Numata, M.
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ON A CHARACTERIZATION OF A CLASS OF THE REGULAR GRAPHS OF DIAMETER 2

MINORU NUMATA

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1. Introduction

S.S. Schrikhande [2], [3], W.S. Conner [4] and A.J. Hoffman [5] determined all of the graphs with intersection matrices $B(2t-2, t-2, 2)$ for $t\geqq 2$ and $B(2t-4, t-2, 4)$ for $t\geqq 4$.

The lattice graphs of dimension 2 with intersection matrices $B(2t-2, t-2, 2)$ for $t\geqq 3$ and the triangle graphs with intersection matrices $B(2t-4, t-2, 4)$ for $t\geqq 6$ have the remarkable property that for any three vertices which are not joined to each other, no vertex is joined to all of these three vertices.

The purpose of this paper is to prove the following.

Theorem 1. *Let* Γ *be the regular graph of diameter* 2 *satisfying the following conditions:*

1. *For any two vertices which are joined to each other, the number of the vertices joined to them is constant.*

2. *There exist three vertices which are not joined each other.*

3. *For any three vertices which are not joined to each other, no vertex is joined to all of these three vertices.*

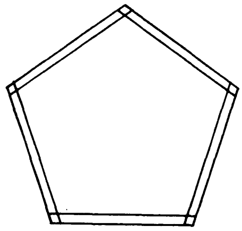
Then, Γ *is one of the following graphs:*

a) L_2*-graphs, that is, the lattice graphs of dimension* 2 *with intersection matrices* $B(2t-2, t-2, 2)$ *for* $t\geqq 3$.

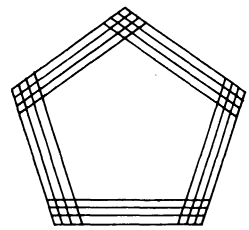
b) T_2*-graphs, that is, the triangle graphs with intersection matrices* $B(2t-4, t-2, 4)$ *for* $t\geqq 6$.

c) *the graph defined by* 27 *lines on cubic surface with intersection matrix* $B(16, 10, 8)$.

d) L_2'*-graphs, that is, the graphs such that for any two vertices which are not joined to each other, the number of the vertices joined to them is one or two. The number of the vertices of the graph is* $5\cdot h^2$ *for* $h\geqq 2$, *and the graph exists uniquely for any integer* $h\geqq 2$.



L_2'-graph when $h=2$



L_2'-graph when $h=4$

2. General definitions and notation

An undirected linear graph Γ is a pair (V, B) where V is a finite set and B is a subset of the unordered pair of the elements of V. The elements of V are called vertices. We say that a vertex α is joined to a vertex β if the unordered pair of α and β is an element of B.

We define for any vertex α of V

$$\Delta(\alpha) = \{\beta \in V \mid (\beta, \alpha) \in B\},\ \Gamma(\alpha) = V \backslash \Delta(\alpha) \backslash \{\alpha\}.$$

$\Gamma=(V, B)$ is regular if the number of the vertices of $\Delta(\alpha)$ is constant for any vertex α, and is of diameter 2 if for any two vertices α and γ such that γ is not joined to α, the intersection of $\Delta(\alpha)$ and $\Delta(\gamma)$ is not empty and B is a proper subset of the set of all the unordered pairs of V. The regular graph Γ is strongly regular when the number of vertices joined to each of two distinct vertices α and β depends only on whether or not α is joined to β.

For the strongly regular graph $\Gamma=(V, B)$ of diameter 2, we put

$$|\Delta(\alpha)| = k, \qquad |\Delta(\alpha) \cap \Delta(\beta)| = \begin{cases} a & \text{for } \beta \in \Delta(\alpha) \\ c & \text{for } \beta \in \Gamma(\alpha) \end{cases}.$$

We call the matrix with three parameters k, a and c

$$B(k, a, c) = \begin{pmatrix} 0 & 1 & 0 \\ k & a & c \\ 0 & k-a-1 & k-c \end{pmatrix}$$

the intersection matrix of the strongly regular graph.

The following condition is necessary for the existance of the strongly regular graph of diameter 2 with intersection matrix $B(k, a, c)$;

(i) $k=2c$, $a=c-1$, or

(ii) $(a-c)^2+4(k-c)=s^2$ for some integer s, and

$$m=\frac{k}{2cs}\{(k-1+c-a)(s+c-a)-2c\}$$

is a positive integer.

The matrix $B(k, a, c)$, $c \geqq 1$, is feasible if three parameters satisfy the condition (i) or (ii), and is realizable if there is the strongly regular graph of diameter 2 with intersection matrix $B(k, a, c)$.

3. Proof of the Theorem 1

Let $\Gamma=(V, B)$ be a regular graph of diameter 2 satisfying the conditions of Theorem 1. We put

$$|\Delta(\alpha)| = k, \quad |\Delta(\alpha) \cap \Delta(\beta)| = a \text{ for } \beta \in \Delta(\alpha) \text{ and}$$
$$\mathfrak{S} = \{ |\Delta(\alpha) \cap \Delta(\gamma)| \mid \alpha, \gamma \in V, \ \gamma \in \Gamma(\alpha)\}.$$

Let δ be the vertex of $\Delta(\alpha) \cap \Delta(\gamma)$, where $\gamma \in \Gamma(\alpha)$. Then, we have by the condition 3 of the Theorem 1,

$$\Delta(\delta) = \{\Delta(\delta) \cap \Delta(\gamma)\} \cup \{\Delta(\delta) \cap \Delta(\alpha) \backslash \Delta(\gamma)\} \cup \{\alpha, \gamma\}.$$

Therefore, if we set $|\Delta(\delta) \cap \Lambda(\alpha) \backslash \Delta(\gamma)| = x$, then we have

(1) $k = a+x+2$

In addition, we have

(2) $|\Delta(\alpha) \cap \Delta(\gamma)| \geqq a-x+1, \ |\Delta(\alpha) \cap \Delta(\gamma) \cap \Delta(\delta)| = a-x$

I. At first, we show that $\mathfrak{S}=\{a-x+2\}$ or $\{a-x+1, a-x+2\}$.

If $\mathfrak{S}=\{a-x+1\}=\{c\}$, then $k=2a-c+3$. From the condition that the intersection matrix $B(2a-c+3, a, c)$ is feasible, we have

i) $2a-c+3 = 2c, \ a+1 = c$ or

ii) $(a-c)^2+4(2a-c+3-c) = (a-c+4)^2-4 = s^2$, for some integer s.

These cases collapse.

We assume that for the vertex γ_1 of $\Gamma(\alpha)$, $|\Delta(\alpha) \cap \Delta(\gamma_1)| = c > a-x+1$ and $\Gamma(\alpha) \cap \Gamma(\gamma_1)$ is not empty.

There is at most $c/2$-vertices of $\Delta(\alpha) \cap \Delta(\gamma_1)$ which are joined to each other. Hence, for any vertex δ of $\Delta(\alpha) \backslash \Delta(\gamma_1)$,

$$|\Delta(\alpha) \cap \Delta(\gamma_1) \cap \Delta(\delta)| \geqq \frac{1}{2} |\Delta(\alpha) \cap \Delta(\gamma_1)| .$$

If δ is joined to all vertices of $\Delta(\alpha) \backslash \Delta(\gamma_1) \backslash \{\delta\}$, then we have

$$\begin{aligned} a-x &= 2(a+1)-k = 2|\Delta(\alpha) \cap \Delta(\delta)|+2-k \\ &\geqq |\Delta(\alpha) \backslash \Delta(\gamma_1)| \geqq |\Delta(\alpha) \cap \Delta(\gamma_2)| \geqq a-x+1 \\ &(\gamma_2 \in \Gamma(\alpha) \cap \Gamma(\gamma_1)) \end{aligned}$$

This is impossible.

Therefore, for any vertex δ of $\Delta(\alpha)\backslash\Delta(\gamma_1)$, there is the vertex δ' of $\Delta(\alpha)\backslash\Delta(\gamma_1)\backslash\{\delta\}$ such that δ' is nlot jointed to δ. From the condition 3 of the Theorem 1, we have

$$\Delta(\alpha)\cap\Delta(\gamma_1)\cap\Delta(\delta)\cap\Delta(\delta')=\phi$$

Hence, we obtain

$$|\Delta(\alpha)\cap\Delta(\gamma_1)\cap\Delta(\delta)|=\frac{1}{2}|\Delta(\alpha)\cap\Delta(\gamma_1)|\,.$$

Counting in two ways quadrilaterals $(\alpha, \gamma_1, \delta, \beta)$ $\beta\in\Delta(\alpha)\cap\Delta(\gamma_1)\cap\Delta(\delta)$, we have

$$|V|\cdot(|V|-k-1)(k-c)c/2=|V|\cdot(|V|-k-1)\cdot c\cdot x\,,$$

that is,

$$2x-k=c\,.$$

From (1), we conclude

$$c=a-x+2\,.$$

II. We suppose that $\mathfrak{S}=\{a-x+2\}=\{c\}$.

If $c=2$, the graphs with intersection matrices $B(2a+2, a, 2)$ satisfying the conditions of the Theorem 1 are only the lattice graphs of dimension 2 with intersection matrices $B(2t-2, t-2, 2)$ for $t\geqq 3$ from [2].

If $c=4$, the graphs with intersection matrices $B(2a, a, 4)$ satisfying the conditions of the Theorem 1 are only the triangle graphs with intersection matrices $B(2t-4, t-2, 4)$ for $t\geqq 6$ from [4], [3], [5].

Now, we assume that $c\geqq 6$.

Let β_1, β_2 be vertices of $\Delta(\alpha)$ such that β_2 is not joined to β_1. Then, we have

$$|\Delta(\alpha)\cap\Delta(\beta_1)\cap\Delta(\beta_2)|=2a+2-k=2a+2-(2a-c+4)=c-2\,.$$

Therefore, $|\Delta(\beta_1)\cap\Delta(\beta_2)\cap\Gamma(\alpha)|=1$.
Let $\{\gamma\}=\Delta(\beta_1)\cap\Delta(\beta_2)\cap\Gamma(\alpha)$. Then, $\Delta(\alpha)\cap\Delta(\gamma)=\{\Delta(\alpha)\cap\Delta(\beta_1)\cap\Delta(\beta_2)\}\cup\{\beta_1, \beta_2\}$ and for each vertex ξ of $\Delta(\alpha)\cap\Delta(\beta_1)\cap\Delta(\beta_2)$, there is only one vertex ξ' of $\Delta(\alpha)\cap\Delta(\beta_1)\cap\Delta(\beta_2)$ such that ξ' is not joined to ξ.

Let δ be the vertex of $\Delta(\alpha)\cap\Gamma(\beta_1)\backslash\{\beta_2\}$, then we have

$$|\Delta(\alpha)\cap\Delta(\beta_1)\cap\Delta(\beta_2)\cap\Delta(\delta)|=\frac{c}{2}-1,\ |\Delta(\alpha)\cap\Gamma(\beta_2)\cap\Delta(\delta)|=\frac{c}{2}-1\,.$$

Furthermore, all vertices of $\Delta(\alpha)\cap\Delta(\beta_1)\cap\Delta(\beta_2)\cap\Delta(\delta)$ are joined to each other. Hence, let η be a vertex of $\Delta(\alpha)\cap\Delta(\beta_1)\cap\Delta(\beta_2)\cap\Delta(\delta)$, then we have $\Delta(\alpha)\cap\Delta(\delta)\cap\Gamma(\beta_2)\cap\Delta(\eta)\neq\phi$ and $\Delta(\alpha)\cap\Delta(\delta)\cap\Gamma(\beta_2)\cap\Gamma(\eta)\neq\phi$. Therefore,

$$\Delta(\alpha)\cap\Gamma(\beta_1)\cap\Gamma(\eta)\neq\phi \text{ and } \Delta(\alpha)\cap\Gamma(\beta_2)\cap\Gamma(\eta)\neq\phi\,.$$

Since $|\Delta(\alpha)\cap\Gamma(\beta_1)\cap\Gamma(\eta)|+|\Delta(\alpha)\cap\Gamma(\beta_2)\cap\Gamma(\eta)|=a-c+3-1=a-c+2$, we may assume $|\Delta(\alpha)\cap\Gamma(\beta_2)\cap\Gamma(\eta)|\geqq(a-c+2)/2$.

Let ρ be a vertex of $\Delta(\alpha)\cap\Gamma(\beta_1)\cap\Gamma(\eta)$, then ρ is joined to all vertices of $\Delta(\alpha)\cap\Gamma(\beta_2)\cap\Gamma(\eta)$. Hence, $|\Delta(\alpha)\cap\Gamma(\beta_2)\cap\Gamma(\eta)|\leqq\frac{c}{2}-1$

Thus we have

$$(a-c+2)/2\leqq|\Delta(\alpha)\cap\Gamma(\beta_1)\cap\Gamma(\eta)|\leqq\frac{c}{2}-1\,,$$

that is

(3) $a\leqq2(c-2)$

Hence we have

$$k=2a-c+4\leqq3c-4<3c\,.$$

This means that there are not four vertices which are not joined to each other. Let γ_1, γ_2 be vertices of $\Gamma(\alpha)$, $\gamma_2\in\Gamma(\gamma_1)$. Then, we have

$$\Gamma(\alpha)=\{\Gamma(\alpha)\cap\Delta(\gamma_1)\}\cup\{\Gamma(\alpha)\cap\Delta(\gamma_2)\}\cup\{\gamma_1,\gamma_2\}\,.$$

Hence,

$$(2a-c+4)(a-c+3)/c=2(2a-2c+4)-c+2\,.$$

Thus we have

$$2a-3c+4=0 \quad\text{or}\quad a-2c+3=0$$

From (3), when we set $c=2d$, we have

$$a=\frac{3}{2}c-2=3d-2,\ \ k=4d\,.$$

We note that the complement of the intersection matrix $B(4d, 3d-2, 2d)$ is $B(2d+2, 1, d+1)$. We show that the intersection matrices $B(2d+2, 1, d+1)$ for $d>4$ are not realized.

Let γ be a vertex of $\Gamma(\alpha)$, then vertices of $\Delta(\alpha)\cap\Delta(\gamma)$ are not joined to each other and $|\Delta(\alpha)\cap\Delta(\gamma)|=d+1$. Therefore, vertices of $\Delta(\gamma)\cap\Gamma(\alpha)$ are not joined to each other and $|\Delta(\gamma)\cap\Gamma(\alpha)|=d+1$. Let β be a vertex of $\Delta(\gamma)\cap\Gamma(\alpha)$, then β is joined to only one vertex of $\Delta(\gamma)\cap\Delta(\alpha)$. Thus, let β_1, β_2 be two vertices of $\Delta(\gamma)\cap\Gamma(\alpha)$, then we have

$$d\geqq|\Delta(\beta_1)\cap\Delta(\beta_2)\cap\Delta(\alpha)|\geqq d-1\,.$$

Therefore

$$|\Delta(\beta_1)\cap\Delta(\beta_2)\cap\Gamma(\alpha)\backslash\{\gamma\}| = 0 \text{ or } 1.$$

Let β_i be vertices of $\Delta(\gamma)\cap\Gamma(\alpha)$, $i=1,2,3,4$. Then, $\bigcup_{i=1}^{4}(\Delta(\beta_i)\cap\Gamma(\alpha)\backslash\{\gamma\}) \subseteq\Gamma(\alpha)\cap\Gamma(\gamma)$. Thus, we have

$$(d+1-1)+(d+1-1)-1+(d+1-1)-2+(d+1-1)-3\leqq 3d-2,$$

that is,

$$4d-6\leqq 3d-2.$$

This is impossible for $d>4$.

When $d=4$, the intersection matrix $B(16, 10, 8)$ is realized and satisfies the conditions of the Theorem 1 from [1].

When $d=3$, the intersection matrix $B(12, 7, 6)$ is not feasible.

When $d=2$ or 1, the graphs with intersection matrix $B(8, 4, 4)$ or $B(4, 1, 2)$ do not satisfy the condition 2 of the Theorem 1.

III. We suppose that $\mathfrak{S}=\{a-x+1, a-x+2\}$.

Step 1. We show that $a-x+1\leqq\frac{k}{3}$.

Assume that $a-x+1>\frac{k}{3}$. Then, it is trivial that there are not four vertices which are not joined to each other. Let α, γ_1 and γ_2 be not joined to each other, and $|\Delta(\alpha)\cap\Delta(\gamma_1)|=a-x+2$. We note that all vertices of $\Delta(\alpha)\backslash\Delta(\gamma_1)\backslash\Delta(\gamma_2)$ are joined to each other. If $|\Delta(\alpha)\cap\Delta(\gamma_2)|=a-x+1$, for any vertex δ of $\Delta(\alpha)\backslash\Delta(\gamma_1)\backslash\Delta(\gamma_2)$ and for any vertex η of $\Delta(\alpha)\cap\Delta(\gamma_2)$, we have $|\Delta(\alpha)\cap\Delta(\gamma_2)\cap\Gamma(\delta)|=|\{\Delta(\alpha)\backslash\Delta(\gamma_1)\backslash\Delta(\gamma_2)\}\cap\Gamma(\eta)|>0$.
Therefore,

$$|\Delta(\alpha)\cap\Delta(\gamma_2)| = |\Delta(\alpha)\backslash\Delta(\gamma_2)\backslash\Delta(\gamma_1)|.$$

Hence we have

$$a-x+1 = |\Delta(\alpha)\cap\Delta(\gamma_1)|<\frac{k}{3}.$$

This is a contradiction.

Thus, we obtain that $|\Delta(\alpha)\cap\Delta(\gamma_2)|=a-x+2$ and $|\Delta(\gamma_1)\cap\Delta(\gamma_2)|=a-x+2$. From the fact that there are not four vertices which are not joined to each other, we have

$$\begin{aligned}|V|-k-1 = {} & |\Delta(\gamma_1)|+|\Delta(\gamma_2)|-|\Delta(\gamma_1)\cap\Delta(\gamma_2)|-|\Delta(\gamma_1)\cap\Delta(\alpha)| \\ & -|\Delta(\gamma_1)\cup\Delta(\alpha)|+2.\end{aligned}$$

Therefore, $|\Delta(\eta_1)\cap\Delta(\alpha)|+|\Delta(\alpha)\cap\Delta(\eta_2)|+|\Delta(\eta_2)\cap\Delta(\eta_1)|$ are constant for any two vertices η_1, η_2 of $\Gamma(\alpha)$ such that η_1 is not joined to η_2.

Hence, for the vertex γ_0 of $\Gamma(\alpha)$ such that $|\Delta(\alpha)\cap\Delta(\gamma_0)|=a-x+1$, γ_0 is joined to all vertices of $\Gamma(\alpha)\backslash\{\gamma_0\}$. Therefore, we have

$$\Gamma(\alpha)=\{\gamma_1, \gamma_2\}\cup\{\Delta(\gamma_1)\cap\Delta(\gamma_2)\}\,.$$

From $|\Delta(\gamma_1)\cap\Delta(\gamma_2)|=a-x+2$, for any vertex γ of $\Delta(\gamma_1)\cap\Delta(\gamma_2)$, there is the vertex γ' of $\Delta(\gamma_1)\cap\Delta(\gamma_2)\backslash\{\gamma\}$ such that γ' is not joined to γ. This contradicts the existance of γ_0.

Step 2. Now, we assume that for some vertex γ of $\Gamma(\alpha)$, $|\Delta(\alpha)\cap\Delta(\gamma)|=c=a-x+1\leqq\dfrac{k}{3}$.

From (1), (2), we have

$$x+1\geqq\frac{k}{3}+\frac{1}{2}\,.$$

Let β_1, β_2 be vertices of $\Delta(\alpha)\backslash\Delta(\gamma)$, $\beta_2\in\Gamma(\beta_1)$. Then, from the condition 3 of the Theorem 1, we have

$$\Delta(\alpha)\cap\Delta(\gamma)\cap\Delta(\beta_1)\cap\Delta(\beta_2)=\phi \quad\text{and}$$
$$\{\Delta(\alpha)\cap\Delta(\gamma)\cap\Delta(\beta_1)\}\cup\{\Delta(\alpha)\cap\Delta(\gamma)\cap\Delta(\beta_2)\}=\Delta(\alpha)\cap\Delta(\gamma)\,.$$

Each vertex of $\Delta(\alpha)\cap\Gamma(\beta_1)\backslash\Delta(\gamma)$ is joined to all vertices of $\Delta(\alpha)\cap\Gamma(\beta_1)\cap\Delta(\gamma)$, and not joined to any vertex of $\Delta(\alpha)\cap\Gamma(\beta_2)\cap\Delta(\gamma)$. Let δ be the vertex of $\Delta(\alpha)\cap\Delta(\beta_1)\cap\Delta(\beta_2)$. If there is the vertex ξ of $\Delta(\alpha)\cap\Gamma(\beta_1)\backslash\Delta(\gamma)$ such that ξ is not joined to δ, then δ is not joined to any vertex of $\Delta(\alpha)\cap\Gamma(\beta_1)\cap\Delta(\gamma)$ and joined to all vertices of $\Delta(\alpha)\cap\Gamma(\beta_2)\cap\Delta(\gamma)$.

If δ is joined to all vertices of $\{\Delta(\alpha)\cap\Gamma(\beta_1)\backslash\Delta(\gamma)\}\cup\{\Delta(\alpha)\cap\Gamma(\beta_2)\backslash\Delta(\gamma)\}$, there is the vertex η of $\Delta(\alpha)\cap\Delta(\beta_1)\cap\Delta(\beta_2)\backslash\{\delta\}$ such that η is not joined to δ. From $|\Delta(\alpha)\cap\Gamma(\beta_1)\backslash\Delta(\gamma)|+|\Delta(\alpha)\cap\Gamma(\beta_2)\backslash\Delta(\gamma)|=2x+2-c>c$ there is the vertex ρ of $\{\Delta(\alpha)\cap\Gamma(\beta_1)\backslash\Delta(\gamma)\}\cup\{\Delta(\alpha)\cap\Gamma(\beta_2)\backslash\Delta(\gamma)\}$ such that ρ is not joined to η.

If ρ is the vertex of $\Delta(\alpha)\cap\Gamma(\beta_1)\backslash\Delta(\gamma)$, then η is not joined to any vertex of $\Delta(\alpha)\cap\Gamma(\beta_1)\cap\Delta(\gamma)$ and joined to all vertices of $\Delta(\alpha)\cap\Gamma(\beta_2)\cap\Delta(\gamma)$. If ρ is the vertex of $\Delta(\alpha)\cap\Gamma(\beta_2)\backslash\Delta(\gamma)$, then η is not joined to any vertex of $\Delta(\alpha)\cap\Gamma(\beta_2)\cap\Delta(\gamma)$ and joined to all vertices of $\Delta(\alpha)\cap\Gamma(\beta_1)\cap\Delta(\gamma)$.

Hence, δ is not joined to any vertex of $\Delta(\alpha)\cap\Gamma(\beta_2)\cap\Delta(\gamma)$ and joined to all vertices of $\Delta(\alpha)\cap\Gamma(\beta_1)\cap\Delta(\gamma)$, or δ is joined to all vertices of $\Delta(\alpha)\cap\Gamma(\beta_2)\cap\Delta(\gamma)$ and not joined to any vertex of $\Delta(\alpha)\cap\Gamma(\beta_1)\cap\Delta(\gamma)$.

From the fact mentioned above, each vertex of $\Delta(\alpha)\backslash\Delta(\gamma)$ is joined to all vertices of $\Delta(\alpha)\cap\Gamma(\beta_1)\cap\Delta(\gamma)$ and not joined to any vertex of $\Delta(\alpha)\cap\Gamma(\beta_2)\cap\Delta(\gamma)$, or joined to all vertices of $\Delta(\alpha)\cap\Gamma(\beta_2)\cap\Delta(\gamma)$ and not joined to any vertex of $\Delta(\alpha)\cap\Gamma(\beta_1)\cap\Delta(\gamma)$.

If $\Delta(\alpha)\cap\Delta(\gamma)\cap\Gamma(\beta_i)$ is not empty, for $i=1, 2$, let η_i be a vertex of $\Delta(\alpha)\cap$

$\Delta(\gamma)\cap\Gamma(\beta_i)$, for $i=1, 2$. Then we have

$$\begin{aligned} k-c &= |\Delta(\alpha)\backslash\Delta(\gamma)| = |\{\Delta(\alpha)\backslash\Delta(\gamma)\}\cap\Delta(\eta_1)|+|\{\Delta(\alpha)\backslash\Delta(\gamma)\}\cap\Delta(\eta_2)| \\ &= 2x\,. \end{aligned}$$

Thus, from (1) we obtain

$$c = a-x+2\,.$$

This is the contradiction.

Therefore, each vertex of $\Delta(\alpha)\backslash\Delta(\gamma)$ is joined to all vertices of $\Delta(\alpha)\cap\Delta(\gamma)$ or not joined to any vertex of $\Delta(\alpha)\cap\Delta(\gamma)$. We define $\sum$ the set of the vertices of $\Delta(\alpha)\backslash\Delta(\gamma)$ which are joined to all vertices of $\Delta(\alpha)\cap\Delta(\gamma)$. Then, all vertices of $\sum$ are joined to each other. Hence, all vertices of $\sum\cup\{\Delta(\alpha)\cap\Delta(\gamma)\}$ are joined to each other and $|\sum\cup\{\Delta(\alpha)\cap\Delta(\gamma)\}| = a+1$. Therefore, we have

$$k = 2(a+1), \qquad \mathfrak{S} = \{1, 2\}\,.$$

Step 3. We assume that $\mathfrak{S}=\{1, 2\}$, $k=2a+2$.

We define the largest subset L of V line such that all vertices of L are joined to each other. Then, we have the following properties

i) each vertex is contained within exactly two lines

ii) each line consists of exactly $a+2$-vertices

iii) two vertices are contained within at most one line

iv) there are not three lines such that any two of them intersect.

v) if for two vertices α, β there is not the line containing them, there are two lines A, B such that A contains α, B contains β, and A intersects with B.

Now, we choose the line $L_\infty=\{l_0, l_1, \cdots, l_{a+1}\}$ and the line L_i which contains l_i, $L_i\neq L_\infty$, $i=0, \cdots, a+1$, and the line B_j which intersect with L_0, $B_j\neq L_\infty$, $j=1, \cdots, a+1$. Let $n_i=|B_i\backslash\bigcup_{j=0}^{a+1}L_j|$ and $n_{i_0}=\max\{n_i\,|\,i=1, \cdots, a+1\}$.

We put

$$B_{i_0}\backslash\bigcup_{j=0}^{a+1}L_j = \{x_1, \cdots, x_{n_{i_0}}\} \text{ and } B_{i_0}\cap L_{j_k} = \phi,\ k = 1, \cdots, n_{i_0}\,.$$

Any vertex is contained within at least one of $L_\infty, L_0, \cdots, L_{a+1}, B_1, \cdots, B_{a+1}$ from the property v).

From $\mathfrak{S}=\{1, 2\}$, we have

$$n_{i_0}>0\,.$$

We choose the line X_i containing x_i, $X_i\neq B_{i_0}$, $i=1, \cdots, n_{i_0}$, and show that any line is one of $L_\infty, L_0, \cdots, L_{a+1}, B_1, \cdots, B_{a+1}, X_1, \cdots, X_{n_{i_0}}$.

We assume that there exists a line X different from the lines above. Then, there exists the line B_{i_1} which intersect with X. Let $\{x\}=X\cap B_{i_1}$. Then, the vertex x is not joined to the vertex x_i, $i=1, \cdots, n_{i_0}$. From $B_{i_1}\cap B_{i_0}=X\cap B_{i_0}=X\cap X_i=\phi$, $i=1, \cdots, n_{i_0}$, by the property v), we have

$$B_{i_1}\cap X_i\neq\phi,\ i=1, \cdots, n_{i_0}.$$

Therefore,

$$|B_{i_1}\backslash\bigcup_{j=0}^{a+1}L_j|\geqq n_{i_0}+1 .$$

This contradicts the maximality of n_{i_0}.

The vertex x_i is not joined to the vertex l_{jk}, $i=1, \cdots, n_{j_0}$, $k=1, \cdots, n_{i_0}$. From the fact that $L_\infty\cap B_{i_0}=L_\infty\cap X_i=L_{jk}\cap B_{i_0}=\phi$, we have $L_{j_k}\cap X_i\neq\phi$, $i=1, \cdots, n_{i_0}$, $k=1, \cdots, n_{i_0}$, by the property v).

On the other hand if $L_j\neq L_{j_k}$, $k=1, \cdots, n_{i_0}$, then $L_j\cap B_{i_0}\neq\phi$, $B_{i_0}\cap X_i\neq\phi$. Therefore, we have $L_j\cap X_i=\phi$ by the property (iii). Hence, we have

$$|X_i\cap\bigcup_{j=1}^{a+1}B_j| = a+2-n_{i_0},\ \ i = 1, \cdots, n_{i_0}.$$

Change the places of L and L_0, we have $a+2-n_{i_0}=n_{i_0}$, that is

$$n_{i_0}=\frac{a+2}{2}=\frac{a}{2}+1 .$$

For any positive even number a, the graph satisfying the conditions exists uniquely. Automorphism group of the graph is transitive, imprimitive and rank 6 on the vertices of the graph. The number of the vertices of the graph is equal to $5\cdot\left(\frac{a+2}{2}\right)^2$. q.e.d.

Theorem 2. *Let $\Gamma=(V, B)$ be the connected regular graph satisfying the conditions* 1, 2 *and* 3 *of the Theorem* 1 *and the diameter of Γ is at least three. Then we have*

1). $k=2(a+1)$, $\mathfrak{S}=\{1\}$ *or*
2). $k=2(a+1)$, $\mathfrak{S}=\{1, 2\}$ *or*
3). Γ *is isomorphic to the graph induced by the vertices and the edges of the icosahedron.*

(We put

$$k = |\Delta(\alpha)|,\ a = |\Delta(\alpha)\cap\Delta(\beta)| \ \text{ for }\ \beta\in\Delta(\alpha),$$
$$\mathfrak{S} = \{|\Delta(\alpha)\cap\Delta(\gamma)|\ |d(\alpha, \gamma) = 2, \alpha, \gamma\in V\}.)$$

Proof of Theorem 2.

Let α, γ and δ be the vertices such that $d(\alpha, \delta)=3$, $d(\alpha, \gamma)=2$ and $d(\gamma, \delta)=1$, and ε be the vertex of $\Delta(\alpha)\cap\Delta(\gamma)$. Since any vertex of $\Delta(\alpha)\cap\Delta(\gamma)$ is not joined to δ, all vertices of $\Delta(\alpha)\cap\Delta(\gamma)$ are joined to each other. Therefore we have

$$|\Delta(\alpha)\cap\Delta(\gamma)| = 2a-k+3 ,$$

Next time, we have

$$\Delta(\varepsilon)\cap\Delta(\delta)\subseteq\{\Delta(\gamma)\backslash\Delta(\alpha)\}\cup\{\gamma\}\cap\Delta(\varepsilon)\,.$$

Therefore,

$$2a-k+3\leq k-a-1\,,$$

that is,

$$k\geq\frac{3}{2}a+2\,.$$

Case I. We suppose that $k>\frac{3}{2}a+2$.

Then, we have

$$|\Delta(\alpha)\backslash\Delta(\gamma)|=k-(2a-k+3)>a+1,\ \text{and}$$
$$|\Delta(\alpha)\backslash\Delta(\gamma)|-2=k-(2a-k+3)-2>2a-k+2\,.$$

By the same way to the proof of the step 2 of the Theorem 1, we conclude that

$$k=(2a+2),\ \mathfrak{S}=\{1\}\ \text{or}\ \mathfrak{S}=\{1,2\}\,.$$

Case II. We assume that $k=\frac{3}{2}a+2$.

When $a=2$, it is easy to prove that the graph Γ is isomorphic to the graph induced by the vertices and the edges of the icosahedron.

We assume that $a>2$.

Since $|\Delta(\Gamma)\backslash\Delta(\gamma)|=a+1$, $\Delta(\alpha)\backslash\Delta(\gamma)$ is not complete. Let β and β' be two vertices of $\Delta(\alpha)\backslash\Delta(\gamma)$ which are not joined to each other. From that $a>2$, we have

$$\Delta(\alpha)\cap\Delta(\beta)\cap\Delta(\beta')\subsetneqq\Delta(\alpha)\backslash\Delta(\gamma)\backslash\{\beta,\beta'\}\,.$$

By the same way to the proof of the step 2 of the Theorem 1, we conclude that each vertex of $\Delta(\alpha)\backslash\Delta(\gamma)$ is joined to all vertices of $\Delta(\alpha)\cap\Delta(\gamma)\cap\Delta(\beta)$ and not joined to any vertex of $\Delta(\alpha)\cap\Delta(\gamma)\cap\Delta(\beta')$, or joined to all vertices of $\Delta(\alpha)\cap\Delta(\gamma)\cap\Delta(\beta')$ and not joined to any vertex of $\Delta(\alpha)\cap\Delta(\gamma)\cap\Delta(\beta)$, or not joined to any vertex of $\Delta(\alpha)\cap\Delta(\gamma)$. It is easy to prove that $\Delta(\alpha)\cap\Delta(\gamma)\cap\Delta(\beta)$ and $\Delta(\alpha)\cap\Delta(\gamma)\cap\Delta(\beta')$ are not empty from that $k=\frac{3}{2}a+2$.

Let η be the vertex of $\Delta(\alpha)\cap\Delta(\gamma)\cap\Delta(\beta)$ and η' be the vertex of $\Delta(\alpha)\cap\Delta(\gamma)\cap\Delta(\beta')$. We have

$$|\Delta(\alpha)\backslash\Delta(\gamma)\backslash\Delta(\eta)\backslash\Delta(\eta')|=1\,.$$

Let ρ be the vertex of $\Delta(\alpha)\backslash\Delta(\gamma)\backslash\Delta(\eta)\backslash\Delta(\eta')$. ρ is joined to all vertices of

$\Delta(\alpha)\backslash\Delta(\gamma)\backslash\{\rho\}$ and not joined to any vertex of $\Delta(\alpha)\cap\Delta(\gamma)$.
Now, we show that $\mathfrak{S}=\{2a-k+3\}$. We define $\Gamma(\alpha)$ the set of the vertices which are distant 2 from α. Let γ' be the vertex of $\Gamma(\alpha)$. If $\gamma'=\gamma$, $|\Delta(\alpha)\cap\Delta(\gamma')|=2a-k+3$. If γ' is the vertex of $\Gamma(\alpha)\cap\Delta(\gamma)$, since γ' is joint to δ, δ is not joined to any vertex of $\Delta(\alpha)\cap\Delta(\gamma')$. therefore, we have

$$|\Delta(\alpha)\cap\Delta(\gamma')| = 2a-k+3\,.$$

If γ' is the vertex of $\Gamma(\alpha)\backslash\Delta(\gamma)\backslash\{\gamma\}$, we have

$$\Delta(\alpha)\cap\Delta(\gamma')\subseteqq\Delta(\alpha)\backslash\Delta(\gamma)$$

By the existance of ρ, we obtain

$$|\Delta(\alpha)\cap\Delta(\gamma')| = 2a-k+3 = k-a-1\,.$$

Thus, we conclude that $|\Gamma(\alpha)|=k$. If there exist two vertices ξ and ξ' which are not joined to each other and $|\Delta(\xi)\cap\Delta(\xi')|=2a-k+4$, then we have

$$V = \{\xi\}\cup\Delta(\xi)\cup\Gamma(\xi) \text{ and } |\Gamma(\xi)|<k\,.$$

This is impossible.
Count by two ways the number of the pair of the vertex of $\Delta(\alpha)\cap\Delta(\eta)\backslash\Delta(\gamma)$ and the vertex of $\Delta(\alpha)\cap\Delta(\eta')\backslash\Delta(\gamma)$ which are not joined to each other. Then we have

$$|\Delta(\alpha)\cap\Delta(\gamma)\cap\Delta(\beta')| = \Delta(\alpha)\cap\Delta(\gamma)\cap\Delta(\beta)| = \frac{1}{2}(2a-k+3)\,.$$

We divide the set $\Delta(\alpha)\cap\Delta(\eta)$ into three subsets $\Delta(\alpha)\cap\Delta(\beta)\cap\Delta(\gamma)$, $\Delta(\alpha)\cap\Delta(\eta)\backslash\Delta(\gamma)$ and $\Delta(\alpha)\cap\Delta(\gamma)\backslash\Delta(\beta)$. Each vertex of $\Delta(\alpha)\cap\Delta(\eta)\backslash\Delta(\gamma)$ is not joined to any vertex of $\Delta(\alpha)\cap\Delta(\gamma)\backslash\Delta(\beta)$, and $|\Delta(\alpha)\cap\Delta(\gamma)\backslash\Delta(\beta)|<|\Delta(\alpha)\cap\Delta(\eta)\backslash\Delta(\gamma)|$ because that $a>2$. Now, ρ is the vertex of $\Delta(\alpha)\cap\Delta(\beta)$ and joined to all vertices of $\{\Delta(\alpha)\cap\Delta(\beta)\backslash\Delta(\gamma)\backslash\{\rho\}\}\cup\{\beta\}$ and not joined to any vertex of $\Delta(\alpha)\cap\Delta(\beta)\cap\Delta(\gamma)$. Therefore there exists not vertex of $\Gamma(\alpha)$ such that is joined to β and ρ.
For any vertex of $\Delta(\alpha)\Delta(\gamma)\backslash\{\rho\}$, the above holds. Thus, let σ be the vertex of $\Delta(\rho)\backslash\Delta(\alpha)\backslash\{\alpha\}$, then we have

$$\Delta(\alpha)\cap\Delta(\sigma) = \{\rho\}\,.$$

This is a contradiction. q.e.d.

EXAMPLE.

Let $G=S^{\Omega}$ be the symmetric group on $\Omega=\{0, 1, \cdots, 2r\}$ for $r\geqq 2$, and the subgroup $H=S_r\int z_2$ of G be the wreath product of the symmetric group of degree or and the group of order 2 and act on $\{1, \cdots, 2r\}$.

We choose the subset E of the unordered pair of $\{Hx\}_{x\in G}$ such that (Hx, Hy) is the element of E if and only if xy^{-1} is the element of $H\sigma H$

$$(0^\sigma = 1,\ 1^\sigma = 0,\ i^\sigma = i,\ i = 2, \cdots, 2r)\,.$$

Then, $\Gamma=(\{Hx\}_{x\in G}, E)$ satisfies the conditions 1, 2, 3 of the Theorem 1 and the diameter of Γ is at least three and $\mathfrak{S}=1$.

OSAKA UNIVERSITY

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