n^* = B_0 \oplus B_{n+1} \oplus B_{n+2} \oplus \cdots$. By the modular law and Lemma 5.6, we obtain:

$$\begin{aligned} M_P \cap (B_n^* + MP^n) &= [M_P \cap (B_0 + MP^n)] + (B_{n+1} \oplus B_{n+2} \oplus \cdots) \\ &= M_P P^n + (B_{n+1} \oplus B_{n+2} \oplus \cdots) . \end{aligned}$$

Thus $M_P = (B_1 \oplus \cdots \oplus B_n) \oplus [M_P P^n + (B_{n+1} \oplus B_{n+2} \oplus \cdots)]$, and so by Lemma 1.3 of [11], B_P is a basic submodule of M_P.

(ii) Let $B_0 = \sum_{\alpha \in \Lambda} \oplus U_\alpha$, where U_α is a uniform right ideal of R and Λ is an index set. Then $B_0/B_0P = \sum_\alpha \oplus U_\alpha/U_\alpha P$. By Lemma 5.7, $U_\alpha/U_\alpha P$ is a simple R-module, and so it suffice to prove that B_0/B_0P is an invariant for M. Since $M \supseteqq B_0 + (M_P + MP) \supseteqq MP + B = M$, and $B_0 \cap (M_P + MP) = B_0 P$ by Lemma 5.6, we obtain $B_0/B_0P \cong B_0/[B_0 \cap (M_P + MP)] \cong M/(M_P + MP)$. Thus B_0/B_0P is an invariant for M.

Lemma 5.9. *Assume that $\dim R = \dim R/P$.*

(i) *Let $M = \sum_{i=1}^\infty \oplus U_i$, where U_i is a uniform right ideal of R. If R is bounded and P is a unique prime ideal of R, then M contains a P-basic submodule different from M.*

(ii) *Let M be not P-divisible with $M_P = 0$. If R is not bounded or has a prime ideal different from P, then M has at least two P-basic submodules.*

Proof. (i) By assumption, R is a g-discrete valuation ring and so $R = (D)_k$, where D is a discrete valuation ring. Let e_{11} be the matrix unit with 1 in the

(1, 1) position and zeros elsewhere. We note that a submodule B of a given module N is basic if and only if Be_{11} is basic of Ne_{11} as a D-module. Now, $Me_{11}=\sum_{i=1}^{\infty}\oplus U_i e_{11}$ and $U_i e_{11}$ is a uniform right ideal of D and so Me_{11} contains a basic submodule B_0 different from Me_{11} by the same argument as in Lemma 35.1 of [4]. Thus M contains a P-basic submodule B_0R different from M

(ii) Let B be a P-basic submodule of M. Then B is a direct sum of uniform right ideals of R. First assume that R has a prime ideals P' different from P, then $BP'\neq B$, $(B/BP')_P=0$ and B/BP' is P-divisible, because $P+P'=R$. Thus BP' is a P-basic submodule of M different from B. Next assume that R is not bounded. Then R has a completely faithful right ideal. It is evident that $0=\cap I$, where I ranges over all completely faithful maximal right ideals of R (cf. Propositions 3.1 and 3.2 of [8]). So for any uniform right ideal U, there is a completely faithful maximal right ideal I such that $U\not\subseteq I$, and so $0\neq U/(U\cap I)$ is C-primary, because $R/I\cong U/(U\cap I)$. Hence we have a submodule B' of B such that $0\neq B/B'$ is C-primary. Thus B' is a P-basic submodule different from M.

Lemma 5.10. *Assume that $\dim R=R/P$ and that R is bounded with unique prime ideal P. Let M be P-reduced and torsion-free, and let B be a P-basic submodule of M with $\dim B=n<\infty$. If B is only one P-basic submodule of M, then $M=B$.*

Proof. If $B\cap V=0$, where $0\neq V$ is a uniform submodule of M, then, by Lemma 5.1, $V^*=\{x\mid x\in M,\ xP^n\subseteq V$ for some $n\}$ has a P-basic submodule U. Thus there exists a P-basic submodule of M containing U, which is a contradiction, because $V^*\cap B=0$. Hence M is an essential extension of B and so M/B is torsion. Since $M_P=0$ and B is P^{ω}-pure, we have $(M/B)_P=(M/B)_T=0$, because R is a g-discrete valuation ring. Hence $M=B$.

Theorem 5.11. *Assume that $\dim R=\dim R/P$. Let M be a module. Then*

(i) *If P is a unique maximal ideal of R and R is bounded, then M has exactly one P-basic submodule if and only if M is either of the following three types;*

(ii) *If R is not bounded or R has a prime ideal different from P, then M has exactly one P-basic submodule if and only if M is either (a) or (b);*

(a) *M is P-divisible,*

(b) *$M=N\oplus T$, where N is P-divisible with $N_P=0$ and T is a P-primary module with bouunded order,*

(c) *$M=N\oplus T$, where N is projective with finite dimension and T is a P-primary module with bounded order* (cf. Theorem 35.3 of [4]).

Proof. First we note that a P-primary module has only one P-basic submodule if and only if it is either divisible or bounded (cf. Theorem 31.3 of [3]). From this fact and Theorem 5.8, we get that if (a) holds, then 0 is the only

P-basic submodule, and that if (b) holds, then T is the only P-basic submodule. Assume that (c) holds and that P is a unique prime ideal of R and R is bounded. Then for any P-basic submodule B of M, we have $M/B=(M/B)P^n$ $=(NP^n+B)/B$ for some large n, and so M/B is finitely generated and divisible. Thus $M/B=0$ and so $M=B$.

Conversely, assume that M has only one P-basic submodule B. Then $B=B_0\oplus B_P$, where B_0 is a projective module with finite dimension by Lemma 5.9 and B_P is a P-primary module with bounded order. If $B_0=B_P=0$, then we obtain (a). If $B_0=0$ and $B_P\neq 0$, then $M_P=B_P$ is of bounded order. Let $M_PP^n=0$. Then from the P^ω-purity of the sequence $0\to M_P\to M\to M/M_P\to 0$ we obtain $MP^n\cap M_P=M_PP^n=0$. Let $\bar{M}=M/MP^n$ and let $\bar{M}_P=(MP^n\oplus M_P)/MP^n$. Then it is clear that $\bar{M}_P$ is P^ω-pure in $\bar{M}$ and so $\bar{M}=\bar{M}_P\oplus\bar{N}$ by Lemma 1.1. Let N be the inverse image in M of $\bar{N}$. Then we get $M=M_P\oplus N$ and N is P-divisible with $N_P=0$. Finally if $B_0\neq 0$, then we have $M=M_P\oplus N$, where $N_P=0$. It is evident that N has only one P-basic submodule L. Now, if R is bounded or has a prime ideal different from P, then N has P-basic submodules more than two by Lemma 5.9. This is a contradiction. If R is bounded and P is a unique prime ideal of R, i.e., R is a g-discrete valuation ring, then N is torsion-free, because $N_P=N_T$, and NP^∞ is divisible. Hence $\bar{N}=N/NP^\infty$ is also torsion-free. It is evident that $\bar{L}=L\oplus NP^\infty/NP^\infty$ is only one P-basic submodule of $\bar{N}$. Thus, by Lemmas 5.9 and 5.10, we have $\bar{N}=\bar{L}$, i.e., $N=NP^\infty\oplus L$. So it suffices to prove that $NP^\infty=0$. Assume that $0\neq NP^\infty$. Let $L=U_1\oplus\cdots\oplus U_n$, where U_i is a uniform right ideal of R and let $U_1=u_1R+\cdots+u_kR\,(u_i\in U_1)$. Since N is torsion-free, NP^∞ contains a uniform right ideal $V=v_1R+\cdots+v_kR$ such that $U_1\overset{f}{\cong}V$ and $f(u_i)=v_i$. We put $V_1=\sum_{i=1}^k(u_i+v_i)R$ and put $L_1=V_1+U_2+\cdots+U_n$. Then it is clear that $V_1\cong U_1$. Further we can easily prove that $V_1\cap(U_2\oplus\cdots\oplus U_n)=0$ and that $N=NP^\infty\oplus L_1$. Thus L_1 is P-basic of N and $L_1\neq L$. This is a contradiction and so $NP^\infty=0$. Thus we get $M=M_P\oplus N$, where N is finitely generated, projective. If $B_P=M_P$, then we obtain (c). If $B_P=0$ and $M_P\neq 0$, then M_P is divisible.Since U_1/U_1P is a direct sum of simple and P-primary modules, there is a nonzero map $f\colon U_1\to M_P$. We put $f(u_i)=v_i$ $(1\leqq i\leqq k)$, $W_1=\sum_{i=1}^k(u_i+v_i)R$ and $N_1=W_1+U_2+\cdots+U_n$. Then we obtain $M=M_P\oplus N_1$, $N_1\cong N$ and $N_1\neq N$. It is clear that N_1 is P-basic of M. This is a contradiction. This completes the proof of Theorem 5.11.

We denote the cardinal number of a set M by $|M|$.

Theorem 5.12. *Assume that $\dim R=\dim R/P$ for every prime ideal P of R and that R is bounded. If M is a reduced module and if B_P is a P-basic submodule of M for every P, then*

$$|M|\leqq(\textstyle\sum_P|B_P|)^{|R|}.$$

Proof. Let $B=\sum_P B_P$. Then M/B is divisible, and so $M/B=\sum_{\alpha\in\Lambda}\oplus M_\alpha/B$, where $M_\alpha/B\cong R(P^\infty)$ for some prime ideal P of R or M_α/B is isomorphic to a minimal right ideal of Q and Λ is an index set. In both cases, there is an epimorphism $f_\alpha\colon Q\to M_\alpha/B$. Since $Q=\sum_{i\in I}c_i^{-1}R$ where c_i ranges over all regular elements of R and I is an index set. Now we define $i\geqq j$ $(i, j\in I)$ to mean $c_j^{-1}R\supseteqq c_i^{-1}R$. We put $f_\alpha(c_i^{-1})=\bar{x}_{\alpha i}$. For $i\geqq j$ $(i, j\in I)$, there exists a regular element $d_{ij}\in R$ such that $c_i^{-1}=c_j^{-1}d_{ij}$, and so $\bar{x}_{\alpha i}=\bar{x}_{\alpha j}d_{ij}$. Hence there is an element $b_{\alpha,i,j}\in B$ with $x_{\alpha i}-x_{\alpha j}d_{ij}=b_{\alpha,i,j}$. If for $\alpha\neq\beta$ $(\alpha, \beta\in\Lambda)$, the vectors $(\cdots, b_{\alpha,i,j}, \cdots)$ and $(\cdots, b_{\beta,i,j}, \cdots)$ are equal, then we have $x_{\alpha i}-x_{\beta i}=(x_{\alpha j}-x_{\beta j})d_{ij}$ for $i\geqq j$. Let N be the submodule of M which is generated by the elements $\{x_{\alpha i}-x_{\beta i}\,|\,i\in I\}$. To prove that N is divisible, we let $x=(x_{\alpha 1}-x_{\beta 1})r_1+\cdots+(x_{\alpha p}-x_{\alpha p})r_p$ be any element of N and let c be any regular element of R. Since $c_i^{-1}=(cc_i)^{-1}c$, we get $(x_{\alpha i}-x_{\beta i})=(x_{\alpha k_i}-x_{\beta k_i})c$, where $cc_i=c_{k_i}$ $(k_i\in I)$, and so $x\in NcR$. Since R is bounded, N is divisible and thus $N=0$. Hence $|\Lambda|$ does not exceed the cardinality of the set of vectors $(\cdots, b_{\alpha,i,j}, \cdots)$ in B. It is evident that the cardinality of the latter set does not exceed $|B|^{|R|}$. Thus we have

$$|M|=|M/B|\cdot|B|\leqq|\Lambda|\cdot|R|\cdot|B|\leqq|B|^{|R|}\cdot|R|\cdot|B|=|B|^{|R|}.$$

On the other hand, $|B|\leqq|\sum_P\oplus B_P|\leqq(\sum_P|B_P|)^{|R|}$, because the cardinal number of the set of prime ideals of R does not exceed $|R|$. Hence $|B|^{|R|}\leqq(\sum_P|B_P|)^{|R|\cdot|R|}=(\sum_P|B_P|)^{|R|}$, and so $|M|\leqq(\sum_P|B_P|)^{|R|}$.

Appendix

We shall present, in this appendix, some elementary facts about cotorsion modules which are obtained by modifying the methods used in the corresponding ones in abelian groups (cf. [4] and [13]).

(A.1) An epimorphic image of a cotorsion module is cotorsion.

(A.2) A direct product $\prod_\alpha G_\alpha$ is cotorsion if and only if every summand G_α is cotorsion.

(A.3) Let G be reduced and cotorsion. Then a submodule H of G is cotorsion if and only if G/H is reduced.

(A.4) Let S be any one of the set $\{P, C, T\}$. If G is S-reduced and S^∞-pure injective, then $G\cong\mathrm{Ext}(K_S, G)$. In case $S=T$, $G\cong\mathrm{Ext}(K_C, G)\oplus\prod_P\mathrm{Ext}(K_P, G)$.

(A.5) Let M be an (R, R)-bimodule such that M is torsion as a left R-module and let N be a module. Then $\mathrm{Ext}(M, N)$ is reduced and cotorsion.

(A.6) Let M be torsion-free. Then $\hat{M}\cong\mathrm{Hom}(K, M\otimes K)\cong\mathrm{Ext}(K, M)$ and $\hat{M}_P\cong\mathrm{Ext}(K_P, M)$ (cf. also, Theorem 5.4 of [15]).

A reduced and cotorsion module is called *adjusted* if it has no nonzero torsion-free direct summands. Let M be a reduced module. Then the exactness of $0 \to M_T \to M \to M/M_T \to 0$ gives the exactness of

$$(1) \qquad 0 \to \mathrm{Ext}(K, M_T) \to \mathrm{Ext}(K, M) \to \mathrm{Ext}(K, M/M_T) \to 0 .$$

Now $\mathrm{Ext}(K, M/M_T)$ is torsion-free, cotorsion, and $\mathrm{Ext}(K, M_T)$ is adjusted (cf. §55 of [4]). Thus we have

$$(2) \qquad \mathrm{Ext}(K, M) \cong \mathrm{Ext}(K, M_T) \oplus \mathrm{Ext}(K, M/M_T) .$$

For adjusted modules, we have

(A.7) The mapping

$$(3) \qquad T \to \mathrm{Ext}(K, T) = G$$

gives a one-to-one correspondence between all reduced, torsion modules T and all adjusted modules G. The inverse of (3) is given by the correspondence: $G \to G_T$ (cf. Theorem 55.6 of [4]).

For the rest of this appendix we assume that R is bounded.

(A.8) Every algebraically compact module is cotorsion.

(A.9) Let G be a torsion module. Then G is cotorsion if and only if G is of bounded order.

(A.10) Let R be not cotorsion as a right R-module, and let M be a finitely generated module. Then M is cotorsion if and only if it is a torsion module.

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