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Osaka J. Math.
11 (1974), 473–481

LINEAR $SU(n)$-ACTIONS ON COMPLEX PROJECTIVE SPACES

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(Received December 19, 1973)

0. Introduction

Let U_* be the bordism ring of weakly complex manifolds and let G be a compact Lie group. Denote by $SF(G)$, an ideal in U_* of those bordism classes represented by a weakly complex manifold on which the group G acts smoothly without stationary points and the action preserves a weakly complex structure.

For a compact abelian Lie group G the ideal $SF(G)$ was computed by tom Dieck [8]. Such ideals are similarly defined in the bordism ring Ω_* of oriented manifolds and those were computed for certain abelian groups by Floyd [3] and Stong [7]. But it seems that there is no useful method to compute the ideal $SF(G)$ for a non-abelian Lie group G.

First we give an upper bound and a lower bound of $SF(G)$ for any compact Lie group G. To state our result precisely we introduce some notations as follows. Denote by $I(G)$, a set of positive integers such that $n\in I(G)$ if and only if there is an n-dimensional complex G-vector space without G-invariant one-dimensional subspaces, by $m(G)$ the maximum dimension of proper closed subgroups of G, and put

$$n(G) = \dim G - m(G) .$$

It is known that the bordism ring $U_*=\sum_{k\geqslant 0} U_{2k}$ is generated by a set of bordism classes

$$\{[P_n(\boldsymbol{C})],\ [H_{p,q}(\boldsymbol{C})];\ n\geqslant 0,\ p\geqslant q>0\}$$

as a ring. Now we define ideals $L(G)$, $M(G)$ in U_* as follows. Let $L(G)$ be an ideal in U_* generated by a set

$$\{[P_n(\boldsymbol{C})],\ [H_{m+n,n}(\boldsymbol{C})];\ n+1\in I(G),\ m\geqslant 0\}$$

and let

$$M(G) = \sum_{2k\geqslant n(G)} U_{2k} .$$

Then we have following results.

Theorem 0.1. *For any compact Lie group G,*

$$L(G)\subset SF(G)\subset M(G)\,.$$

Corollary. $SF(SU(2))=SF(U(2))=\sum_{n>0} U_{2n}$.

For each positive integer n, $P_n(\boldsymbol{C})$ admits a linear $SU(2)$-action without stationary points, but for example $P_3(\boldsymbol{C})$ does not admit a linear $SU(3)$-action without stationary points. Thus we next consider $SU(3)$-actions on $P_3(\boldsymbol{C})$ and we have a following result. Denote by $hP_3(\boldsymbol{C})$, a compact smooth 6-dimensional manifold homotopy equivalent to $P_3(\boldsymbol{C})$.

Theorem 0.2. (a) *Any smooth $SU(3)$-action on $hP_3(\boldsymbol{C})$ has at least one stationary point.* (b) *Any non-trivial smooth $SU(3)$-action on $hP_3(\boldsymbol{C})$ is equivariantly diffeomorphic to a linear $SU(3)$-action on $P_3(\boldsymbol{C})$.*

1. Weakly complex G-manifolds without stationary points

Let G be a Lie group and V be an n-dimensional complex G-vector space. Denote by $P(V)$ the complex projective space $P_{n-1}(\boldsymbol{C})$ with an induced G-action. We call such a G-action on $P_{n-1}(\boldsymbol{C})$ a linear G-action. Then $P(V)$ is a weakly complex G-manifold in the sense of Conner-Floyd [1]. Denote by $[v]$ a point of $P(V)$ represented by a non-zero vector v of V. Then

Lemma 1.1. *A point $[v]$ of G-manifold $P(V)$ is a stationary point if and only if the vector v spans a G-invariant one-dimensional subspace of V.*

Lemma 1.2. *Any smooth G-action on a manifold M is trivial, if* dim $M<n(G)=\dim G-m(G)$. *Here $m(G)$ is the maximum dimension of proper closed subgroups of G.*

Proof. If $x\in M$ is not a stationary point, then the isotropy subgroup G_x at x is a proper closed subgroup of G, the orbit $G\cdot x$ is a submanifold of M, and $G\cdot x$ is diffeomorphic to the homogeneous space G/G_x. Then

$$\dim M\geqslant\dim(G\cdot x)=\dim G-\dim G_x\geqslant\dim G-m(G)\,.$$

Remark. The integer $m(G)$ was calculated by Mann [5] for compact connected simple Lie groups G, by making use of Dynkin's work [2].

Proof of Theorem 0.1. Let V be an n-dimensional complex G-vector space and W be an m-dimensional complex G-vector space. The canonical G-action on the dual space $V^*=\mathrm{Hom}_{\boldsymbol{C}}(V,\boldsymbol{C})$ is defined by

$$(g\cdot u)(v)=u(g^{-1}\cdot v);\, g\in G,\, u\in V^*,\, v\in V\,.$$

Define

$$H(V\oplus W,\ V^*) = \{([v+w],\ [u])\in P(V\oplus W)\times P(V^*): u(v)=0\}\ ,$$

then $H(V\oplus W,V^*)$ is a manifold $H_{m+n-1,n-1}(\boldsymbol{C})$ with a weakly complex G-action. If V has no G-invariant one-dimensional subspaces, then the G-action on $H(V\oplus W,V^*)$ has no stationary points by Lemma 1.1. Therefore the inclusion $L(G)\subset SF(G)$ is proved. Next the inclusion $SF(G)\subset M(G)$ follows from Lemma 1.2. This completes the proof of Theorem 0.1.

Next we consider the case for $G=SU(n)$, the special unitary group. Let $I(G)$ be the set of positive integers defined in the introduction. Then by definition

(1.3) $$n_1,\ n_2\in I(G) \text{ implies } n_1+n_2\in I(G)\ .$$

Lemma 1.4. *Any binomial coefficient* $\binom{n+k-1}{k}$ *is contained in* $I(SU(n))$ *for* $n\geqslant 2$ *and* $k\geqslant 1$.

Proof. Denote by V_n the complex vector space $\boldsymbol{C}^n$ with the standard $SU(n)$-action. Then the k-th symmetric product $S_k(V_n)$ is irreducible as a complex $SU(n)$-vector space for each $k\geqslant 1$ and

$$\dim{}_{\boldsymbol{C}} S_k(V_n) = \binom{n+k-1}{k}\,.$$

Corollary 1.5. $SF(SU(2))=SF(U(2))=\sum_{n>0} U_{2n}$.

Proof. Since $I(SU(2))=I(U(2))$ consists all positive integers $n\geqslant 2$ by Lemma 1.4,

$$L(SU(2)) = L(U(2)) = \sum_{n>0} U_{2n}\ .$$

On the other hand,

$$M(SU(2)) = M(U(2)) = \sum_{n>0} U_{2n}$$

by the connectivity of $SU(2)$ and $U(2)$.

2. $SU(3)$-actions on $P_3(\boldsymbol{C})$

Let us first recall some basic facts in differentiable transformation groups.

(i) Let G be a compact Lie group acting on a manifold M. Then by averaging an arbitrary given Riemannian metric on M, we may have a G-invariant Riemannian metric on M.

(ii) Let $x\in M$, then the isotropy subgroup G_x acts on a normal vector space N_x of the orbit $G\cdot x$ at x orthogonally; we call it the normal representation of G_x at x and denote by ρ_x.

(iii) (The differentiable slice theorem) Let $E(\nu)$ be the normal bundle of the orbit $G\cdot x=G/G_x$. Then

$$E(\nu)=G\times_{G_x}N_x$$

where G_x acts on N_x via ρ_x. We note that G acts naturally on $E(\nu)$ as bundle mappings and we may choose small positive real number ε such that the exponential mapping gives an equivariant diffeomorphism of the ε-disk bundle of $E(\nu)$ onto an invariant tubular neighborhood of $G\cdot x$. ([6], Lemma 3.1)

(iv) Let $H\subset G$ be a closed subgroup. Denote by (H), the set of all subgroups of G which are conjugate to H in G. We introduce the following partial ordering relation "$<$" by defining $(H_1)<(H_2)$ if and only if there exist $H_1\in(H_1)$ and $H_2\in(H_2)$ such that $H_1\subset H_2$. If M is connected, then there exists an absolute minimal (H) among the conjugate classes in $\{G_x\,|\,x\in M\}$, moreover the set

$$M_{(H)}=\{x\in M\,|\,G_x\in(H)\}$$

is a dense open submanifold. The conjugate class (H) is called the type of principal isotropy subgroups. ([6], (2.2) and (2.4))

Combining (iii) and (iv), we have a following lemma.

Lemma 2.1. *If M is connected, then the normal representation of G_x at $x\in M$ is trivial if and only if G_x is a principal isotropy subgroup.*

Now we consider $SU(3)$-actions. Let H be a closed subgroup of $SU(3)$. Denote by $N(H)$ the normalizer of H in $SU(3)$.

Lemma 2.2. (a) *Let H be a closed connected proper subgroup of $SU(3)$ with* dim $H\geqslant 3$, *then H is conjugate to $SU(2)$, $SO(3)$ or $N(SU(2))$.* (b) *There are isomorphisms, $N(SU(2))/SU(2)\cong S^1$, the circle group; $N(SO(3))/SO(3)\cong Z_3$, the cyclic group of order 3; $N(N(SU(2)))=N(SU(2))$, as the subgroups of $SU(3)$.* (c) *$N(SU(2))$ does not contain any subgroup which is conjugate to $SO(3)$.*

Proof. (a) is proved by considering the structure of Lie algebra of $SU(3)$ and the 3-dimensional unitary representations of $SU(2)$. (b) is proved by direct calculation. (c) is true since $N(SU(2))\subset SU(3)$ is not irreducible but $SO(3)\subset SU(3)$ is irreducible.

REMARK. dim $SU(3)=8$ and dim $SU(2)=$dim $SO(3)=3$.

Lemma 2.3. *Let M be an orientable connected 6-dimensional manifold with smooth $SU(3)$-action. If an isotropy subgroup $SU(3)_x$ is of 3-dimensional, then $SU(3)_x$ is a principal isotropy subgroup.*

Proof. First we may prove that the homogeneous space $SU(3)/SU(3)_x$ is an orientable 5-manifold by Lemma 2.2. Thus the normal bundle $E(\nu)$ of

$SU(3)/SU(3)_x$ is a trivial line bundle, since M and $SU(3)/SU(3)_x$ are orientable. But if the normal representation of $SU(3)_x$ at $x\in M$ is non-trivial, then the normal bundle $E(\nu)$ is non-orientable. This is a contradiction. Therefore the result follows from Lemma 2.1.

Now we consider non-trivial smooth $SU(3)$-actions on $hP_3(\boldsymbol{C})$, a compact 6-dimensional manifold with the homotopy type of $P_3(\boldsymbol{C})$.

Lemma 2.4. (a) *Any isotropy subgroup is of dimension* $\geqslant 3$. (b) $hP_3(\boldsymbol{C})$ *does not admit only one type* (H) *of isotropy subgroups for any proper subgroup* H *of* $SU(3)$.

Proof. If $\dim SU(3)_x \leqslant 1$, then the 6-dimensional manifold $hP_3(\boldsymbol{C})$ contains a submanifold $SU(3)/SU(3)_x$ of dimension $\geqslant 7$. This is a contradiction. Next if $\dim SU(3)_x=2$, then $SU(3)/SU(3)_x$ is an open and closed submanifold of $hP_3(\boldsymbol{C})$. Therefore

$$hP_3(\boldsymbol{C}) = SU(3)/SU(3)_x .$$

By an exact sequence of homotopy groups

$$\pi_2(SU(3)) \to \pi_2(SU(3)/SU(3)_x) \to \pi_1(SU(3)_x) \to \pi_1(SU(3)) ,$$

we obtain $\pi_1(SU(3)_x)=Z$, an infinite cyclic group, since $SU(3)$ is 2-connected. On the other hand, since $\dim SU(3)_x=2$, the identity component of $SU(3)_x$ is isomorphic to a 2-dimensional toral group, and hence $\pi_1(SU(3)_x)=Z\oplus Z$. This is a contradiction. Next we prove (b). It is sufficient to consider the case

$$\dim H = 3 \text{ or } 4 ,$$

by (a) and Lemma 2.2. If $hP_3(\boldsymbol{C})$ admits only one type (H) of isotropy subgroups, then there is a differentiable fibering

$$SU(3)/H \to h_3P(\boldsymbol{C}) \xrightarrow{p} h_3P(\boldsymbol{C})/SU(3) ,$$

and the orbit space $hP_3(\boldsymbol{C})/SU(3)$ is a compact manifold without boundary, by the differentiable slice theorem (iii). First if $\dim H=3$, then the orbit space is of one-dimensional and hence

$$hP_3(\boldsymbol{C})/SU(3) = S^1 .$$

By exact sequences

$$\pi_3(S^1) \to \pi_2(SU(3)/H) \to \pi_2(hP_3(\boldsymbol{C})) \xrightarrow{p_*} \pi_2(S^1) ,$$
$$\pi_2(SU(3)) \to \pi_2(SU(3)/H) \to \pi_1(H) \to \pi_1(SU(3)) ,$$

we obtain $\pi_1(H)=Z$. On the other hand $\pi_1(H)=0$ or Z_2, since $\pi_1(SU(2))=0$

and $\pi_1(SO(3))=Z_2$. This is a contradiction. Next if dim $H=4$, then H is conjugate to $N(SU(2))$ and the orbit space $hP_3(\boldsymbol{C})/SU(3)$ is of 2-dimensional. Since

$$SU(3)/N(SU(2))=P_2(\boldsymbol{C}),$$

there is an exact sequence

$$\pi_1(hP_3(\boldsymbol{C})) \to \pi_1(hP_3(\boldsymbol{C})/SU(3)) \to \pi_0(P_2(\boldsymbol{C})).$$

Thus the orbit space is a simply connected 2-dimensional compact manifold without boundary. Therefore

$$hP_3(\boldsymbol{C})/SU(3) = S^2.$$

Then there is a contradiction in the following exact sequence

$$\pi_4(hP_3(\boldsymbol{C})) \to \pi_4(hP_3(\boldsymbol{C})/SU(3)) \to \pi_3(P_2(\boldsymbol{C})),$$

since $\pi_4(S^2)=Z_2$.

REMARK 2.5. By the above consideration, if there is a smooth $SU(3)$-action on $hP_3(\boldsymbol{C})$ without stationary points, then $hP_3(\boldsymbol{C})$ admits just two types (H) and $(N(SU(2)))$ of isotropy subgroups, where the identity component of H is $SU(2)$.

Proof of Theorem 0.2 (a). If there is a smooth $SU(3)$-action on $hP_3(\boldsymbol{C})$ with just two types (H) and $(N(SU(2)))$ of isotropy subgroups, where the identity component of H is $SU(2)$, then $hP_3(\boldsymbol{C})$ is a special $SU(3)$-manifold in the sense of Hirzebruch-Mayer [4]. Therefore the orbit space $hP_3(\boldsymbol{C})/SU(3)$ is a compact smooth manifold with boundary, and hence

$$hP_3(\boldsymbol{C})/SU(3) = [0,1].$$

Let $p:hP_3(\boldsymbol{C})\to[0,1]$ be a projection and

$$X_0 = p^{-1}\left(\left[0, \frac{1}{2}\right]\right), X_1 = p^{-1}\left(\left[\frac{1}{2}, 1\right]\right).$$

Then X_0 and X_1 are diffeomorphic to the disk bundle of n-fold tensor product of the canonical complex line bundle over $P_2(\boldsymbol{C})$ for certain positive integer n, by the differentiable slice theorem (iii). Therefore $X_0 \cap X_1$ is a 5-dimensional rational homology sphere. Then there is a contradiction in the following exact sequence of cohomology groups with rational coefficients,

$$H^1(X_0 \cap X_1) \to H^2(hP_3(\boldsymbol{C})) \to H^2(X_0)\oplus H^2(X_1) \to H^2(X_0 \cap X_1).$$

Therefore any smooth $SU(3)$-action on $hP_3(\boldsymbol{C})$ has at least one stationary point, by Remark 2.5.

Lemma 2.6. *Consider a non-trivial smooth $SU(3)$-action on a connected 6-dimensional manifold M. Let $x \in M$ be a stationary point. Then the normal representation $SU(3) \to O(6)$ is equivalent to the standard inclusion $SU(3) \subset O(6)$, and $SU(2)$ is a principal isotropy subgroup.*

Proof. This follows from the fact that non-trivial 6-dimensional real representation of $SU(3)$ is isomorphic to the real restriction of the standard 3-dimensional complex representation.

REMARK 2.7. Denote by V_3, the 3-dimensional complex vector space $\boldsymbol{C}^3$ with the standard $SU(3)$-action. Then $P(\boldsymbol{C}^1 \oplus V_3)$ is the complex projective space $P_3(\boldsymbol{C})$ with a non-trivial linear $SU(3)$-action, where the $SU(3)$-action on $\boldsymbol{C}^1$ is trivial. Denote by D^6 the unit disk in V_3. Then there is an equivariant decomposition

$$P(\boldsymbol{C}^1 \oplus V_3) = (SU(3) \underset{N(SU(2))}{\times} D^2) \underset{h}{\cup} D^6,$$

where the $N(SU(2))$-action on D^2 is induced from the standard action of $N(SU(2))/SU(2) = S^1$ on D^2 and h is an equivariant diffeomorphism on boundaries.

Lemma 2.8. *Any equivariant diffeomorphism on ∂D^6 is extendable to an equivariant diffeomorphism on D^6.*

Proof. Since the $SU(3)$-action on ∂D^6 is transitive, it is easy to prove that any equivariant diffeomorphism on ∂D^6 is given by a scalar multiplication

$$(z_1, z_2, z_3) \to (uz_1, uz_2, uz_3),$$

where $(z_1, z_2, z_3) \in \partial D^6$, $u \in \boldsymbol{C}$ and $|u| = 1$. Such a diffeomorphism is canonically extended to an equivariant diffeomorphism on D^6.

Proof of Theorem 0.2 (b). Let $hP_3(\boldsymbol{C})$ admit a non-trivial smooth $SU(3)$-action. Then we can use Lemma 2.6, via Theorem 0.2 (a). Thus $SU(2)$ is a principal isotropy subgroup, and hence the possible types of isotropy subgroups are

$$(SU(2)),\ (N(SU(2)))\ \text{and}\ (SU(3)),$$

by Lemma 2.2 and Lemma 2.3. In any case, $hP_3(\boldsymbol{C})$ becomes a special $SU(3)$-mainfold with the orbit spacc $[0,1]$. If the type $(N(SU(2)))$ does not appear, then $hP_3(\boldsymbol{C})$ is diffeomorphic to $D^6 \cup D^6$. This is a contradiction. Therefore $hP_3(\boldsymbol{C})$ has isotropy subgroups of type $(N(SU(2)))$ and of type $(SU(3))$. Hence, by the differentiable slice theorem (iii), there is an equivariant decomposition

$$hP_3(\boldsymbol{C}) = (SU(3) \underset{N(SU(2))}{\times} D^2) \underset{k}{\cup} D^6,$$

where k is an equivariant diffeomorphism on boundaries. Moreover there is an equivariant diffeomorphism from $hP_3(\boldsymbol{C})$ to $P(\boldsymbol{C}^1 \oplus V_3)$, by making use of Lemma 2.8 and Remark 2.7.

3. Concluding remarks

3.1. If $G=T^n$, the n-dimensional toral group, then it is known that for any smooth G-action on an oriented compact manifold M without boundary, each connected component of the stationary point set M^G is canonically oriented and the index formula

$$I(M) = I(M^G)$$

holds. Thus we ask whether the above is true or not when G is a compact connected Lie group. The answer is no as follows. Denote by $S_k(V_n)$ the k-th symmetric product of V_n which is $\boldsymbol{C}^n$ with the standard $SU(n)$-action. If $n \geqslant 2$ and $n-1<2^a$, then

$$t = \dim_{\boldsymbol{C}} S_{2^a}(V_n)$$

is odd, and there is a linear $SU(n)$-action on $P_{s+t}(\boldsymbol{C})$ with $P_s(\boldsymbol{C})$ as the stationary point set for each integer s. This example shows that the index formula is false for $SU(n)$-actions in general. Similarly we can construct linear $SO(n)$-actions on $P_{s+t}(\boldsymbol{R})$ with $P_s(\boldsymbol{R})$ as the stationary point set. This example shows that there are smooth $SO(n)$-actions for which the stationary point sets are not orientable.

3.2. Let V_n be as above, then $SU(n)$-manifold $P(\boldsymbol{C}^1 \oplus V_n)$ has only one stationary point for each $n \geqslant 2$. Such a phenomenon does not appear for compact G-manifold without boundary when G is an abelian group such as a toral group or a finite cyclic group of prime order.

3.3. Let G be a compact Lie group. Denote by F_A the family of all closed subgroups of G, and by F_P the family of all closed proper subgroups of G. Then there is an exact sequence of bordism modules of weakly complex G-manifolds,

$$\cdots \to U_*(G; F_P) \xrightarrow{i_*} U_*(G; F_A) \xrightarrow{j_*} U_*(G; F_A, F_P) \xrightarrow{\partial_*} U_*(G; F_P) \to \cdots .$$

It is known that i_* is trivial for $G=T^n$ and almost trivial for G a finite cyclic group of prime order. On the other hand, we can prove that i_* is injective when G is a compact connected semi-simple Lie group, by making use of projective space bundles associated to complex G-vector bundles.

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