Araki, S.
Osaka J. Math.
**11** (1974), 347–352

# $[p]$-TYPICAL FORMAL GROUPS AND THE HOMOMORPHISM $\Omega_*^U \to \Omega_*^{SO}$

SHÔRÔ ARAKI

(Received October 30, 1973)

In the present note we construct a $[p]$-typical formal group $F_{[p]}$, $p$ a prime, which is universal for $[p]$-typical formal groups over arbitrary ground rings; and we study structure of the ground ring of $F_{[p]}$ (Corollary 5). Using $F_{[2]}$ we describe the kernel of the forgetful homomorphism $S$: $\Omega_*^U \to \Omega_*^{SO}$ of complex structures (Corollary 7).

Our basic reference is [1] and we use the notations of [1] freely.

## 1. Universal $[p]$-typical formal group

Let $U$ be the Lazard ring and $F_U$ the universal one-dimensional formal group defined over $U$. As usual we identify $U$ with the complex cobordism ring $\Omega_*^U$. Then $U$ is graded by non-negative even degrees (or by non-positive even degrees when we regard $\Omega_*^U$ as $U^*(pt)$).

Let $p$ be a prime, $R$ a commutative ring with 1 and $F$ a (commutative one-dimensional) formal group over $R$. By the terminology of [1] $F$ is $[p]$-typical iff $\boldsymbol{f}_{p,F}\gamma_0=0$, where $\boldsymbol{f}_p$ is the Frobenius operator and $\gamma_0$ is the identity curve.

Let $F$ be $[p]$-typical and $u$: $U \to R$ the unique unitary homomorphism of rings such that $u_* F_U = F$. By the notation of [1] we put

$$(\boldsymbol{f}_{p,U}\gamma_0)(T) = \sum_{n\geq 1}{}^{F_U}(v_{np-1}^{(p)}T^n)\,.$$

Then $v_{np-1}^{(p)} \in U_{2(np-1)}$. Now

$$u_*(\boldsymbol{f}_{p,U}\gamma_0) = \boldsymbol{f}_{p,F}\gamma_0 = 0\,.$$

Hence

$$\sum_{n\geq 1}{}^{F}(u(v_{np-1}^{(p)})T^n) = 0\,,$$

and by [1], Proposition 2.10, we obtain

$$(1.1) \qquad u(v_{np-1}^{(p)}) = 0 \qquad \text{for all } n\geq 1\,.$$

Let

$$J_p = (v_{p-1}^{(p)}, v_{2p-1}^{(p)}, \cdots, v_{np-1}^{(p)}, \cdots)\,,$$

the ideal of $U$ generated by $v_{np-1}^{(p)}$, $n \geq 1$. By (1.1) $u$ factorizes as the composition of the sequence

$$U \xrightarrow{\pi_p} U/J_p \xrightarrow{u_p} R . \tag{1.2}$$

Define

$$V_{[p]} = U/J_p \quad \text{and} \quad F_{[p]} = \pi_{p*}F_U .$$

Then $u_{p*}F_{[p]}=F$ and the homomorphism $u_p$: $V_{[p]} \to R$, $u_{p*}F_{[p]}=F$, is unique by the uniqueness of $u$. And obviously $\boldsymbol{f}_{p,F_{[p]}}\gamma_0=0$. Thus we obtain

**Proposition 1.** *$F_{[p]}$ is a $[p]$-typical formal group over $V_{[p]}$ and universal for $[p]$-typical formal groups.*

In [1] we observed a $[p]$-typical formal group over a $Z[1/p]$-algebra which is universal for $[p]$-typical formal groups over $Z[1/p]$-algebras. The present $[p]$-typical formal group $F_{[p]}$ differs from that of [1] as it is universal for $[p]$-typical formal groups over arbitrary commutative rings with unities.

## 2. Structure of $\hat{U}/\hat{J}_p$

Let

$$\log_U T = \sum_{k\geq 0} m_k T^{k+1}, \quad m_0 = 1 ,$$

the logarithm of $F_U$ over $U \otimes Q$ and put

$$\hat{U} = Z[m_1, m_2, \cdots, m_k, \cdots]$$

as in [1]. As usual we can identify $\hat{U}$ with $H_*(MU)$. Then the inclusion map

$$U \subset \hat{U} \tag{2.1}$$

is identified with the Hurewicz homomorphism

$$\pi_*(MU) \to H_*(MU) .$$

Let $p$ be a prime and put

$$\hat{J}_p = (v_{p-1}^{(p)}, v_{2p-1}^{(p)}, \cdots, v_{np-1}^{(p)}, \cdots)_{\hat{U}} ,$$

the ideal of $\hat{U}$ generated by $v_{np-1}^{(p)}$, $n \geq 1$. In this section we observe structure of the quotient ring $\hat{U}/\hat{J}_p$.

Recall the relation (6.2) of [1]:

$$pm_{np-1} = v_{np-1}^{(p)} + \sum_{\substack{ij=n \\ 1\leq i<n}} m_{j-1}(v_{ip-1}^{(p)})^j . \tag{2.2}$$

This is the basic relation we use here. This shows that

(2.3) $$pm_{np-1} \in \hat{J}_p$$

on one hand, and by an induction on $n$,

(2.4) $$(1/p)v^{(p)}_{np-1} \in \hat{U}$$

on the other hand.

Let $p$ and $q$ be different primes. For each integer $k \geq 1$ we have

(2.5) $$pv^{(q)}_{pq^k-1} \in \hat{J}_p$$

Proof by induction on $k$. By (2.2) we have

$$qm_{pq-1} = v^{(q)}_{pq-1} + m_{p-1}(v^{(q)}_{q-1})^p .$$

Hence

$$pv^{(q)}_{pq-1} = qpm_{pq-1} - pm_{p-1}(v^{(q)}_{q-1})^p \in \hat{J}_p$$

by (2.3). Thus (2.5) is true for $k=1$. Now assume that $pv^{(q)}_{pq^j-1} \in \hat{J}_p$ for $j<k$. Then by (2.2) we have

$$pv^{(q)}_{pq^k-1} = pqm_{pq^k-1} - \sum_{\substack{ij=pk \\ 1 \leq i < pk}} pm_{j-1}(v^{(q)}_{q^i-1})^j .$$

$pqm_{pq^k-1} \in \hat{J}_p$ by (2.3). For each term under the summation, if $p|j$ then $pm_{j-1} \in \hat{J}_p$, and if $p \not| j$ then $p|i$ and $pv^{(q)}_{q^i-1} \in \hat{J}_p$ by induction hypothesis. Thus

$$pv^{(q)}_{pq^k-1} \in \hat{J}_p , \qquad \text{Q.E.D.}$$

Here we recall Milnor basis of $U$. Let $s_n$ denote the Chern number corresponding to $\sum t_i^n$. As is well-known a series of elements $u_n \in U_{2n}$, $n \geq 1$, forms a polynomial basis of $U$ if it satisfies

$$\begin{aligned} s_n(u_n) &= q \ \text{ when } \ n = q^s - 1 \ \text{ for some prime } \ q , \\ &= 1 \ \text{ otherwise.} \end{aligned}$$

Such a basis is called Milnor basis. We shall choose a Milnor basis in a specific form.

By (2.2) we see that

(2.6) $$s_{nq-1}(v^{(q)}_{nq-1}) = q$$

for any prime $q$ and $n \geq 1$. First we choose

(2.7) $$u_n = v^{(q)}_{q^s-1} \ \text{ when } \ n = q^s - 1, \ q \text{ a prime.}$$

Now let $p$ be the specified prime. When $p|n+1$ and $n+1$ is not a power of $p$, choosing the smallest prime $q$ dividing $n+1$ and differing from $p$, we can express $n$ as $n=pqk-1$, $k$ a pitive integer. In such a case we put

(2.8) $$u_{qpk-1} = sv^{(q)}_{pq^k-1} + tv^{(p)}_{pq^k-1} ,$$

where $s$ and $t$ are integers such that $sq+tp=1$. Then

$$s_{pqk-1}(u_{pqk-1}) = 1$$

by (2.6).

For remaining $n$, i.e., $p \nmid n+1$ and $n+1$ is not a prime power, we choose $u_n$ arbitrarily so that $s_n(u_n)=1$.

Hereafter we use only the above special choice of Milnor basis. First of all we have

$$pu_{pqk-1} \in \hat{J}_p \tag{2.9}$$

for elements of type (2.8), which follows from (2.5).

Put

$$\begin{aligned} m'_n &= (1/q)u_n \quad \text{when } n+1=q^s,\ q \text{ a prime}, \\ &= u_n \qquad\quad\ \text{when } n+1 \text{ is not a prime power.} \end{aligned} \tag{2.10}$$

These are well defined elements of $\hat{U}$ by (2.4) and

$$\hat{U} = Z[m'_1, m'_2, \cdots, m'_k, \cdots] \tag{2.11}$$

since $s_k(m'_k)=1$.

For degrees of type (2.8) we observe the elements $pm'_{pqk-1} - v^{(p)}_{pqk-1}$. These belong to $\hat{J}_p$ by (2.9) and are decomposable in $\hat{U}$ since $s_n$-numbers are zero. Thus by induction on $qk$ we can replace the ideal basis elements $v^{(p)}_{pqk-1}$ of $\hat{J}_p$ by $pm'_{pqk-1}$ for such degrees and we obtain

**Proposition 2.** $\hat{J}_p = (pm'_{pn-1},\ n \geq 1)$.

**Corollary 3.** *$\hat{U}/\hat{J}_p$ is a direct sum of copies of $Z$ and $Z/pZ$ of which each direct summand is generated by a monomial of $m'_k$'s. A monomial is of order $p$ when it contains an element $m'_k$ with $p \mid k+1$ as a factor, and otherwise of infinite order.*

## 3. Structure of $V_{[p]}$

Under our special choice of Milnor basis of $U$ we could choose a polynomial basis of $\hat{U}$ so that its each element is a constant multiple of the corresponding element of the Milnor basis (cf., (2.10)—(2.11)).

**Theorem 4.** $J_p = (u_{p^k-1},\ k \geq 1,\ pu_{np-1},\ n \neq p^s)$.

Proof. Inductively on $n$ we replace generators $v^{(p)}_{np-1}$ of $J_p$ by the elements stated in Theorem. Since $u_{p-1} = v^{(p)}_{p-1}$ the replacement is already done for $n=1$. Assume the replacement is done for $k<n$. When $n=p^s$ it is done already. Suppose $n$ is not a power of $p$ Since $pu_{pn-1} - v^{(p)}_{pn-1}$ is decomposable we can

express it as a polynomial of $u_k$'s such that $1\leq k<pn-1$, say, $P$. The polynomial expression $P'$ of $P$ in $\hat{U}$ can be obtained by replacing each monomial in $P$ by the corresponding monomial of $m'_k$ multiplied with a non-zero integer. Now $pu_{pn-1}-v^{(p)}_{pn-1}\in\hat{J}_p$ by (2.9). Then by Proposition 2 each summand of $P'$ belongs to $\hat{J}_p$. This implies that each monomial in $P$ with non-zero coefficient contains a $u_m$ with $m=ps-1$ as a factor and, when it contains no $u_m$ with $m=p^j-1$ as a factor, then $p$ divides its coefficient. Hence each summand of $P$ belongs to $J_p$ and $pu_{pn-1}-v^{(p)}_{pn-1}\in J_p$. Now we can replace $v^{(p)}_{pn-1}$ by $pu_{pn-1}$ in the system of generators of $J_p$, Q.E.D.

**Corollary 5.** *$V_{[p]}$ is a direct sum of copies of $Z$ and $Z/pZ$ of which each direct summand is generated by a monomial of $u_n$'s such that $n\neq p^s-1$. A monomial is of order $p$ when it contains an element $u_k$ with $p\,|\,k+1$ as a factor, and otherwise of infinite order.*

## 4. The forgetful homomorphism $\Omega^U_*\to\Omega^{SO}_*$

Let

$$S:\ \Omega^U_*\to\Omega^{SO}_* \quad\text{and}\quad \Psi:\ \Omega^{SO}_*\to\mathfrak{N}_*$$

be the forgetful homomorphisms of complex structures and orientations respectively. Milnor [2] observed that

$$(\Psi\circ S)(\Omega^U_*)=(\mathfrak{N}_*)^2\,,$$

where $(\mathfrak{N}_*)^2$ is the subalgebra of $\mathfrak{N}_*$ consisting of bordism classes of manifold squares $N\times N$. Let $M$ be a weakly complex $2n$-manifold and $\Psi\circ S(M)=[N\times N]$. Then the Milnor's result shows that

$$s_n(M)\equiv s'_n(N) \mod 2\,,$$

where $s'_n$ denotes the Whitney number corresponding to $\sum t_i{}^n$. Thus we have a polynomial basis $\{x_n,\ n\neq 2^h-1\}$ of $\mathfrak{N}_*$ such that

$$(\Psi\circ S)(u_n)=x_n{}^2,\quad n\neq 2^h-1\,.$$

and $\Psi\circ S$ induces an isomorphism

$$\Omega^U_*/(u_{2^k-1},\ k\geq 1)\otimes Z/2Z\cong(\mathfrak{N}_*)^2\,. \tag{4.1}$$

As we remarked in [1], §5, the oriented cobordism $\Omega^*(\ \ )$ is complex-oriented, [2]-typical and $S_*F_U=F_{SO}$. Thus $S$ factorizes as the composition of the sequence

$$\Omega^U_*=U\xrightarrow{\pi_2}V_{[2]}\xrightarrow{\Phi}\Omega^{SO}_*\,,$$

By Corollary 5 we have

(4.2) $$V_{[2]}\otimes Z/2Z=Z/2Z[u_n,\ n\neq 2^h-1]\,.$$

By (4.1) and (4.2) we see that $\Psi\circ\Phi$ induces the isomorphism

(4.3) $$V_{[2]}\otimes Z/2Z\cong(\mathfrak{N}_*)^2\,.$$

By Corollary 5 we have

$$V_{[2]}/\mathrm{Tors}=Z[u_{2n},\ n\geq 1]\,.$$

Then by [3], p. 180, we conclude that

(4.4) $$\Phi/\mathrm{Tors}\colon V_{[2]}/\mathrm{Tors}\cong\Omega_*^{SO}/\mathrm{Tors}.$$

Finally by (4.3) and (4.4) we obtain

**Theorem 6.** *$\Phi\colon V_{[2]}\to\Omega_*^{SO}$ is an injection.*

**Corollary 7.** Ker $S=J_2$, Im $S\cong V_{[2]}$.

OSAKA CITY UNIVERSITY

## References

[1] S. Araki: Typical Formal Groups in Complex Cobordism and K-Theory, Lectures in Mathematics, Kyoto Univ., **6**, Kinokuniya Book-Store, 1973.

[2] J. Milnor: *On the Stiefel-Whitney numbers of complex manifolds and of spin manifolds* Topology **3** (1965), 223–230.

[3] R.E. Stong: Notes on Cobordism Theory, Princeton Univ. Press, 1968.