Bannai, E.
Osaka J. Math.
11 (1974), 401–411

ON MULTIPLY TRANSITIVE PERMUTATION GROUPS I

EIICHI BANNAI*)

(Received October 8, 1973)

Introduction

In [5], M. Hall determined 4-ply transitive permutation groups whose stabilizer of 4 points is of odd order. (See also Nagao [11].) On the other hand, in Bannai [1] and Miyamoto [9], t-ply transitive finite permutation groups in which the stabilizer of t points is of order prime to an odd prime p have been determined for $t=p^2+p$ and $3p$ respectively. The purpose of this seies of notes is to strengthen those results. In this first note, we will improve Lemma 2.1 in Miyamoto [9]. Namely, we will prove the following result.

Theorem 1. *Let p be an odd prime. Then there exists no permutation group G on a set $\Omega=\{1, 2, \cdots, n\}$ which satisfies the following three conditions:*
(i) *G is $(p+2)$-ply transitive, and $n\equiv 2 \pmod p$,*
(ii) *a Sylow p subgroup P_0 of $G_{1,2,\cdots,p+2}$ is semiregular on $\Omega-\{1, 2, \cdots, p+2\}$, and*
(iii) $|P_0|\geqslant p^2$.

Corollary to Theorem 1. *Let p be an odd prime. Let G be a $(2p+2)$-ply transitive permutation group on a set $\Omega=\{1, 2, \cdots, n\}$. If the order of $G_{1,2,\cdots,2p+2}$ is not divisible by p, then G must be $S_n(2p+2\leqslant n\leqslant 3p+1)$ or $A_n(2p+4\leqslant n\leqslant 3p+1)$.*

This corollary is immediately proved by combining Theorem 1 with a result of Miyamoto [9]. To be more precise, if the order of $G_{1,2,\cdots,p+2}$ is not divisible by p^2, then the $2p$-ply transitive group $G_{1,2}$ on $\Omega-\{1, 2\}$ must contain $A^{\Omega-\{1,2\}}$ by the result of Miyamoto [9, §1], and so G must be one of the groups listed in the conclusion of the corollary. If the order of $G_{1,2,\cdots,p+2}$ is divisible by p^2, then the $(p+2)$-ply transitive group $G_{1,2\cdots,i}$ on $\Omega-\{1, 2, \cdots, i\}$ (if $n\equiv i+2 \pmod p$ with $0\leqslant i\leqslant p-1$) satisfies the three conditions of Theorem 1, and we have a contraidction.

In our proof of Theorem 1, the following result is very important. This result is a kind of generalization of a result of Jordan [8, Chap. IV], and will be of independent interest.

*) Supported in part by the Sakkokai Foundation.
Present address: The Ohio State University.

Theorem A. *Let p be an odd prime. Then A_{p+2} (hence S_{p+2}) is not involved in $GL(p, p)$.*

Theorem A will be proved in §1 by exploiting the theory of modular representations of the symmetric groups due to Nakayama [12] together with some other results (theory of projective representations of the symmetric groups due to Schur [13], theory of p groups and so on).

In Appedix, we will discuss some partial generalization of Theorem 1.

Notation. Our notation will be standard. S^{Δ} and A^{Δ} denote the symmetric and alternating groups on a set Δ. If $|\Delta|$, the cardinality of Δ, is m, we denote them by S_m and A_m instead of S^{Δ} and A^{Δ}. If X is a permutation group on a set Ω, and if Δ is a subset of Ω which is fixed as a whole by X, we denote by X^{Δ} the restriction of X to Δ. For a subset $\Delta=\{1, 2, \cdots, i\}$ of Ω, we denote by $X_{1,2,\dots,i}$ the pointwise stabilizer of Δ in X. $GL(m, K)$ denotes the general linear group of dimension m over a field K. $PGL(m, K)$ denotes the projective linear group of dimension m over K, $PGL(m, K)=GL(m, K)/Z(GL(m, K))$, where $Z(GL(m, K))$ denotes the center of $GL(m, K)$. When K is of cardinality p, we denote $GL(m, K)$ by $GL(m, p)$. For a group X, $\mathrm{Aut}(X)$ denotes the automorphism group of X.

1. A_{p+2} is not involved in $GL(p, p)$

The purpose of this section is to prove Theorem A that A_{p+2} is not involved in $GL(p, p)$.

We first remark the following lemma.

Lemma 1. *Theorem A is true for $p=3$ and 5.*

Proof. A_5 is not involved in $GL(3, 3)$, because the order of $GL(3, 3)$ is not divisible by 5. Similarly, A_7 is not involved in $GL(5, 5)$, because the order of $GL(5, 5)$ is not divisible by 7.

From now on, we always assume that $p\geqslant 7$. In case of $p\geqslant 7$, we can prove Theorem A in a little stronger from as in Lemma 4 mentioned later.

Lemma 2. *Let $p\geqslant 7$. Then S_{p+2} is not a subgroup of $GL(p, K)$, where K is an algebraically closed field of characteristic p.*

Proof. We have only to prove that S_{p+2} has no faithful p-modular (absolutely) irreducible representation of degree $\leqslant p$ over K. Lemma 2 will be proved through the following steps (1) and (2).

(1) The degree of any not 1 dimensional ordinary irreducible representation of $S_k (k\geqslant 5)$ is $\geqslant k-1$. Therefore, the degree of any irreducible p-modular representation of S_{p+2} over K which is contained in a p-block of defect 0 is more than p.

The first assertion is immediately proved by using the Schur's recursive formula (a special case of Murnaghan-Nakayama's recursive formula) (see [13, §44]). The last assertion is obvious from an elementary properties of a p-block of defect 0.

(2) The degree of any not 1 dimensional p-modular irreducible representation of S_{p+2} over K which is contained in a p-block of defect 1 is more than p.

By Nakayama [12], we obtain that there exist just two p-blocks of defect 1 for S_{p+2}. Moreover, one block (say B_0) with p-core of type [2] consists of p ordinary irreducible representations $T_{0,r}$ with $0\leqslant r\leqslant p-1$, where $T_{0,r}$ is the representation associated with the Young diagram of type $[p+2]$ (for $r=0$), $[p-r, 3, 1^{r-1}]$ (for $1\leqslant r\leqslant p-3$), $[2, 2, 1^{p-2}]$ (for $r=p-2$) and $[2, 1^p]$ (for $r=p-1$). While, the other block (say B_1) with p-core of type $[1^2]$ consists of p ordinary irreducible representations $T_{1,r}$ with $0\leqslant r\leqslant p-1$, where $T_{1,r}$ is the representation associated with the Young diagram which is obtained by transposing that of $T_{0,p-1-r}$. Also by a result of Nakayama [12], $T_{i,r}$ and $T_{i,r+1}$ ($i=0,1$, $r=0, 1, \cdots, p-1$) have just one p-modular irreducible representation (over K) in common, say, let us denote it by $\phi_{i,r}$ ($0\leqslant r\leqslant p-2$), and $T_{i,r}$ and $T_{i,s}$ with $s>r+1$ have no p-modular irreducible representation in common. That is to say, the Brauer graphs associated with the p-blocks B_i ($i=0, 1$) are trees without branches and their nodes are arranged on natural order on r. (For the definition of Brauer graphs, see, e.g., [3, §68].) Therefore, we can calculate the degree $|\phi_{i,r}|$ of $\phi_{i,r}$ inductively for $r=0, 1, 2, \cdots$ (and for $r=p-2$, $p-3, p-4, \cdots$), because the degree $|T_{i,r}|$ of $T_{i,r}$ is given explicitly by the following formula:
$|T_{i,r}|=(p+2)!/$(the product of all hook lengths of the Young diagram of $T_{i,r}$).

In the case of $p=7$, we can immediately calculate all the values of $|\phi_{i,r}|$ ($i=0, 1, r=0, 1, \cdots, 5$), and we obtain that they are all $\geqslant 8>7$ except $|\phi_{0,0}|$ and $|\phi_{1,5}|$ which are equal to 1. Thus, in the following we may assume that $p\geqslant 11$. Now, we obtain that $|T_{0,0}|=1$, $|T_{0,1}|=\dfrac{(p+2)(p+1)(p-3)}{6}\geqslant p^2$, $|T_{0,p-3}|=\dfrac{(p+2)(p+1)(p-2)(p-3)}{12}\geqslant p^2$, $|T_{0,p-2}|=\dfrac{(p+1)(p-1)}{2}$ and $|T_{0,p-1}|=p+1$. Therefore, we obtain $|\phi_{0,0}|=1$, $|\phi_{0,p-3}|>p$, $|\phi_{0,p-2}|=p+1>p$. Moreover, when $1\leqslant r\leqslant p-4$, we obtain that $|T_{0,r}|/|T_{0,r+1}|=\dfrac{r(r+3)(p-r-2)}{(r+2)(p-r)(p-3-r)}$. Now, we obtain that $\dfrac{1}{p}<\dfrac{r(r+3)(p-r-2)}{(r+2)(p-r)(p-3-r)}<p$ for any $r=1, 2, \cdots, p-2$, and $\dfrac{r(r+3)(p-r-2)}{(r+2)(p-r)(p-3-r)}<1$ when $r\leqslant\dfrac{p-1}{2}$, and >1 when $r\geqslant\dfrac{p+1}{2}$. Therefore, we obtain that $|T_{0,1}|<|T_{0,2}|<\cdots<|T_{0,(p+1)/2}|$, and $|T_{0,(p+1)/2}|>\cdots>|T_{0,p-2}|>|T_{0,p-1}|$. Hence, we obtain that $|\phi_{0,r}|>p$ (for $r=1, 2, \cdots, p-4$), because of the fact that the Brauer graph of the block B_0 is a tree without branches and of natural order on r. Therefore, we obtain that $|\phi_{0,r}|>p$ for any $r\neq 0$ $(1\leqslant r\leqslant p-2)$. Since $|T_{0,r}|=|T_{1,p-1-r}|$ and $|\phi_{0,r}|=|\phi_{1,p-2-r}|$ for any r, we also obtain that $|\phi_{1,r}|>p$ for any $r\neq p-2$ $(0\leqslant r\leqslant p-3)$ and $|\phi_{1,p-2}|=1$. Thus, we have proved the assertion of (2).

Since any p-block of S_{p+2} is either of defect 0 or 1, we have completed the proof of Lemma 2 by (1) and (2).

We also have

Lemma 2′. *Let $p\geqslant 7$. Then A_{p+2} is not a subgroup of $GL(p, K)$, where K is an algebraically closed field of characteristic p.*

Proof. The assertion corresponding to step (1) in Lemma 2 is easily obtained similarly by using the Schur's recursive formula for the characters of the symmetric groups. Namely, we have

(1′) The degree of any (p-modular) irreducible representation of A_{p+2} over K which is contained in a p-block of defect 0 is more than p.

Since $T_{i,r}$ $(i=0, 1, r=0, 1, \cdots, p-1)$ are all irreducible representations of A_{p+2}, we immediately obtain the following assertion.

(2′) The degree of not 1 dimensional (p-modular) irreducible representation of A_{p+2} over K which is contained in a p-block of defect 1 is more than p.

Thus, we have proved Lemma 2′.

Lemma 3. *Let p be an odd prime$\geqslant 7$. Then S_{p+2} is not a subgroup of $PGL(p, K)$, where K is an algebraically closed field of characteristic p.*

Proof. We have only to prove that S_{p+2} has no not 1 dimensional projective irreducible representation of degree $\leqslant p$ over K. Since we have already proved in Lemma 2 that S_{p+2} is not a subgroup of $GL(p, K)$, we have only to prove that S_{p+2} has no projective representation of degree $\leqslant p$ over K which is not a linear representation. As is easily seen from a result of Schur (and a slight extension of it) (cf. Yamazaki [15, §3.3, Corollary 1]), there is a finite group (which is a central extension of S_{p+2} and is called a representation group of S_{p+2} over K) such that any projective representation of S_{p+2} is induced by a linear representation of the representation group. Movreover, by Yamazaki [15, §3, e.g., Proposition 3.3, 2) and Proposition 3.5], we may take as a representation group of S_{p+2} over K the following group T_{p+2} defined by the generators

$$\{J, X_i\ (i=1, 2, \cdots, p+1)\}$$

with the defining relations

$$J^2=1\,,$$
$$X_\alpha^2=J\ (\alpha=1, 2, \cdots, p+1)\,,$$

$$(X_\beta X_{\beta+1})^3 = J\ (\beta = 1, 2, \cdots, p) \quad \text{and}$$
$$X_\gamma X_\delta = J X_\delta X_\gamma\ (\gamma = 1, 2, \cdots, p-1,\ \delta=\gamma+2, \cdots, p+1)\,.$$

(Note that $Z(T_{p+2})=\langle J\rangle$ (which is contained in the commutator subgroup of T_{p+2}) is a cyclic group of order 2, and $T_{p+2}/Z(T_{p+2})=S_{p+1}$. T_{p+2} is the group denoted $\mathfrak{T}_{p+2}$ in Schur [13]. Also note that $H^2(S_{p+2}, K^*)=H^2(S_{p+2}, \boldsymbol{C}^*)=Z_2$.)

The ordinary irreducible representations of T_{p+2} were completely determined by Schur [13]. As in [13], let us call an ordinary irreducible representation of T_{p+2} is of the first kind (resp. of the second kind) if the kernel of the representation contains $Z(T_{p+2})$ (resp. does not contain $Z(T_{p+2})$). The proof of Lemma 3 will be done through the following steps (1), (2) and (3).

(1) The degree of any ordinary irreducible representation of T_{p+2} of the second kind is more than $2^{[(p+1)/2]}$. Moreover $2^{[(p+1)/2]}>p$.

The degree of any ordinary irreducible representations of T_{p+2} of the second kind is given as follows (Schur [13]):

$$f_{\nu_1,\nu_2,\cdots,\nu_m} = 2^{[(p+2-m)/2]} g_{\nu_1,\nu_2,\cdots,\nu_m}\,,$$

with

$$g_{\nu_1,\nu_2,\cdots,\nu_m} = \frac{(p+2)!}{\nu_1!\,\nu_2!\cdots\nu_m!} \prod_{\alpha<\beta} \frac{\nu_\alpha-\nu_\beta}{\nu_\alpha+\nu_\beta}\,,$$

where $\nu_1+\nu_2+\cdots+\nu_m=p+2$ and $\nu_1>\nu_2>\cdots>\nu_m>0$. Moreover, by Schur [13, §44], it is proved that

$$f_{\nu_1,\nu_2,\cdots,\nu_m} \geqslant 2^{[(p+2-1)/2]} = 2^{[(p+1)/2]}$$

for any $f_{\nu_1,\nu_2,\cdots,\nu_m}$. Thus we obtain the first assertion. The last assertion is clear, because $p\geqslant 7$.

(2) The degree of any ordinary irreducible representation of T_{p+2} of the second kind which is not divisible by p is divisible by $2^{[(p-1)/2]}$. Moreover, $2^{[(p-1)/2]}>p$.

Since $f_{\nu_1,\nu_2,\cdots,\nu_m}$ is not divisible by p, we obtain that $m\leqslant 3$, by noticing the formula of $f_{\nu_1,\nu_2,\cdots,\nu_m}$. Since $f_{\nu_1,\nu_2,\cdots,\nu_m}=2^{[(p+2-m)/2]}g_{\nu_1,\nu_2,\cdots,\nu_m}$ and $g_{\nu_1,\nu_2,\cdots,\nu_m}$ is an integer (Schur [13, §40], we obtain the first assertion. The last assertion is clear, because $p\geqslant 7$.

(3) The degree of any not 1 dimensional (p-modular) irreducible representation of T_{p+2} over K is more than p.

Let ϕ be an irreducible representation of T_{p+2} over K of degree >1. If ϕ is contained in a p-block of defect 0 of T_{p+2}, then by step (1) and the step (1) in Lemma 2, we obtain that the degree of ϕ is more than p. Now, let us assume that ϕ is contained in a p-block of defect 1. Since any block of defect 1 contains at most p ordinary irreducible representations in general (and in this case) (cf. [3, §68]), B_0 and B_1 (p-blocks of S_{p+2}) themselves also become p-blocks of T_{p+2} of defect 1 (all representation of S_{p+2} are naturally regarded as representations of T_{p+2}). Therefore, any ordinary irreducible representation of T_{p+2} which is contained in a p-block of defect 1 and not contained in B_0 and B_1 (as blocks of T_{p+2}) must be of the second kind. Therefore, the degree of any ordinary irreducible representation of T_{p+2} contained in a p-block of defect 1 and not contained in B_0 and B_1 must be divisible by $2^{[(p-1)/2]}$ by step (2). Since p is to the first power in the order of T_{p+2}, the Brauer graph of any p-block of defect 1 of T_{p+2} must be a tree (cf. [3, §68]), and so the degree of any irreducible representation of T_{p+2} over K is divisible by $2^{[(p-1)/2]}>p$. Thus, we obtain the assertion of (3).

Thus, we have completed the proof of Lemma 3.

We also have

Lemma 3′. *Let $p\geqslant 7$. Then A_{p+2} is not a subgroup of $P\,GL(p, K)$, where K is an algebraically closed field of characteristic p.*

Proof. The commutator subgroup T_{p+2}' of T_{p+2} with index 2 becomes a representation group of A_{p+2} over K.

(1′) The degree of any ordinary irredubicle representation of T_{p+2}' of the second kind is more than $2^{[(p+1)/2]-1}>p$.

Proof is clear.

(2′) The degree of any ordinary irreducible representation of $T_p{}'_{+2}$ of the second kind which is not divisible by p is divisible by $2^{[(p-1)/2]-1}$ and divisible by 8 if $p=7$. Moreover, $2^{[(p-1)/2]-1}>p$ when $p\geqslant 11$.

Proof of the first assertion is clear. The second assertion for $p=7$ is proved directly and easily.

(3′) The degree of not 1 dimensional (p-modular) irreducible representation of T_{p+2}' over K is more than p.

The proof is quite the same as that of step (3) in Lemma 3.

Thus, we have proved Lemma 3′.

Lemma 4. *A_{p+2} is not involved in a finite subgroup of $GL(p, K)$, where K is an algebraically closed field of characteristic p.*

Proof. Let us assume that l is the smallest integer $\leqslant p$ such that A_{p+2} is involved in a finite subgroup X of $GL(l, K)$. Moreover, let us take X being of the least order among them, then X contains a normal subgroup Y such that $X/Y=A_{p+2}$. Now, we will derive a contradiction. By the assumption, we may assume that X is an irreducible subgroup of $GL(l, K)$, and moreover that X is a primitive subgroup of $GL(l, K)$, because A_{p+2} is obviously not involved in S_l. (Cf. Dixon [2, §4], see also [2] for some fundamental properties of (finite) linear groups). By Lemma 2 and Lemma 3, we may assume that Y is not contained in $Z(GL(l, K))$. Thus, there exists a Sylow q subgroup Q (for some prime

q) of Y such that Q is not contained in $Z(GL(l, K))$. By the theorem of Sylow (Frattini argument), and since A_{p+2} is not involved in Y by the minimality of the order of X, we obtain that X normalizes the Sylow q subgroup Q which is not contained in $Z(GL(l, K))$. The proof of Lemma 4 will be completed through the following steps (1) to (6).

(1) $p \neq q$.

Otherwise, X becomes not irreducible as a subgroup of $GL(l, K)$, and this contradicts the minimality of l. (Cf. Dixon [2, §§2.2 and 2.8, or §4.2]).

(2) Q does not contain any characteristic abelian subgroup of rank $\geqslant 2$.

Otherwise, X becomes imprimitive or not irreducible as a subgroup of $GL(l, K)$, and this contradicts the minimality of l. (Cf. Dixon [2, §4.2].)

(3) Q is a central product of groups Q_1 and Q_2, where Q_1 is either 1 or extraspecial q group, say of order q^{2r+1}, and Q_2 is either cyclic or $q=2$ and isomorphic to one of dihedral, generalized quaternion and semidihedral groups of order $\geqslant 2^4$.

Since Q contains no characteristic abelian subgroup of rank $\geqslant 2$, we obtain the assertion by a result of P. Hall (cf. Gorenstein [4, Theorem 5.4.9]).

Next, we utilize the following important result of Jordan.

Lemma of Jordan ([8, Chap. (V, page 56, (3)]). *Let q be a prime. If r is a prime such that $r \neq q$ and $r \leqslant k-2$, then A_k is not involved in $GL(r-2, q)$.*

As a special case of Lemma of Jordan, we obtain the following assertion.

(4) A_{p+2} is not involved in $GL(p-2, q)$, where q is a prime different from p.

(5) Let x be an element of $GL(l, K)$ which is of order prime to p and not lying in $Z(GL(l, K))$. Then A_{p+2} is not involved in $C_{GL(l,K)}(x)$.

This assertion is well known and immediately proved, e.g., by Dixon [2, §4.2], because $C_{GL(l,K)}(x)$ becomes either not irreducible or imprimitive as a subgroup of $GL(l, K)$.

(6) A_{p+2} is not involved in $\mathrm{Aut}(Q)$.

We obtain that all irreducible components of the natural representation of Q in $GL(l, K)$ are equivalent (cf. [2, §4.2]), and so it is a faithful representation of Q. Now, any faithful ordinary absolutely irreducible representation of Q (and hence any faithful absolutely irreducible representation of Q over a field of characteristic $p \neq q$ (cf. Dixon [2, §3.8]) is) either of degree q^r (when Q_1 is extraspecial of order q^{2r+1} and Q_2 is cyclic) or q^{r+1} (when Q_1 is extraspecial of order q^{2r+1} and Q_2 is one of dihedral, generalized quaternion and semidihedral and $q=2$), or $\leqslant 2$ (when $Q_1=1$) (cf. Gorenstein [4, Theorem 5.5.5 and Theorem 3.7.2]). If $Q_1=1$, then we easily have that A_{p+2} is not involved in $\mathrm{Aut}(Q)$, and so in the following we assume that $Q_1 \neq 1$. Thus, we obtain in every case that $q^r \leqslant l$ ($\leqslant p$) or $q^{r+1} \leqslant l$ ($\leqslant p$). Now, investigating the structures of the group

Q in every posible case, we obtain that Q contains a series of characteristic subgroups $Q_{(i)}$ such that

$$Q = Q_{(0)} > Q_{(1)} > \cdots > Q_{(k)} = 1\,,$$

and $Q_{(i)}/Q_{(i+1)}$ ($i=0, 1, \cdots, k-1$) are elementary abelian q subgroups of rank $\leqslant 2r$. Here, note that in every case $Q/Z(Q)$ is a direct product of $Q_1/Z(Q)$ (an elementary abelian group of order q^{2r}) and a group $Q_2/Z(Q)$ which is either trivial or one of cyclic subgroups of order $\geqslant q^2$ (since, if of order q then Q becomes an extraspecial q group of order q^{2r+2}, and this is a contradiction) or $q=2$ and dihedral group of order $\geqslant 2^3$. Therefore, in any way, since $q^r \leqslant p$ or $q^{r+1} \leqslant p$, we obtain that $p-2 \geqslant 2r$ whenever $p \geqslant 7$. Therefore, in order that A_{p+2} is involved in $\mathrm{Aut}(Q)$, A_{p+2} must be involved in $GL(2r, q)$, because $\mathrm{Aut}(Q)$/(the stabilizer group of the above chain oi characteristic subgroups) is a subgroup of the direct product of $GL(l_i, q)$'s with $l_i \leqslant 2r$, and the stabilizer group of the chain is a q group (cf. Gorenstein [4, §5.3]). But, since $p-2 \geqslant 2r$, this contradict the assertion of (4). Thus, we have obtained the assertion of (6).

Now, we will complete the proof of Lemma 4. Since A_{p+2} is not involved in $C_{GL(l,K)}(Q)$ by step (5), and since $\mathrm{Aut}(Q)$ is a subgroup of $N_{GL(l,K)}/C_{GL(l,K)}(Q)$, we obtain that A_{p+2} is not involved in $N_{GL(l,K)}(Q)$. But this is a contradiction, and we have completed the proof of Lemma 4.

Thus, we have completed the proof of Theorem A.

REMARK 1. Theorem A improves Lemma of Jordan (stated preceding step (4) in Lemma 4) a little. That is, we can omit the assumption that $r \neq q$ in Lemma of Jordan.

REMARK 2. Since it will be not easy for us to follow the proof of Lemma of Jordan along the original paper [8] of Jordan, because of its old fahsionedness of its way of description and its terminologies (but not of its context), we give a sketch of an alternative proof.

(a) Let q be a prime $\neq p$. Then A_{p+2} is not a subgroup of $GL(p-2, F)$, where F is an algebraically closed field of characteristic q.

A_{p+2} contains a Frobenius group H of order $p(p-1)$ whose any Sylow subgroups are cyclic. Since the Schur multipliers of any cyclic subgroups are trivial, $H^2(H, K^*)$ also becomes trivial (cf. Yamazaki [15, §3]). Therefore, we obtain the assertion by Lemma 1.4 in Harris and Hering [6].

The next assertion will be of independent interest.

(b) Let G be a finite simple group which is not involved in $A_8 \simeq GL(4, 2)$[1]. If the degree of any not 1 dimensional projective (including linear) irreducible

1) The assumption that G is not involved in A_8 is unnecessary in practice, as we can easily see by the case by case considerations of such simple groups.

representation over any (algebraically closed) field of any characteristic is more than t, then G is not involved in a finite subgroup of $GL(t, K)$, where K is any (algebraically closed) field of any characteristic.

Proof. Let $l\,(<t)$ be the smallest integer such that G is contained in a finite subgroup X of $GL(l, K)$ with some algebraically closed field K of characteristic, say s. Among them, let us take X to be of the least order. Because of the assumption, we obtain by quite the same argument as used in the proof of Lemma 3, that X contains a nontrivial normal Sylow q $(\neq s)$ subgroup Q which is not contained in $Z(GL(l, K))$, and that X is not involved in $C_{GL(l,K)}(Q)$. Moreover, since G must be involved in $\mathrm{Aut}(Q)$, G must be involved in $GL(2r, q)$, where $l \geqslant q^r$ (or q^{r+1}) holds. From the minimality of l, $q^r \leqslant 2r$. This asserts that $q=2$ and $r=2$ and $l=4$. Hence G must be involved in $GL(4, 2)$.

Proof of Lemma of Jordan follows immediately from steps (a) and (b) together with Lemma 3′ and Lemma 1.

2. Proof of Theorem 1

Let us assume that G satisfies the three conditions of Theorem 1. Now, we will derive a contradiction.

There is an element a of G of order p such that

$$a=(1)\,(2)\,(3, \cdots, p+2)(p+3)\cdots(2p+2)\cdots,$$

i.e., a fixes $p+2$ points. Then there exists a Sylow p subgroup of $G_{1,2,\cdots,p+2}$ which is normalized by the element a. We may denote it by P_0 without loss of generality. Now, let us set P be the subgroup generated by a and P_0. Then P is a Sylow p subgroup of G.

(1) P is of maximal class (in the sense of Blackburn). Therefore, $|Z(P)|=p$.

Since we obtain that $|C_{P_0}(a)|=p$ from the semiregularity of P_0 on $\Omega-\{1, 2, \cdots, p+2\}$ (cf. Lemma of Nagao [11]), we have $|C_P(a)|=p^2$, and so we have the first assertion (cf. [7, Kapital III, Satz 14.23]). The last assertion is immediate from the assumption that $|P_0| \geqslant p^2$.

(2) $N_G(P_0)^{\{1,2,\cdots,p+2\}}=S^{\{1,2,\cdots,p+2\}}$.

This assertion is an immediate consequence of Lemma of Witt (cf. [14, Theorem 9.3]).

(3) $C_G(P_0)^{\{1,2,\cdots,p+2\}} \geqslant A^{\{1,2,\cdots,p+2\}}$.

Otherwise, $C_G(P_0)^{\{1,2,\cdots,p+2\}}=1$ (because $p+2 \geqslant 5$), and S_{p+2} must be involved in $\mathrm{Aut}(P_0)$, because $N_G(P_0)/C_G(P_0)$ is a subgroup of $\mathrm{Aut}(P_0)$. Now, P_0 has an automorphism σ (induced from the element a) such that the following condition (*) is satisfied:

$$(*) \qquad \sigma \text{ is of order } p \text{ and } |C_{P_0}(\sigma)|=p\,.$$

If a p group X has an automorphism σ satisfying the condition $(*)$, then any σ-invariant subgroup of X and any factor group X/Y for a σ-invariant normal subgroup Y of X have the automorphism (naturally induced by σ) satisfying the condition $(*)$ provided σ acts nontrivially on them (cf. Huppert [7, Kapital III, §14], or the argument in Zassenhaus [16, pp. 18–19]), because the map τ of X to X defined by $\tau(x)=x^{-1}x^{\sigma}$ is p to 1, and if $(xY)^{\sigma}=xY$ then $\tau(x)$ is contained in Y. Moreover, by a lemma of Ito in Nagao [10], an elementary abelian p group which has an automorphism with the property $(*)$ is of rank $\leqslant p$. Thus, if we take a chain of Frattini subgroups $\Phi^{(i)}(P_0)$ of P_0:

$$P_0>\Phi^{(1)}(P_0)>\Phi^{(2)}(P_0)>\cdots>\Phi^{(k)}(P_0)=1\,,$$

where $P_0=\Phi^{(0)}(P_0)$ and $\Phi^{(1)}(P_0)$ is the Frattini subgroup of P_0 and $\Phi^{(i+1)}(P_0)=\Phi^{(1)}(\Phi^{(i)}(P_0))$ for $i\geqslant 2$, then $\Phi^{(i)}(P_0)/\Phi^{(i+1)}(P_0)$ is an elementary abelian p group of rank $r_i\leqslant p$ $(i=0, 1, \cdots, k-1)$. Therefore, we obtain that

$$\mathrm{Aut}(P_0)/(\text{the stabilizer group of the above chain})$$

is a subgroup of the direct product of the groups $GL(r_i, p)$ with $r_i\leqslant p$ $(i=0, 1, \cdots, k-1)$, and the stabilizer group of the chain is a p group. Therefore, since S_{p+2} is not involved in $GL(p, p)$ by Theorem A, we obtain that S_{p+2} is not involved in $\mathrm{Aut}(P_0)$. But, this is a contradiction.

Since $C_G(P_0)^{\{1,2,\cdots,p+2\}}\geqslant A^{\{1,2,\cdots,p+2\}}$ we obtain that $|Z(P)|\geqslant p^2$. But, this contradict the fact (1) that P is of maximal class.

Thus, we have completed the proof of Theorem 1.

Appendix

In this appendix, we will prove the following result.

Theorem 2. *Let p be an odd prime $\geqslant 11$. Let G be a permutation group on a set $\Omega=\{1, 2, \cdots, n\}$ which satisfies the following conditions:*
(i) *G is $(p+1)$-ply transitive, and $n\equiv 1$ (mod p),*
(ii) *a Sylow p subgroup P_0 of $G_{1,2,\cdots,p+1}$ is semiregular on $\Omega-\{1, 2, \cdots, p+1\}$,* and
(iii) $|P_0|\geqslant p^2$.
Then we obtain that P_0 is an elementary abelian p group of order p^p and that a Sylow p subgroup P of G is isomorphic to $\boldsymbol{Z}_p \int \boldsymbol{Z}_p$ (wreathed product).

The next Theorem B is proved by quite the same argument as in Theorem A, and so we omit the proof.

Theorem B. *Let p be an odd prime $\geqslant 11$. Then S_{p+1} is not involved in $GL(p-1, p)$.*

Proof of Theorem 2. Let P be a Sylow subgroup of G which contains P_0. Then P is of maximal class. We obtain that $|P_0/\Phi(P_0)| \leqslant p^p$, because of Lemma of Ito in Nagao [10]. Since S_{p+1} must be involved in $\mathrm{Aut}(P_0)$ (cf. the proof of Theorem 1) and since S_{p+1} is not involved in $GL(p-1, p)$ by Theorem B, we obtain that $|P_0/\Phi(P_0)| = p^p$, because of a result of Burnside (cf. Gorenstein 4, Theorem 5.1.4.) (The use of the result of Burnside simplifies the argument of the proof of Theorem 1 a little, i.e., in step (3) we have only to show that S_{p+1} is not involved in $\mathrm{Aut}(P_0/\Phi(P_0))$.) Now, $P/\Phi(P_0)$ is a homomorphic image of P and is isomorphic to $\boldsymbol{Z}_p \int \boldsymbol{Z}_p$. Therefore, by a result of Blackburn (cf. Huppert [7, Kapital III, Satz 14.20]) we obtain that $\Phi(P_0)=1$, and so we obtain the assertion of Theorem 2.

UNIVERSITY OF TOKYO

References

[1] E. Bannai: *A note on multiply transitive permutation groups* (to appear in J. Algebra).

[2] J.D. Dixon: The Structure of Linear Groups, Van Nostrand Reinhold Math. Studies 37, London, 1971.

[3] L. Dornhoff: Group Representation Theory, Part B, Dekker, New York, 1972.

[4] D. Gorenstein: Finite Groups, Harper and Row, New York, 1968.

[5] M. Hall, Jr.: *On a theorem of Jordan*, Pacific. J. Math. **4** (1954), 219–226.

[6] M.E. Harris and C. Hering: *On the smallest degrees of projective representations of the groups PSL(n, q)*, Canad. J. Math. **28** (1971), 90–102.

[7] B. Huppert: Endliche Gruppen I, Springer, Berlin-Heidelberg-New York, 1967.

[8] C. Jordan: *Sur la limite de transitivité des groupes non alternés*, Bull. Soc. Math. France **1** (1875), 40–71.

[9] I. Miyamoto: *Multiply transitive permutation groups and odd primes*, Osaka J. Math. **11** (1974), 9–13.

[10] H. Nagao: *On multiply transitive groups*, Nagoya Math. J. **27** (1966), 15–19.

[11] H. Nagao: *On multiply transitive groups* V, J. Algebra **9** (1968), 240–248.

[12] T. Nakayama: *On some modular properties of irreducible representations of a symmetric group* II, Japanese J. Math. **17** (1940), 411–423.

[13] I. Schur: *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, Crelle J. **139** (1911), 155–250.

[14] H. Wielandt: Finite Permutation Groups, Academic Press, New York and London, 1964.

[15] K. Yamazaki: *On projective representations and ring extensions of finite groups*, J. Fac. Sci. Univ. Tokyo **10** (1963–4), 147–195.

[16] H. Zassenhaus: *Kennzeichnung endlicher linearer Gruppen als Permutationsgruppen*, Abh. Math. Sem. Univ. Hamburg, **11** (1934), 17–40.