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Osaka J. Math.
11 (1974), 307–312

WOVEN KNOTS ARE SPUN KNOTS

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(Received November 10, 1971)
(Revised December 1, 1973)

Given a knotted 1-sphere, k, in R^3 it is possible to find a knotted 2-sphere, K, in R^4 such that $\Pi_1(R^3-k)$ is isomorphic to $\Pi_1(R^4-K)$. In [1], Artin constructs one such example called a spun knot; in [3], Yajima also gives an example which we will refer to as a woven knot. The object of this paper is to show that these knots are, in fact, the same; that is, given k, the corresponding spun knot and the woven knot constructed from the mirror image of k are ambiently isotopic.

By a knotted n-sphere in R^{n+2}, we will mean an ambient isotopy class of embeddings of S^n into R^{n+2}. Sometimes, in order to avoid proliferation of notations we will use the same letter to denote a map and the image of that map. We will also generalize this construction to other types of spinnings of higher dimensional knots.

We will use PL spheres in our constructions. We will use the following notion of general position: if γ is a PL n-sphere in R^{n+2}, we will say γ is in general position if for each vertex, v, and k-simplex σ of γ, with v not a vertex of σ, γ is not contained in the k-plane of R^{n+2} determined by σ.

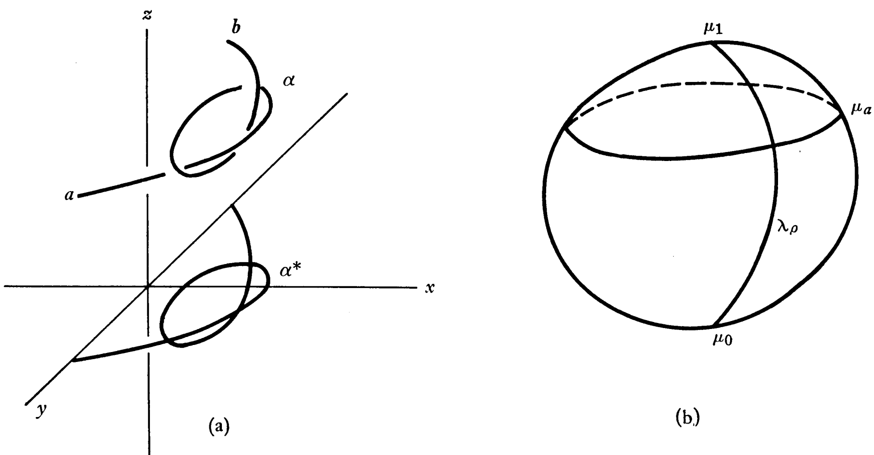


Figure 1

Suppose γ is an n-sphere in R^{n+2}; let $R_+^{n+2}=\{(x_1, \cdots, x_{n+2})\in R^{n+2}$ with $x_1\geqq 0\}$, let $\partial R_+^{n+2}=\{(x_1, \cdots, x_{n+2})\in R^{n+2}$ with $x_1=0\}$. Also, define $h: R_+^{n+2}\to R^1$ by $h(x_1, \cdots, x_{n+2})=x_{n+2}$; we may think of h as a height function. Without loss of generality, we may assume that γ is the union of two n-disks α and β such that $\alpha\cap\beta$ is an $(n-1)$-sphere, and (1) $\gamma(S^n)\subseteq R_+^{n+2}$, such that $h\circ\gamma>0$ (i.e., γ lies above the half-$(n+1)$-plane in R_+^{n+2} given by $x_{n+2}>0$; (2) $\gamma(S^n)\cap\partial R_+^{n+2}=\beta$; (3) if p: $R_+^{n+2}\to R_+^{n+1}$ is given by $p(x_1, \cdots, x_{n+1}, x_{n+2})=(x_1, \cdots, x_{n+1})$, then we will require that $p|\beta$ is an embedding (all that we will ever use is that $p|\partial\beta=p|\partial\alpha$ is an embedding); (μ) γ is in general position. If γ is a circle in R^3, α is an arc as in figure 1 (a).

To describe the spun knot, we will write points of $R^{n+k+2}\approx R^{k+1}\times R^{n+1}$ in the form $(z\rho, x_{k+2}, \cdots, x_{n+k+2})$ where ρ is a unit vector in the first $(k+1)$-coordinates and $z\geqq 0$. For each ρ, let H_ρ denote the half-$(n+2)$-hyperplane of all points of the form $(z\rho, x_{k+2}, \cdots, x_{n+k+2})$. Then the maps h_ρ defined by $h_\rho(x_1, \cdots, x_{n+2})=(x_1\,\rho, x_2 \cdots, x_{n+2})$ are embeddings of R_+^{n+2} into R^{n+k+2}, and $\bigcup_\rho h_\rho(R_+^{n+2})=R^{n+k+2}$.

We will need the following notations for subsets of the $(n+k)$-sphere. We will consider S^{n+k} to be the unit sphere $R^{n+k+1}\approx R^{k+1}\times R^n$ and denote points by $(z\rho, x_{k+2}, \cdots, x_{n+k+1})$ where ρ is a unit vector in the first $k+1$ coordinates, $z\geqq 0$; we will consider D^n to be the unit disk in R^{n+1}. Let $\lambda\rho$ be the n-disk in S^{n+k} which is the image of the map $\lambda\rho(x_1, \cdots, x_n)=(\sqrt{1-\sum x_i^2}\,\rho, x_1, \cdots, x_n)$; $\lambda\rho$ is the intersection of S^{n+k} with the set of all points of the form $(z\rho, x_{k+2}, \cdots, x_{n+k+1})$. For each point $a\in D^n$, $a=(a_1, \cdots, a_n)$, define a map μ_a: $S^k\to S^{n+k}$ by $\mu_a(x_1, \cdots, x_{k+1})=(\eta_a x_1, \eta_a x_2, \cdots, \eta_a x_{k+1}, a_1, \cdots, a_n)$ where $\eta_a=\sqrt{1-\sum a_i^2}$. Thus μ_a is the intersection of S^{n+k} with the set of points $(x_1, \cdots, x_{k+1}, a_1, \cdots, a_n)$; also we may see that μ_a is a k-sphere of radius η_a if $a\in\operatorname{Int} D^n$, μ_a is a point if $a\in\partial D^n$. If we are spinning an arc, then S^{n+k} is a 2-sphere, and $\lambda\rho$ is a longitudinal are, μ_a is a meridian circle, or a pole, see figure 1(b).

We will now define an embedding S_α^k: $S^{n+k}\to R^{n+k+2}$ by requiring for each ρ, $S_\alpha^k\circ\lambda_\rho=h_\rho\circ\alpha$. The isotopy class of S_α^k will be called the knot obtained by k-spinning α. We remark that if α and α' are two n-disks in R^{n+2} and α_t is an isotopy with $\alpha_0=\alpha$, $\alpha_1=\alpha'$ and for all t, $0\leqq t\leqq 1$, $\alpha_t\cap R_+^{n+2}=\alpha_t(\partial D^n)$, then there is an isotopy, K_t, between the sphere obtained k-spinning α and that obtained by k-spinning α'; the isotopy is defined so that for all t, $h_\rho(\alpha_t)=K_t(\lambda_\rho)$.

We will want to examine the projection of S_α^k by projection along the last coordinate, x_{n+k+2}. Let Π be this projection; $\Pi(z\rho, x_{k+2}, \cdots, x_{n+k+1}, x_{n+k+2})=(z\rho, x_{k+2}, \cdots, x_{n+k+1})$. Let p: $R_+^{n+2}\to R_+^{n+1}$ be as before; let $\alpha^*=p(\alpha)$. For each ρ, we may define embeddings h_ρ': $R_+^{n+1}\to R^{n+k+1}$ by $h_\rho'(x_1, \cdots, x_{n+1})=(x_1\rho, x_2, \cdots, x_{n+1})$. Since $\Pi\circ h_\rho=h_\rho'\circ p$, $\Pi(S_\alpha^k)=\Pi(\bigcup_\rho h_\rho(\alpha))\ \bigcup_\rho \Pi h_\rho(\alpha)=\bigcup_\rho h_\rho'(\alpha^*)$. We may state this as follows: The projection of the k-spinning of α is the same as the k-

spinning of the projection of α (for the spinning of the arc of figure 1, see figure 2; figure 2(b) shows $\Pi(S^1_\alpha)$ with $\bigcup_\psi h_\psi'(\alpha^*)$ removed where $0<\psi<\Pi/2$). We may also describe $\Pi(S^k_\alpha)$ as follows; if $b\in\alpha$ with $b=\alpha(a)$ with $a\in D^n$, let $A_b=\bigcup_\rho h_\rho(b)$, A_b will be a k-sphere if $a\in \operatorname{Int} D^n$, a point if $a\in\partial D^n$, let $A^*_b=\Pi(A_b)=\bigcup_\rho h_\rho'(b)$, then $\Pi(S^k_\alpha)=\bigcup_{b\in\alpha} A^*_b$. If M_r is the set of points of multiplicity r of α under p, that is, $M_r=\{x\in\alpha^*$ such that $p^{-1}(x)\cap\alpha$ consists of exactly r points$\}$, and if M_r' is the set of points of multiplicity r of S^k under Π, $M_r'=\{x\in\Pi(S^k_\alpha)$ such that $\Pi^{-1}(x)\cap S^k_\alpha$ consists of exactly r points$\}$, then M_r' is obtained by k-spinning M_r, i.e., $M_r'=\{\bigcup_\rho h_\rho'(x)$ where $x\in M_r\}$. In the case of spinning a 1-sphere, each double point of the projection will correspond to a circle of double points of the spun knot. Furthermore, suppose that $b,b'\in\alpha$ with $p(b)=p(b')$ and $h(b)<h(b')$, then for all ρ, the x_{n+k+2}-coordinate of $h_\rho(b)$ will be less than the x_{n+k+2}-coordinate of $h_\rho(b')$ (since these will be equal to $h(b)$ and $h(b')$, respectively), denote this by $A_b<A_b'$.

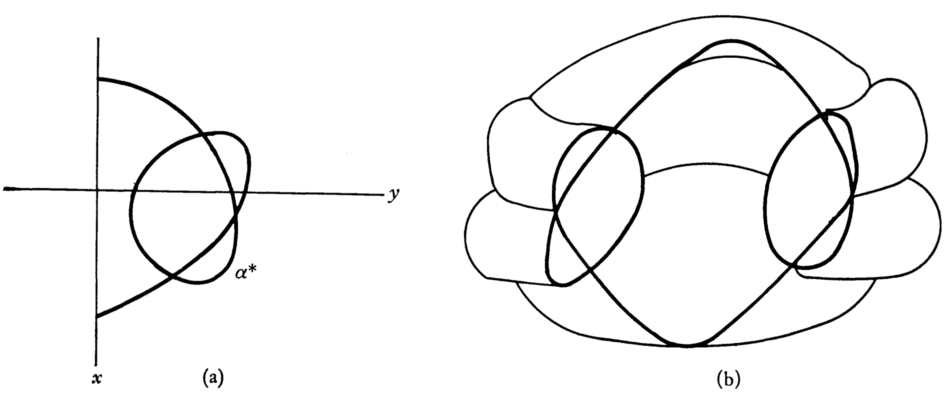


Figure 2

We next describe another embedding of S^{n+k} into R^{n+k+2}, the woven knot. As before, we begin with α. Recall that $h(b)>0$ for all $b\in\alpha$; let M be a number such that $M>h(b)$ for all $b\in\alpha$. By our general position, we may find an ε such that if v is a vertex of α, σ a k-simplex of α with $v\notin\sigma$, then ε is less than the distance between v and the k-plane of R^{n+2} determined by σ. Now suppose that α is given by $\alpha(a)=(x_1(a),\cdots,x_{n+2}(a))$, let $x_1'(a)=x_1(a)(1+(\varepsilon x_{n+2}(a))/M$, and for t, $0\le t\le 1$, $(x_1)_t(a)=x_1(a)(1+(t\varepsilon x_{n+2}(a))/M)$. Next define $\alpha'(a)=(x_1'(a), x_2(a),\cdots,x_{n+2}(a))$, $\alpha_t(a)=((x_1)_t(a), x_2(a),\cdots,x_{n+2}(a))$, then $\alpha_t(a)$ is an isotopy in R^{n+2}_+ from α to α' fixed on $\partial\alpha$. If $a\in D^n$, $a=(a_1,\cdots,a_n)$, let H_a be the $(k+1)$-hyperplane of $R^{n+k+1}=R^{k+1}\times R^n$ of the form $(x_1,\cdots,x_{k+1},a_1,\cdots,a_n)$, then $\mu_a=S^{n+k}\cap H_a$. Let k_a: $H_a\to R^{n+k+2}$ be the map which takes H_a to a hyperplane of R^{n+k+2} by a map which takes μ_a to a circle of radius $x_1'(a)$ defined as follows:

let $\nu_a = x_1'(a)/\eta_a$ if $\eta_a \neq 0$, $\nu_a = 0$ if $\eta_a = 0$ (i.e., if $a \in \partial D^n$), then define $k_a(x_1, \cdots, x_{k+1}, a_1, \cdots, a_n) = (\nu_a x_1, \cdots, \nu_a x_{k+1}, x_2(a), x_3(a), \cdots, x_{n+1}(a), x_1(a))$. Note that the last coordinate is given by $x_1(a)$.

Now we define an embedding W^k_α: $S^{n+k} \to R^{n+k+2}$ by requiring that $W^k_\alpha \circ \mu_a = k_a \circ \mu_a$, or $W^k(\mu_a) = k_a(\mu_a)$. The isotopy class of W^k_α will be called the k-woven knot corresponding to γ.

We will now discuss the special case of 1-weaving a 1-sphere, illustrating with the particular example of the trefoil knot of figure 1(a). In this case, α' can be described as a slight distortion of α which, above the doublepoints of α^*, bends α on the overpasses away from ∂R^3_+ more than on the underpasses. Thus $(\alpha')^*$ looks like figure 3(a). If $\alpha(a) = (x_1(a), x_2(a), x_3(a))$, with $a \in D^1$, $\alpha^*(a) = (x_1(a), x_2(a))$. Let P^3 be the hyperplane in R^4 with last coordinate zero. Let R_α be the set of points of the form $(0, y, x_1(a), x_2(a))$ with $|y| \leq x_1'(a)$, see figure 3(b). Then R_α is a ribbon in P^3 and if Π', Π': $R^4 \to P^3$, is defined by $\Pi'(x_1, x_2, x_3, x_4) = (0, x_2, x_3, x_4)$ then $\Pi'(W^1_\alpha) = R_\alpha$. In fact, we may see that W^1_α is the symmetric ribbon knot of R_α, see Yajima [4]. Furthermore, it is clear from the discussion in Yajima [4], page 137, that W^1_α is the same as the 2-sphere similar to the knot γ, defined in Yajima [3]. From the discussion which is to follow, we will see that W^1_α will be a spun knot; thus the knots defined in Yajima [3] are all spun knots.

For convenience we will describe Yajima's construction [3] and illustrate it with the trefoil knot. Given a knot γ and the corresponding knotted arc, α, we construct a self-intersecting tube around the projection, α^*, of α, narrowing the tube along the arc at the underpasses and closing off the tube at the end points of α^* (see figure 3). This describes the projection of a knotted 2-sphere; to

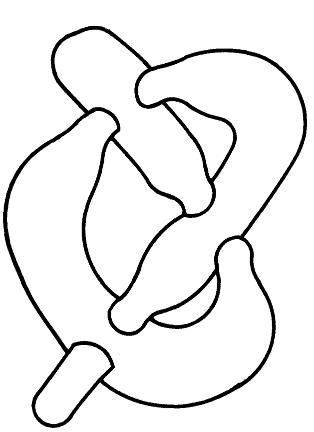
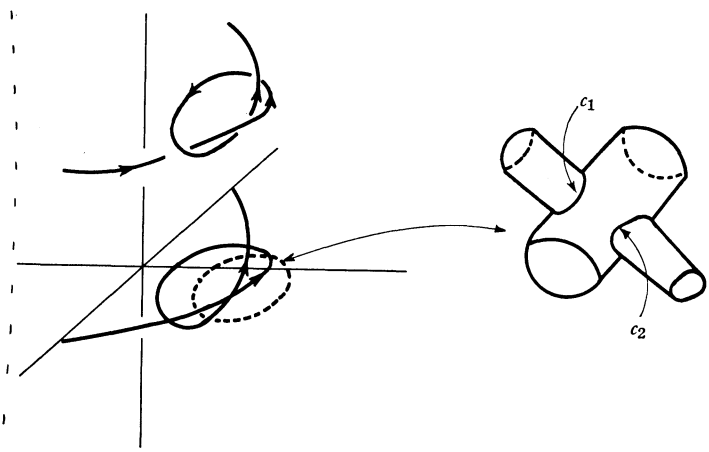


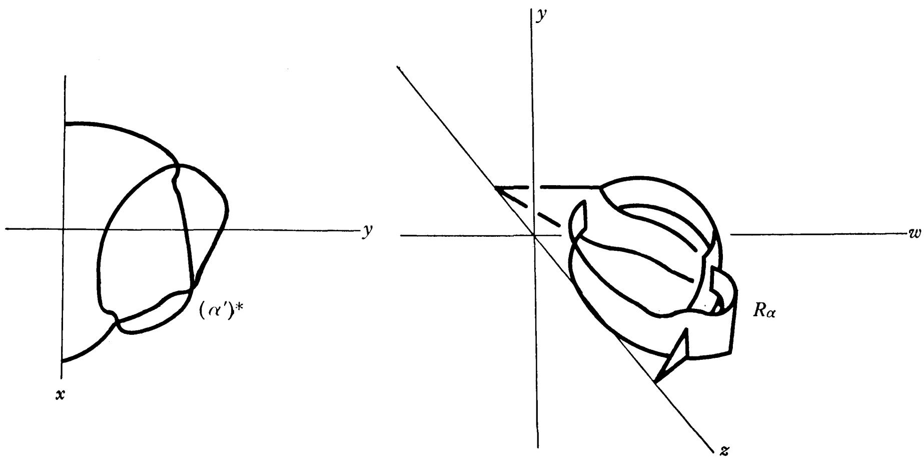
Figure 3



(a) (b)

Figure 4

determine the height relations at the double points we use the following rule: choose a direction for α indicated by arrows, if the crossing at a point of α^* is as in figure 4a, then the double point set consists of two circles c_1 and c_2 and we will define our embedded sphere so that the smaller tube passes *under* the large one at c_1 and the smaller tube passes over the large tube at c_2; the projection of these tubes will look like figure 4b. (This over-under alternation at each crossing point accounts for our choice of the term "weaving" to describe this knot and its generalizations.)



(a) (b)

Figure 5

We now wish to examine the projection $\Pi(W^1_{\alpha'})$. For each $b\in\alpha'$, with $b=\alpha'(a)$, we define $B_b=W^k(\mu_a)$; then B_b is a k-sphere of radius $x_1'(a)$ if $a\in\text{Int } D^n$, a point if $a\in\partial D^n$. If $A_b'=\bigcup_\rho h_\rho(b)$, $(A_b')^*=\Pi(A_b')$, $B_b^*=\Pi(B_b)$ then we see that for all b, $(A_b')^*=B_b^*$, since each set consists of a k-sphere of radius $x_1'(a)$ in the hyperplane $(x_1, \cdots, x_{k+1}, x_2(a), \cdots, x_{n+1}(a))$ with center $(0, \cdots, 0, x_2(a), \cdots, x_{n+1}(a))$. Thus $\Pi(S^k_{\alpha'})=\Pi(W^k_{\alpha'})$; however, this does not imply that $S^k_{\alpha'}$ is ambiently isotopic to $W^k_{\alpha'}$, we need to check the height relations in the x_{n+k+2} coordinate. We note that for any B_b, the x_{n+k+2} coordinate of points of B_b are the same, namely $x_1(a)$. Now suppose that $B_b^*=B_{b'}^*$ and thus $(A_b')^*=(A_{b'}')^*=B_b^*$, then $(\alpha')^*(b)=(\alpha')(b')$, and thus $x_1'(a)=x_1'(a')$, where $\alpha(a')=b'$. Now suppose that $h(b)<h(b')$, then as we have seen, $A_b'<A_{b'}'$; however, $B_b>B_{b'}$ since the x_{n+k+2} coordiate of points in B_b and $B_{b'}$ is given by $x_1(a)$ and $x_1(a')$, respectively, and from the definition of x_1' we see that if $x_1'(a)=x_1'(a')$ with $h(b)<h(b')$, then $x_1(a)>x_1(a')$. We may summarize this by saying that although $\Pi(S^k_{\alpha'})=\Pi(W^k_{\alpha'})$, the height relations of S^k_α are the opposite of those of $W^k_{\alpha'}$.

Let $-\alpha'$ be the mirror image of α' obtained by reflection in the last coordinate of R^{n+2}_+; $(-\alpha')(a)=(x_1'(a), x_2(a), \cdots, -x_{n+2}(a)+M)$ (we need to add the M to the last coordinate in order that $-\alpha'$ satisfy condition (1) in the definition of α). For mirror images of circles in R^3, see Crowell-Fox, Chapter 1, Section 4 [2]. Now the height relations of $S^k_{-\alpha'}$ are the reverse of those of $S^k_{\alpha'}$, and $\Pi(S^k_{\alpha'})=\Pi(S^k_{-\alpha'})$. Thus $S^k_{-\alpha'}$ is ambiently isotopic to $W^k_{\alpha'}$; in fact, by an ambient isotopy which translates B_b in the x_{n+k+2} coordinate until it coincides with $-A_b'=\bigcup_{\rho} h_\rho(-\alpha'(a))$.

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