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ON THE BASIC G-SPACE IN EQUIVARIANT K-THEORY

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1. Introduction

Let G be a compact, connected Lie group such that $\pi_1(G)$ is torsion free and let $\mathcal{A}_G$ denote the category of compact, locally contractible G-spaces of finite covering dimension and G-maps. Throughout this paper all spaces will be supposed to be in $\mathcal{A}_G$ and K_G^* will denote the equivariant K-theory defined in [5]. We use the following definition by Hodgkin [1].

DEFINITION. A G-space Z is called *a basic G-space* if the following conditions are satisfied.

(i) $K_G^*(Z)$ is projective as an $R(G)$ $(=K_G^*(\text{point}))$-module.

(ii) For any $X\in\mathcal{A}_G$ the external product homomorphism

$$K_G^*(Z)\underset{R(G)}{\otimes}K_G^*(X)\to K_G^*(Z\times X)$$

is an isomorphism.

Using the notation of [1], Snaith [6] proved that if G is a torus then $\Gamma_G^*(\_,\_)$ vanishes.

In this paper we give a simple proof of Snaith's theorem ([6], Theorem 2.11) and show that if G is $SU(n)$, $U(n)$, $Sp(n)$ or G_2 then $\Gamma_G^*(\_,\_)$ vanishes.

Consider the construction of the Künneth formula spectral sequence [1], then we see that the above statements are equivalent to the following

Theorem 1.1 (Snaith [6]). *Let T be a torus and Z a T-space. If $K_T^*(Z)$ is projective as an $R(T)$-module then the T-space Z is a basic T-space.*

Theorem 1.2. *Let G denote the (special) unitary group $(SU(n))$ $U(n)$, the sympletic group $Sp(n)$ or the exceptional group G_2, and let Z be a G-space. If $K_G^*(Z)$ is projective as an $R(G)$-module then the G-space Z is a basic G-space.*

In the following sections we denote by μ the external product homomorphism $K_G^*(X)\underset{R(G)}{\otimes}K_G^*(Y)\to K_G^*(X\times Y)$.

2. Proof of (1.1)

Lemma 2.1. *Let T be the n-dimensional torus and S a closed subgroup of T. If $K_T^*(Z)$ is projective as an $R(T)$-module for a T-space Z then*

$$\mu\colon R(S)\underset{R(T)}{\otimes}K_T^*(Z)\to K_S^*(Z)$$

is isomorphic.

Proof. First we consider the following situation: Let $T=Z_{m_1}\times\cdots\times Z_{m_{r-1}}\times S_r^1\times S_{r+1}^1\times\cdots\times S_n^1$, $S=Z_{m_1}\times\cdots\times Z_{m_{r-1}}\times Z_{m_r}\times S_{r+1}^1\times\cdots\times S_n^1$ where Z_{m_j} is a cyclic group of order m_j and S_k^1 is the circle group, $(1\leq j\leq r,\ r\leq k\leq n)$, such that $Z_{m_r}\subset S_r^1$, and let Z be a T-space such that $K_T^*(Z)$ is $R(T)$-projective.

Let $C(T/S)$ be the cone on T/S. Then $C(T/S)-T/S$ is isomorphic to the representation space V of the m_r-fold tensor product of the canonical 1-dimensional, non-trivial representation t_r of S_r^1 since $T/S=S_r^1/Z_{m_r}$ is isomorphic to S^1.

Consider the exact sequence for the pair $(C(T/S)\times Z,\ T/S\times Z)$ then we get the diagram

$$\begin{array}{ccccc}\to K_T^*(V\times Z) & \xrightarrow{j^*} & K_T^*(Z) \to K_S^*(Z)\to \\ \varphi_*\uparrow & & \\ K_T^*(Z) & & \end{array}$$

where the row is an exact sequence, φ_* is the Thom isomorphism and $j^*\varphi_*(1)=1-t_r^{m_r}$. Since $K_T^*(Z)$ is $R(T)$-projective and $R(S_r^1)$ has no zero divisors we get a short exact sequence

$$0\to K_T^*(Z)\xrightarrow{(1-t_r^{m_r})\cdot}K_T^*(Z)\to K_S^*(Z)\to 0$$

from the above diagram.

Apply the functor $\underset{R(T)}{\otimes}K_T^*(Z)$ to the exact sequence obtained by putting $Z=a$ point in the above short exact sequence then we also have an exact sequence

$$0\to K_T^*(Z)\xrightarrow{(1-t_r^{m_r})\cdot}K_T^*(Z)\to R(S)\underset{R(T)}{\otimes}K_T^*(Z)\to 0$$

Here consider the commutative diagram

$$\begin{array}{ccccccc} 0\to K_T^*(Z) & \xrightarrow{f} & K_T^*(Z) & \longrightarrow & K_S^*(Z) & \longrightarrow & 0 \\ \| & & \| & & \uparrow\mu & & \\ 0\to K_T^*(Z) & \xrightarrow{f} & K_T^*(Z) & \to & R(S)\underset{R(T)}{\otimes}K_T^*(Z) & \to & 0 \end{array}$$

where the rows are exact and $f=(1-t_r^{m_r})\cdot$. Then we see that $\mu\colon R(S)\underset{R(T)}{\otimes}K_T^*(Z)\to K_S^*(Z)$ is an isomorphism by the five lemma.

In the general case we may consider that $T=S_1^1\times\cdots\times S_l^1\times H$, $S=Z_{m_1}\times\cdots\times Z_{m_l}\times H$ and $Z_{m_j}\subset S_j^1$, $(1\leq j\leq l)$, where H is a torus, by Proof of [1], Lemma 7.1 or [6], Lemma 2.3.

Put $S_k = Z_{m_1}\times\cdots\times Z_{m_h}\times S^1_{k+1}\times\cdots\times S^1_l\times H$ for $0\leq k\leq l$. By the preceding discussion we have an isomorphism

$$R(S_k)\underset{R(S_{k-1})}{\otimes} K^*_{S_{k-1}}(Z)\to K^*_{S_k}(Z)$$

for $1\leq k\leq l$ inductively. This completes the proof of Lemma 2.1.

Proof of (1.1). $K^*_T(Z)\underset{R(T)}{\otimes}K^*_T(\_)$ is a cohomology theory since $K^*_T(Z)$ is $R(T)$-projective and $K^*_T(Z\times\_)$ is so. Using the Segal's spectral sequence [5] and the natural transformation μ: $K^*_T(Z)\underset{R(T)}{\otimes}K^*_T(\_)\to K^*_T(Z\times\_)$, compare these cohomology theories. Then Lemma 2.1 shows that μ induces an isomorphism of the E_2-terms of these spectral sequences. Therefore this concludes (1.1).

3. Proof of (1.2)

Let T be a maximal torus of G. According to [6], §3 it suffices to show that

(3.1) $\mu_G=\mu$: $R(T)\underset{R(G)}{\otimes}R(T)\to K^*_T(G/T)$ *is an isomorphism*

for a proof of (1.2). However, from Proof of [6], Theorem 3.6 we see that

(3.2) μ_G *is a monomorphism for any compact, connected Lie group G such that* $\pi_1(G)$ *is free.*

Therefore it suffices to prove that μ_G is an epimorphism.

Now, since $R(T)$ is a projective $R(G)$-module [4], we see by (1.1) that

(3.3) *If* (3.1) *is true then the T-space* G/T *is a basic T-space.*

(1) *Proof for* $U(n)$. This follows from [5], Proposition (3.9) (See [6], Corollary 3.7).

(2) *Proof for* $SU(n)$. Let T be a maximal torus of $U(n)$ and put $ST= T\cap SU(n)$. Then ST is a maximal torus of $SU(n)$ and $SU(n)/ST\cong U(n)/T$ as T-spaces.

By (1) and (3.3), $U(n)/T$ is a basic T-space and so

$$\begin{aligned} K^*_{ST}(U(n)/T) &\cong K^*_T(T/ST\times U(n)/T)\\ &\cong R(ST)\underset{R(T)}{\otimes}K^*_T(U(n)/T)\\ &\cong R(ST)\underset{R(U(n))}{\otimes}R(T)\,. \end{aligned}$$

Hence we get the following commutative diagram

$$\begin{array}{ccc} K^*_{ST}(SU(n)/ST) & \xleftarrow{\cong} & K^*_{ST}(U(n)/T) \\ \mu\uparrow & & \uparrow\cong \\ R(ST)\underset{R(SU(n))}{\otimes}R(ST) & \xleftarrow{1\otimes i^*} & R(ST)\underset{R(U(n))}{\otimes}R(T) \end{array}$$

where i: $SU(n)\to U(n)$ is the inclusion of $SU(n)$, and this shows that μ is surjective for $G=SU(n)$.

(3) *Proof for $Sp(n)$*. We regard $Sp(n)$ as a closed subgroup of $U(2n)$ by the canonical embedding. Then $Sp(1)=SU(2)$ and so the proof for $Sp(1)$ follows from (2). We shall prove the case of (3) by induction on n.

Suppose $Sp(k)$ satisfies (3.1) for $1\leq k\leq n-1$. Then (3.1) is true for $Sp(n-1)\times Sp(1)$. Because

$$\begin{aligned} K^*_{T_1\times T_2}(Sp(n-1)\times Sp(1)/T_1\times T_2) &\cong K^*_{T_1}(Sp(n-1)/T_1)\otimes K^*_{T_2}(Sp(1)/T_2) \\ &\cong R(T_1\times T_2)\underset{R(Sp(n-1)\times Sp(1))}{\otimes}R(T_1\times T_2) \end{aligned}$$

where T_1 and T_2 are maximal tori of $Sp(n-1)$ and $Sp(1)$ respectively, by the inductive hypothesis and [3]. Therefore, by [6], Theorem 3.6 $Sp(n-1)\times Sp(1)/T$ is a basic $Sp(n-1)\times Sp(1)$-space and so

$$R(T)\underset{R(Sp(n-1)\times Sp(1))}{\otimes}K^*_{Sp(n-1)\times Sp(1)}(Sp(n)/T)\cong K^*_T(Sp(n)/T)$$

where T is the standard maximal torus of $Sp(n)$. Hence it suffices to show that

$$R(T)\underset{R(Sp(n))}{\otimes}R(Sp(n-1)\times Sp(1))\to K^*_T(Sp(n)/Sp(n-1)\times Sp(1))$$

is an isomorphism, because of $K^*_T(Sp(n)/Sp(n-1)\times Sp(1))\cong K^*_{Sp(n-1)\times Sp(1)}(Sp(n)/T)$.

Put $R(T)=Z[t_1,\cdots,t_n;t_1^{-1},\cdots,t_n^{-1}]$, then $R(Sp(n))=Z[\sigma_1,\cdots,\sigma_n]$ as a subring where σ_k is the k-th elementary symmetric function in the n variables $t_1+t_1^{-1},\cdots,t_n+t_n^{-1}$ ([2], §13, Theorem 6.1).

Define the ring homomorphism ϕ: $R(Sp(n))[\theta]\to R(Sp(n-1)\times Sp(1))$ by the restriction $R(Sp(n))\to R(Sp(n-1)\times Sp(1))$ and the correspondence $\theta\mapsto t_n+t_n^{-1}$. Then we have

Lemma 3.1. $R(Sp(n))[\theta]/(\sum_{j=0}^n(-1)^j\sigma_j\theta^{n-j})\cong R(Sp(n-1)\times Sp(1))$.

Proof. By the definition of ϕ, ϕ is surjective obviously.

If $\phi(f(\theta))=0$ for $f(\theta)\in R(Sp(n))$ then $(\theta-(t_n+t_n^{-1}))$ divides $f(\theta)$. By symmetry, $(\theta-(t_j+t_j^{-1}))$ divides $f(\theta)$ for $1\leq j\leq n$. Hence $\sum_{j=0}^n(-1)^j\sigma_j\theta^{n-j}$ divides $f(\theta)$. This shows Lemma 3.1.

The following lemma completes the proof for $Sp(n)$ by the preceding discussion.

Lemma 3.2. *μ: $R(T)\underset{R(Sp(n))}{\otimes}R(Sp(n-1)\times Sp(1))\rightarrow K_T^*(Sp(n)/Sp(n-1)\times$ $S(1))$ is an isomorphism for any $n\geq 2$.*

Proof. $Sp(n)/Sp(n-1)\times Sp(1)$ is homeomorphic to the projective space of dimension $n-1$ over the quaternion number field. By the canonical embedding $P^{n-2}(\boldsymbol{Q})\subset P^{n-1}(\boldsymbol{Q})$ we have an equivariant embedding i: $Sp(n-1)/Sp(n-2)\times Sp(1)\subset Sp(n)/Sp(n-1)\times Sp(1)$.

For simplicity we write $P^{n-1}(\boldsymbol{Q})$ for $Sp(n)/Sp(n-1)\times Sp(1)$. Then we have

(a) μ': $R(T)\underset{R(Sp(n-1))}{\otimes}R(Sp(n-2)\times Sp(1))\xrightarrow{\cong}K_T^*(P^{n-2}(\boldsymbol{Q}))$

by the inductive hypothesis and

(b) μ: $R(T)\underset{R(Sp(n))}{\otimes}R(Sp(n-1)\times Sp(1))\rightarrow K_T^*(P^{n-1}(\boldsymbol{Q}))$ is a monomorphism

by the analogous argument to the proof for (3.2). Moreover the T-space $P^{n-1}(\boldsymbol{Q})-P^{n-2}(\boldsymbol{Q})$ is isomorphic to the representation space W of $t_1t_n^{-1}\oplus\cdots\oplus t_{n-1}t_n^{-1}\oplus t_1^{-1}t_n^{-1}\oplus\cdots\oplus t_{n-1}^{-1}t_n^{-1}$.

Consider the exact sequence for the pair $(P^{n-1}(\boldsymbol{Q}), P^{n-2}(\boldsymbol{Q}))$, then by Lemma 3.1, (a) and (b) we obtain the diagram

$$\begin{array}{ccccccc}
0\rightarrow K_T^*(W) & \xrightarrow{j^*} & K_T^*(P^{n-1}(\boldsymbol{Q})) & \xrightarrow{\quad i^*\quad} & K_T^*(P^{n-2}(\boldsymbol{Q}))\rightarrow 0 \\
\varphi_*\uparrow & & \mu\uparrow & & \cong\uparrow\mu' \\
R(T) & & R(T)[\theta]/(\sum_{j=0}^{n}(-1)^j\sigma_j\theta^{n-j}) & & R(T)[\theta']/(\sum_{j=0}^{n-1}(-1)^j\sigma'_j\theta'^{n-j-1}) \\
 & & \uparrow & & \\
 & & 0 & &
\end{array}$$

where the row is an exact sequence, φ_* is the Thom isomorphism and the definition of θ' and σ'_j, $(0\leq j\leq n-1)$, are similar to that of θ and σ_j. In this diagram we see that i^* is surjective from the fact that $i^*(\mu(\theta))=\mu'(\theta')$, and furthermore we can easily check that $j^*\varphi_*(1)=(t_n^{-1})^{n-1}\sum_{j=0}^{n-1}(-1)^j\sigma'_j\mu(\theta)^{n-j-1}$. Therefore we see that μ is surjective. q.e.d.

This completes the induction.

(4) *Proof for G_2.* G_2 contains $SU(3)$ as a closed subgroup of maximal rank and the homogeneous space $G_2/SU(3)$ is homeomorphic to the unit sphere S^6.

Let T denote a maximal torus of $SU(3)$ and put $R(T)=Z[t_1, t_2, t_3;\ t_1^{-1}, t_2^{-1}, t_3^{-1}]/(t_1t_2t_3-1)$. Moreover we denote the representation space of $t_1\oplus t_2\oplus t_3$ by W and the unit sphere in $\boldsymbol{R}\oplus W$ by $S(\boldsymbol{R}\oplus W)$ where $\boldsymbol{R}$ is the real number field.

Then we see easily that

Lemma 3.3. *$G_2/SU(3)$ is homeomorphic to $S(R\oplus W)$ as T-spaces.*

The following lemma completes the proof for G_2 by the same reason as for $Sp(n)$.

Lemma 3.4. $\mu\colon R(T)\underset{R(G_2)}{\otimes} R(SU(3))\to K_T^*(G_2/SU(3))$ *is an epimorphism.*

Proof. Consider the exact sequence for the pair consisting of the unit ball $D(W)$ and the unit sphere $S(W)$ in W, then we have the diagram

$$\begin{array}{ccccccc} 0\to K_T^*(W) & \xrightarrow{j^*} & K_T^*(D(W)) & \xrightarrow{i^*} & K_T^*(S(W))\to 0 \\ \varphi_*\uparrow & & \| & & \\ R(T) & & R(T) & & \end{array}$$

where the row is exact and φ_* is the Thom isomorphism, and then we get

$$K_T^*(S(W))=R(T)/(\lambda_2-\lambda_1)$$

since $j^*\varphi_*(1)=\lambda_2-\lambda_1$ where λ_1 and λ_2 are the ring generators of $R(SU(3))$ as in [2], §13, Theorem 3.1.

Next we divide $S(\boldsymbol{R}\oplus W)$ into two closed T-subspaces $D^\pm$ as follows: Put $D^\pm=\{(r, z_1, z_2, z_3)\in S(\boldsymbol{R}\oplus W);\ r\geq 0 \text{ or } r\leq 0\}$ and then $D^+\cup D^-=S(\boldsymbol{R}\oplus W)$ and $D^+\cap D^-=S(W)$. Consider the diagram obtained by the Mayer-Vietoris exact sequence for the triple $(S(\boldsymbol{R}\oplus W);\ D^+,\ D^-)$ then we obtain the diagram

$$\begin{array}{ccccc} 0\to K_T^*(S(\boldsymbol{R}\oplus W)) & \xrightarrow{(j_+^*,\, j_-^*)} & K_T^*(D^+)\oplus K_T^*(D^-) & \xrightarrow{i_+^*-i_-^*} & K_T^*(S(W))\to 0 \\ \mu\uparrow & & \| & & \wr\| \\ R(T)\underset{R(G_2)}{\otimes} R(SU(3)) & & R(T)\oplus R(T) & & R(T)/(\lambda_2-\lambda_1) \end{array}$$

where the row is exact and $j_\pm\colon D^\pm\to S(\boldsymbol{R}\oplus W)$ and $i_\pm\colon S(W)\to D^\pm$ are the inclusion maps. Then we see that $K_T^*(S(\boldsymbol{R}\oplus W))$ is isomorphic to the submodule of $R(T)\oplus R(T)$ over $R(T)$ generated by (1,1) and $(\lambda_2-\lambda_1, 0)$, and μ satisfies $(j_+^*, j_-^*)\mu(1\otimes 1)=(1, 1)$ and $(j_+^*, j_-^*)\mu(1\otimes\lambda_1)=(\lambda_1, \lambda_2)$. This shows that μ is surjective.

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