Saitō, Y.
Osaka J. Math.
11 (1974), 295–306

THE PRINCIPLE OF LIMITING ABSORPTION FOR THE NON-SELFADJOINT SCHRÖDINGER OPERATOR IN R^2

YOSHIMI SAITŌ

(Received November 13, 1973)

Introduction

The present paper is a continuation of [3] and is devoted to extending the results obtained in [3] to the non-selfadjoint Schrödinger operator in $\boldsymbol{R}^2$.

In the paper [3] we considered the non-selfadjoint Schrödinger operator

$$(0.1) \qquad L = -\sum_{j=1}^{N}\left(\frac{\partial}{\partial x_j}+ib_j(x)\right)^2+Q(x)$$

in $\boldsymbol{R}^N$, where N is a positive integer such that $N\neq 2$, and the complex-valued function $Q(x)$ and the real-valued functions $b_j(x)$ ($j=1, 2, \cdots, N$) are assumed to satisfy some asymptotic conditions at infinity. Among others we have shown the following: Let us define a Hilbert space $L_{2,\beta}=L_{2,\beta}(\boldsymbol{R}^N)$ ($\beta\in\boldsymbol{R}$) by

$$(0.2) \qquad L_{2,\beta} = \{f(x)/(1+|x|)^\beta f(x)\in L_2(\boldsymbol{R}^N)\}$$

with its inner product

$$(0.3) \qquad (f, g)_\beta = \int_{\boldsymbol{R}^N}(1+|x|)^{2\beta}f(x)\overline{g(x)}\,dx$$

and norm

$$(0.4) \qquad \|f\|_\beta = [(f, f)_\beta]^{1/2}\,.$$

If $\kappa\in\boldsymbol{C}_+=\{\kappa\in\boldsymbol{C}/\kappa\neq 0 \text{ and } \mathrm{Im}\,\kappa\geqq 0\}$ does not belong to an exeptional set which is called the set of the singular points of L, then the operator $(L-\kappa^2)^{-1}$ is well-defined as a bounded linear operator from $L_{2,(1+\varepsilon)/2}$ into $L_{2,-(1+\varepsilon)/2}$ ($\varepsilon>0$) with the estimate

$$(0.5) \qquad \|(L-\kappa^2)^{-1}\| = O(|\kappa|^{-1}) \qquad (|\kappa|\to\infty)\,.$$

Here $u=(L-\kappa^2)^{-1}f\in L_{2,-(1+\varepsilon)/2}$ ($f\in L_{2,(1+\varepsilon)/2}$) is a unique solution of the equation

$$(0.6) \qquad (L-\kappa^2)u = f$$

with a sort of "radiation condition", and $\|(L-\kappa^2)^{-1}\|$ means the operator norm of $(L-\kappa^2)^{-1}$ from $L_{2,(1+\varepsilon)/2}$ into $L_{2,-(1+\varepsilon)/2}$[1].

In this paper, modifying the method of [3], we shall show that the estimate (0.5) holds good for L defined in $\boldsymbol{R}^2$ with $b_j(x)=0$, $j=1, 2$. In our case L takes the form

$$(0.7) \qquad L = -\Delta + Q(x).$$

At the same time it will be shown that the other results obtained in [3] also hold for L in $\boldsymbol{R}^2$. Throughout this paper we shall use the same notations as in [3][2]. For example $\partial_j u = \dfrac{\partial u}{\partial x_j}$, $\mathcal{D}_j u = \mathcal{D}_j^{(\kappa)} u = \partial_j u + (\tilde{x}_j/(2r))u - i\kappa\tilde{x}_j u$, $r=|x|$, $\tilde{x}_j = x_j/r$, $\mathcal{D}_r u = (\mathcal{D}_1 u)\tilde{x}_1 + (\mathcal{D}_2 u)\tilde{x}_2$ etc.

1. A priori estimates

Let us define a differential operator L in $\boldsymbol{R}^2$ by (0.7), where $Q(x)$ is a complex-valued function on $\boldsymbol{R}^2$ and L is regarded as an operator from $H_{2,loc}$ into $L_{2,loc}$. We decompose $Q(x)$ as $Q(x)=V_0(x)+V(x)$. Throughout this paper the following is assumed[3]: $V_0(x)$ is a real-valued, measurable function such that the radial derivative exists and

$$(1.1) \qquad |V_0(x)| \leqq C(1+|x|)^{-\delta}, \ \frac{\partial V_0}{\partial |x|} \leqq C(1+|x|)^{-1-\delta} \qquad (x \in \boldsymbol{R}^2).$$

$V(x)$ is a complex-valued, measurable function which satisfies

$$(1.2) \qquad |V(x)| \leqq C(1+|x|)^{-1-\delta} \qquad (x \in \boldsymbol{R}^2).$$

Here C and δ are positive constants.

Now let us note that with no loss of generality $V_0(x)$ can be assumed to satisfy

$$(1.3) \qquad V_0(x) = 0 \qquad (|x| \leqq R)$$

by replacing V_0 and V with αV_0 and $(1-\alpha)V_0+V$, respectively, $\alpha(x)$ being a real-valued, C^∞-function such that

$$(1.4) \qquad \alpha(x) = \begin{cases} 0 & (|x| \leqq R), \\ 1 & (|x| \geqq R+1). \end{cases}$$

Henceforth we assume (1.3) with $R=7$ as well as (1.1) and (1.2).

1) In this regard we note that Ikebe-Saitō [1] has shown the boundedness of $\|(L-\kappa^2)^{-1}\|$ for κ moving in any compact set contained in $\boldsymbol{C}_+$, where L is a self-adjoint Schrödinger operator in $\boldsymbol{R}^N$ and N is an arbitrary positive integer.

2) The list of the notation is given in the end of Introduction of [3].

3) This aptissumon is the same as the one imposed on $Q(x)$ in [3].

Let ε be a positive number such that $0<\varepsilon\leqq 1$ and $0<\varepsilon\leqq\delta/2$. As in Definition 1.2 of [3] we define by $\sum=\sum(L)=\sum(L, \varepsilon)$ the set of the singular points of L. i.e., $\kappa\in\sum$ if and only if $\kappa\in\boldsymbol{C}_+=\{\kappa\in\boldsymbol{C}/\kappa\neq 0, \operatorname{Im}\kappa\geqq 0\}$ and there exists a non-trivial solution u of the equation

$$(1.5)\qquad \begin{cases} (L-\kappa^2)u=0\,, & u\in H_{2,loc}\cap L_{2,-(1+\varepsilon)/2}\,,\\ \|\mathcal{D}u\|_{(-1+\varepsilon)/2,E_1}<\infty^{4)}\,, \end{cases}$$

where $E_1=\{x\in\boldsymbol{R}^2/|x|\geqq 1\}$.

For $\kappa\in\boldsymbol{C}_+$ with $\operatorname{Im}\kappa>0$ and the above ε we put

$$(1.6)\qquad D_{\kappa,\varepsilon}=D_\kappa=\{u\in H_{2,loc}\cap L_{2,-(1+\varepsilon)/2}/(L-\kappa^2)u\in L_{2,(1+\varepsilon)/2}\}\,.$$

As is easily seen, Lemma 2.1 and Proposition 2.3, (i), (ii) of [3] are true in $\boldsymbol{R}^2$, too, and hence we have

Proposition 1.1. *Let $u\in D_\kappa$ with $\kappa\in\boldsymbol{C}_+$ and* $\operatorname{Im}\kappa>0$. *Then u, $\partial_1 u$, $\partial_2 u\in L_{2,(1+\varepsilon)/2}$ and the estimate*

$$(1.7)\qquad \|u\|_{(1+\varepsilon)/2}\leqq C(\|u\|_{-(1+\varepsilon)/2}+\|(L-\kappa^2)u\|_{(1+\varepsilon)/2})$$

holds with a constant $C=C(\kappa, L, \varepsilon)$[5]. As a function of κ, C is bounded when κ moves in a compact set contained in $\{\kappa\in\boldsymbol{C}/\operatorname{Im}\kappa>0\}$.

The purpose of this section is to prove the following estimates for $u\in D_\kappa$.

Theorem 1.2. *Let M be an open set such that $M\subset M_a=\{\kappa\in\boldsymbol{C}/|\kappa|>a$, $\operatorname{Im}\kappa>0\}$ with some $a>0$. and $\bar{M}\cap\sum=\phi$, $\bar{M}$ being the closure of M in $\boldsymbol{C}$. Let $\kappa\in M$ and let $u\in D_\kappa$. Then there exists a constant $C=C(M, L, \varepsilon)$ such that we have the estimates*

$$(1.8)\qquad \|\mathcal{D}u\|_{(-1+\varepsilon)/2,E_1}\leqq C\|f\|_{(1+\varepsilon)/2}\,,$$

$$(1.9)\qquad \|u\|_{-(1+\varepsilon)/2,E_\rho}\leqq\frac{C}{|\kappa|}(1+\rho)^{-\varepsilon/2}\|f\|_{(1+\varepsilon)/2}\qquad(\rho\geqq 0)\,,$$

where $f=(L-\kappa^2)u$ and $E_\rho=\{x\in\boldsymbol{R}^2/|x|\geqq\rho\}$.

REMARK 1.3. Cf. Theorem 2.7 of [3]. In $\boldsymbol{R}^2$ the relation $\mathcal{D}_j u\in L_{2,(-1+\varepsilon)/2}$ for $u\in D_\kappa$ is not necessarily true, because $u/|x|$ $(u\in H_{2,loc})$ is not always square

---

4) As in [3] we put

$$\|\mathcal{D}u\|^2_{(-1+\varepsilon)/2}=\sum_{j=1}^{2}\int_{|x|\geqq 1}(1+|x|)^{-1+\varepsilon}|\mathcal{D}_j u|^2\,dx\,.$$

5) Here and in the sequel we mean by $C=C(A, B, \cdots)$ that C is a positive constant depending only on A, B, $\cdots$.

integrable on a neighborhood of the origin $x=0$. But, of course, for $u\in D_\kappa$ $\mathcal{D}_j u\in L_{2,(1+\varepsilon)/2}(E_r)$ $(r>0)$.

In order to prove Theorem 1.2 we prepare several propositions. Let us first show that the above estimates (1.8) and (1.9) can be easily obtained for $u\in D_\kappa$ if $\mathrm{Im}\,\kappa$ is sufficiently large. Set

$$\beta_0=\max\,[\{2(\sup_x|V_0(x)+V_1(x)|+1)\}^{1/2},\ \sup_x|V_2(x)|] \qquad (V_1(x)=\mathrm{Re}\;V(x),\ V_2(x)=\mathrm{Im}\;\;V(x))\,. \tag{1.10}$$

Then we have

Proposition 1.4. *Let* $u\in D_k$ *with* $\mathrm{Im}\,\kappa\geqq\beta_0$. *Then the estimates*

$$\|u\|_{-\mu,E_\rho}\leqq\frac{C_0}{|\kappa|}(1+\rho)^{-\mu}\|f\| \qquad (\rho\geqq 0,\ \mu\geqq 0) \tag{1.11}$$

and

$$\|\mathcal{D}u\|_{E_1}\leqq C_0\|f\| \tag{1.12}$$

hold with a constant $C_0=C_0(\beta_0)$, *where* $f=(L-\kappa^2)u$ *and* $\|\ \|$ *means the usual* L_2*-norm.*

Proof. Take the real and imaginary part of $((L-\kappa^2)u,u)=(f,u)$ to obtain

$$\sum_{j=1}^2(\partial_j u,\partial_j u)+((\kappa_2^2-\kappa_1^2+V_0+V_1)u,u)=\mathrm{Re}\,(f,u)\,, \tag{1.13}$$

$$((V_2-2\kappa_1\kappa_2)u,u)=\mathrm{Im}\,(f,u)\,, \tag{1.14}$$

where $\kappa_1=\mathrm{Re}\,\kappa$, $\kappa_2=\mathrm{Im}\,\kappa$, and (,) is the L_2-inner product. It follows from (1.10) that

$$\kappa_2^2-\kappa_1^2+V_0(x)+V_1(x)\geqq\kappa_2^2/2\geqq\frac{\beta_0}{2}\kappa_2 \qquad (|\kappa_1|<1,\ \kappa_2\geqq\beta_0) \tag{1.15}$$

and

$$|V_2(x)-2\kappa_1\kappa_2|\geqq|\kappa_1|\kappa_2 \qquad (|\kappa_1|\geqq 1,\ \kappa_2\geqq\beta_0) \tag{1.16}$$

for all $x\in\boldsymbol{R}^2$. By the use of the relations (1.13)~(1.16) we can show

$$\|u\|\leqq\frac{C_1}{|\kappa|}\|f\| \qquad (C_1=C_1(\beta_0))\,. \tag{1.17}$$

In fact, if $|\kappa_1|<1$ and $\kappa_2\geqq\beta_0$, we have from (1.13) and (1.15)

$$\|u\|\leqq(2/(\beta_0\kappa_2))\|f\|\leqq(4/(\beta_0|\kappa|))\|f\|\,, \tag{1.18}$$

where we should note that $|\kappa|\leqq|\kappa_1|+\kappa_2\leqq 2\kappa_2$. If $|\kappa_1|\geqq 1$ and $\kappa_2\geqq\beta_0$, we can see from (1.14) and (1.16) that

$$(1.19)\qquad |\kappa_1|\kappa_2\|u\|\leqq\|f\|,$$

whence we obtain

$$(1.20)\qquad \|u\|\leqq(|\kappa_1|\kappa_2)^{-1}\|f\|\leqq(|\kappa|-1)^{-1}\|f\|\leqq\frac{2}{|\kappa|}\|f\|.$$

(1.11) is a direct consequence of (1.17).

Next let us prove (1.12). Since

$$(1.21)\qquad \|\mathcal{D}u\|_{E_1}\leqq[\sum_{j=1}^{2}\|\partial_j u\|^2]^{1/2}+\left\|\frac{1}{2|x|}u\right\|_{E_1}+\|i\kappa u\|_{E_1}$$

and (1.17) has been established, we have only to show

$$(1.22)\qquad [\sum_{j=1}^{2}\|\partial_j u\|^2]^{1/2}\leqq C_2\|f\|\qquad (C_2=C_2(\beta_0)).$$

This follows from (1.1), (1.2), (1.13) and (1.17). Q.E.D.

In the rest of this section it is enough to consider $u\in D_\kappa$ with $0<\operatorname{Im}\kappa<\beta_0$.

Proposition 1.5. *Let a be a positive number and let $u\in D_\kappa$ with $|\kappa|>a$ and $0<\operatorname{Im}\kappa<\beta_0$, where β_0 is as above. Then the estimate*

$$(1.23)\qquad \|\mathcal{D}u\|_{(-1+\varepsilon)/2,\,E_1}\leqq C\{\|u\|_{-(1+\varepsilon)/2}+\|f\|_{(1+\varepsilon)/2}\}\qquad (f=(L-\kappa^2)u)$$

holds with a positive constant $C=C(a,\beta_0,L,\varepsilon)$.

Proof. It follows from the formula (2.21) given in Lemma 2.5 of [3], which is true in the case $N=2$, too, that

$$(1.24)\qquad \int_{B_{t1}}\left(\frac{\partial\phi}{\partial r}-\frac{\phi}{r}\right)|\mathcal{D}_r u|^2dx+\int_{B_{t1}}\left(\kappa_2\phi+\frac{\phi}{r}-\frac{1}{2}\cdot\frac{\partial\phi}{\partial r}\right)|\mathcal{D}u|^2dx$$
$$+\int_{B_{1T}}\left(\kappa_2\phi+\frac{1}{2}\cdot\frac{\partial\phi}{\partial r}\right)|\mathcal{D}u|^2dx+\int_{B_{1T}}\left(\frac{\phi}{r}-\frac{\partial\phi}{\partial r}\right)(|\mathcal{D}u|^2-|\mathcal{D}_r u|^2)dx$$
$$=\int_{B_{tT}}\frac{1}{4}\left\{\frac{\phi}{r^2}\kappa_2-\frac{1}{2}\cdot\frac{\partial}{\partial r}\left(\frac{\phi}{r^2}\right)\right\}|u|^2dx$$
$$+\int_{B_{tT}}\left\{\frac{1}{2}\left(\frac{\partial\phi}{\partial r}V_0+\phi\frac{\partial V_0}{\partial r}\right)-\kappa_2\phi V_0\right\}|u|^2dx$$
$$+\operatorname{Re}\int_{B_{tT}}\phi Vu(\overline{\mathcal{D}_r u})dx+\operatorname{Re}\int_{B_{tT}}\phi f(\mathcal{D}_r u)dx$$
$$-\frac{1}{2}\left[\int_{S_T}-\int_{S_t}\right]\phi\left\{|\mathcal{D}u|^2-2|\mathcal{D}_r u|^2+\left(V_0-\frac{1}{4r^2}\right)|u|^2\right\}dS,$$

where $0<t<1<T<\infty$, $B_{ps}=\{x\in\boldsymbol{R}^2/p\leqq|x|\leqq s\}$, $r=|x|$, $\phi=\phi(r)$ is a real-valued, piecewise continuously differentiable function on $[0,\infty]$, and we put in

(2.21) of [3] $c_N=c_2=-1/4$ and $B_{jk}(x)=0$. Set

$$\phi(r)=\begin{cases} r^2 & (0\leqq t\leqq 1)\,, \\ \dfrac{1}{2^\varepsilon}(1+r)^\varepsilon & (r>1) \end{cases} \tag{1.25}$$

in (1.24). Then we estimate the both sides of (1.24) as follows:

(1.26) the left-hand side of (1.24)

$$\geqq \int_{B_{t1}} r|\mathcal{D}_r u|^2 dx+\int_{B_{1T}} \frac{\varepsilon}{2^{1+\varepsilon}}(1+r)^{-1+\varepsilon}|\mathcal{D}u|^2 dx$$

and

(1.27) the right-hand side of (1.24)

$$\begin{aligned}
&\leqq \int_{B_{tT}} \frac{1}{4}\left\{\beta_0\frac{\phi}{r^2}-\frac{1}{2}\cdot\frac{\partial}{\partial r}\left(\frac{\phi}{r^2}\right)\right\}|u|^2 dx \\
&+\int_{B_{tT}} \frac{1}{2}\left(\frac{\partial\phi}{\partial r}|V_0|+\phi\frac{\partial V_0}{\partial r}\right)|u|^2 dx+\kappa_2\int_{B_{tT}} \phi|V_0|\,|u|^2 dx \\
&+\int_{B_{tT}} \phi|V|\,|u|\,|\mathcal{D}_r u|\,dx+\int_{B_{tT}} \phi|f|\,|\mathcal{D}_r u|\,dx \\
&+\int_{S_t} \phi(|\mathcal{D}u|^2+|V_0|\,|u|^2)\,dS \\
&+\int_{S_T} \phi\left\{2|\mathcal{D}_r u|^2+\left(|V_0|+\frac{1}{4r^2}\right)|u|^2\right\}dS \\
&=J_1+J_2+J_3+J_4+J_5+J_6+J_7\,.
\end{aligned}$$

Let us estimate each J_k. First we obtain

$$J_k\leqq C_k\|u\|^2_{-(1+\varepsilon)/2} \qquad (k=1,2)\,, \tag{1.28}$$

where $C_1=C_1(\beta_0,\varepsilon)$, $C_2=C_2(L,\varepsilon)$ and we used (1.1). It follows from (1.1) and (1.3) with $R=7$ that

$$\begin{aligned} J_3 &\leqq C_3'\kappa_2\|u\|^2_{-\varepsilon/2,\,E_2} \\ &\leqq C_3'(\kappa_2\|u\|_{(1-\varepsilon)/2,\,E_2})\|u\|_{-(1+\varepsilon)/2} \qquad (C_3'=C_3'(L,\varepsilon))\,. \end{aligned} \tag{1.29}$$

On the other hand in quite a similar way to the one used to prove Proposition 2.3, (iii) of [3] we can show

$$\begin{aligned} \kappa_2\|u\|_{(1-\varepsilon)/2,\,E_2} \leqq C_3''\{&\|u\|_{-(1+\varepsilon)/2} \\ &+\|\mathcal{D}u\|_{(-1+\varepsilon)/2,\,E_1}+\|f\|_{(1+\varepsilon)/2}\} \qquad (C_3''=C_3''(a,L,\varepsilon))\,, \end{aligned} \tag{1.30}$$

which, together with (1.29), yields

$$(1.31)\qquad J_3 \leqq C_3\{\|u\|^2_{-(1+\varepsilon)/2} + \|u\|_{-(1+\varepsilon)/2}\|\mathcal{D}u\|_{(-1+\varepsilon)/2,\,E_1} + \|u\|_{-(1-\varepsilon)/2}\|f\|_{(1+\varepsilon)/2}\} \qquad (C_3 = C_3(a, L, \varepsilon))\,.$$

As to J_4 and J_5 we have, using (1.2),

$$(1.32)\qquad \begin{cases} J_4 \leqq C_4\|u\|\|_{-(1+\varepsilon)/2}(\|r^{1/2}(\mathcal{D}_r u)\|_{B_1} + \|\mathcal{D}u\|_{(-1+\varepsilon)/2,\,E_1})\,, \\ J_5 \leqq C_5\|f\|_{(1+\varepsilon)/2}(\|r^{1/2}(\mathcal{D}_r u)\|_{B_1} + \|\mathcal{D}u\|_{(-1+\varepsilon)/2,\,E_1})\,, \end{cases}$$

where $C_k = C_k(L, \varepsilon)$, $k=4, 5$. Here we should note that $r^{1/2}(\mathcal{D}_r u) \in L_2(\boldsymbol{R}^2)_{loc}$ because $u \in H_2(\boldsymbol{R}^2)_{loc}$ is a continuous function on $\boldsymbol{R}^2$ by the Sobolev lemma. $\lim_{t\to 0} J_6 = 0$ and $\lim_{T\to\infty} J_7 = 0$ follow from the fact that $r(|\mathcal{D}u|^2 + |V_0||u|^2)$ and $r^{\varepsilon}\{2|\mathcal{D}_r u|^2 + (|V_0| + 1/(4r^2))|u|^2\}$ are integrable on B_1 and E_1, respectively. Summing up these estimates and letting $t\to 0$ and $T\to\infty$, we arrive at

$$(1.33)\qquad \begin{aligned} &\|r^{1/2}(\mathcal{D}_r u)\|^2_{B_1} + (\varepsilon/2^{1+\varepsilon})\|\|\mathcal{D}u\|^2_{(-1+\varepsilon)/2,\,E_1} \\ &\leqq C'\{\|u\|^2_{-(1+\varepsilon)/2} + \|u\|_{-(1+\varepsilon)/2}\|f\|_{(1+\varepsilon)/2} \\ &\quad + \|r^{1/2}(\mathcal{D}_r u)\|_{B_1}(\|u\|_{-(1+\varepsilon)/2} + \|f\|_{(1+\varepsilon)/2}) \\ &\quad + \|\mathcal{D}u\|_{(-1+\varepsilon)/2,\,E_1}(\|u\|_{-(1+\varepsilon)/2} + \|f\|_{(1+\varepsilon)/2})\} \qquad (C' = C'(a, \beta_0, L, \varepsilon))\,. \end{aligned}$$

(1.23) is a direct consequence of (1.33). Q.E.D.

REMARK 1.6. In the present paper we assume that the magnetic potentials $b_j(x) = 0$ $(j=1, 2)$. This assumption is used only to prove the above Proposition 1.5. Technically, it is possible to adopt a weaker assumption. For example it is enough to assume that $B_{12}(x) = \partial_1 b_2(x) - \partial_2 b_1(x) = 0$ in a neighborhood of the origin $x=0$.

Proposition 1.7. *Let $u \in D_\kappa$ with $|\kappa| > a$ $(a>0)$ and $0 < \operatorname{Im}\kappa < \beta_0$. Then there exists a constant $C = C(a, \beta_0, L, \varepsilon)$ such that we have*

$$(1.34)\qquad \|u\|^2_{-(1+\varepsilon)/2,\,E_\rho} \leqq C(1+\rho)^{-\varepsilon}\left\{\frac{1}{|\kappa|}\|u\|^2_{-(1+\varepsilon)/2} + \frac{1}{|\kappa|}\|u\|_{-(1+\varepsilon)/2}\|f\|_{(1+\varepsilon)/2} + \frac{1}{|\kappa|^2}\|f\|^2_{(1+\varepsilon)/2}\right\} \qquad (\rho \geqq 0)\,,$$

where $f = (L-\kappa^2)u$.

Proof. The proof will be divided into two steps. In Step I the estimate (1.34) with $\rho \geqq 1$ will be proved, where Proposition 1.5 will be useful. In Step

II we shall show (1.34) with $\rho=0$. From these results we can easily obtain (1.34) for all $\rho\geqq 0$.

Step I. We can find positive numbers b and c such that $\{\kappa=\kappa_1+i\kappa_2/|\kappa|>a, 0<\kappa_2<\beta_0\}\subset K_1\cup K_2$, where

$$\text{(1.35)}\qquad \begin{cases} K_1=\left\{\kappa=\kappa_1+i\kappa_2/|\kappa|>a,\ 0<\kappa_2<\beta_0,\ \kappa_1^2-\dfrac{1}{2}\kappa_2^2\geqq b^2\right\}, \\ K_2=\{\kappa=\kappa_1+i\kappa_2/|\kappa|>a,\ 0<\kappa_2<\beta_0,\ \kappa_2^2-\kappa_1^2\geqq c^2\}, \end{cases}$$

e.g., we may put $b=a/4$ and $c=a/2$. Consider the case $\kappa\in K_1$. Then, proceeding as in the first half of the proof of Proposition 2.6 of [3], we arrive at (1.34) with $\rho\geqq 1$. Next consider the case $\kappa\in K_2$. Then we can proceed as in the second half of the proof of Proposition 2.6 of [3] to obtain (1.34) with $\rho\geqq 1$.

Step II. Set $x_0=(2, 0)\in \boldsymbol{R}^2$ and set for $u\in D_\kappa$

$$\text{(1.36)}\qquad \tilde{u}(x)=u(x-x_0)\,.$$

It follows from the relation $(L-\kappa^2)u=f$ that

$$\text{(1.37)}\qquad (\tilde{L}-\kappa^2)\tilde{u}=\tilde{f}\,,$$

where

$$\text{(1.38)}\qquad \begin{cases} \tilde{L}=-\Delta+\tilde{V}_0(x)+\tilde{V}(x)\,, \\ \tilde{V}_0(x)=V_0(x-x_0),\ \tilde{V}(x)=V(x-x_0)\,, \\ \tilde{f}(x)=f(x-x_0)\,. \end{cases}$$

Obviously $\tilde{V}(x)$ satisfies (1.2) with the same δ and some positive $\tilde{C}$. It can be also shown that $\tilde{V}_0(x)$ satisfies (1.1). In fact we have

$$\text{(1.39)}\qquad \begin{aligned} \frac{\partial\tilde{V}_0(x)}{\partial|x|}&=\frac{V_0(x-x_0)}{\partial|x-x_0|}\cdot\frac{\partial|x-x_0|}{\partial|x|} \\ &=\frac{\partial V_0}{\partial|x|}(x-x_0)\cdot\frac{|x|-|x_0|\cos\theta}{|x-x_0|}\,, \end{aligned}$$

θ being the angle between x and x_0. By (1.3) with $R=7$ $\dfrac{\partial V_0}{\partial|x|}(x-x_0)=0$ for $|x-x_0|\leqq 5$, and for $|x-x_0|>5$ it follows that

$$\text{(1.40)}\qquad 0<\frac{|x|-|x_0|\cos\theta}{|x-x_0|}\leqq\frac{1-(|x_0|/|x|)\cos\theta}{1-(|x_0|/|x|)}<\frac{1+2/3}{1-2/3}=5\,,$$

where we have used the fact that $|x_0|/|x|<2/3$ if $|x-x_0|>5$. Thus we obtain, together with (1.1),

$$(1.41)\qquad \frac{\partial \tilde{V}_0(x)}{\partial |x|} \leqq 5C(1+|x-x_0|)^{-1-\delta}$$
$$\leqq \tilde{C}(1+|x|)^{-1-\delta} \qquad \left(\tilde{C} = 5C\left[\sup_x \left(\frac{1+|x|}{1+|x-x_0|}\right)\right]^{1+\delta}\right))$$

Hence the result obtained in Step I can be applied to $\tilde{L}$ to show

$$(1.42)\qquad ||\tilde{u}||^2_{-(1+\varepsilon)/2,\, E_1} \leqq \tilde{C}\left\{\frac{1}{|\kappa|}||\tilde{u}||^2_{-(1+\varepsilon)/2}\right.$$
$$\left. + \frac{1}{|\kappa|}||\tilde{u}||_{-(1+\varepsilon)/2}||\tilde{f}||_{(1+\varepsilon)/2} + \frac{1}{|\kappa|^2}||\tilde{f}||^2_{(1+\varepsilon)/2}\right\} \qquad (\tilde{C}=\tilde{C}(a,\ \beta_0,\ \tilde{L},\ \varepsilon))\,.$$

Since the unit disc B_1 of $\boldsymbol{R}^2$ is contained in the set $\{x\in\boldsymbol{R}^2/|x+x_0|\geqq 1\}$, we have

$$(1.43)\qquad ||\tilde{u}||^2_{-(1+\varepsilon)/2,\, E_1} = \int_{|x|\geqq 1}(1+|x|)^{-1-\varepsilon}|u(x-x_0)|^2 dx$$
$$= \int_{|x+x_0|\geqq 1}(1+|x-x_0)^{-1-\varepsilon}|u(x)|^2 dx$$
$$\geqq \int_{B_1}(1+|x-x_0|)^{-1-\varepsilon}|u(x)|^2 dx\,.$$

Therefore it folloss from (1.42) and the boundedness of $(1+|x|)/(1+|x-x_0|)$ and $(1+|x-x_0|)/(1+|x|)$ on the whole space $\boldsymbol{R}^2$ that

$$(1.44)\qquad ||u||^2_{-(1+\varepsilon)/2,\, B_1} \leqq C'\left\{\frac{1}{|\kappa|}||u||^2_{-(1+\varepsilon)/2}\right.$$
$$\left. + \frac{1}{|\kappa|}||u||_{-(1+\varepsilon)/2}||f||_{(1+\varepsilon)/2} + \frac{1}{|\kappa|^2}||f||^2_{(1+\varepsilon)/2}\right\} \qquad (C' = C'(a,\ \beta_0,\ L,\ \varepsilon))\,.$$

(1.34) with $\rho=0$ can be easily obtained from (1.44) and (1.34) with $\rho=1$.

Q.E.D.

Proof of Theorem 1.2. Set

$$(1.45)\qquad \begin{cases} M_1 = \{\kappa = \kappa_1 + i\kappa_2/\kappa\in M,\ \kappa_2\geqq\beta_0\} \\ M_2 = \{\kappa = \kappa_1 + i\kappa_2/\kappa\in M,\ 0<\kappa_2\leqq\beta_0\}\,. \end{cases}$$

We have $M=M_1\cup M_2$. For $u\in D_\kappa$ with $\kappa\in M_1$ we have (1.8) and (1.9) from (1.12) and (1.11), respectively. Next suppose that $u\in D_\kappa$ with $\kappa\in M_2$. Then, since the estimates (1.23) and (1.34) have been shown, we can proceed as in the proof of Theorem 2.7 of [3] to obtain

$$(1.46)\qquad ||u||_{-(1+\varepsilon)/2} \leqq \frac{C}{|\kappa|}||f||_{(1+\varepsilon)/2} \qquad (C = C(a,\ \beta_0,\ L,\ \varepsilon))\,.$$

(1.8) and (1.9) for $u\in D_\kappa$ with $\kappa\in M_2$ follow from (1.23), (1.34) and (1.46), which completes the proof. Q.E.D.

2. The limiting absorption principle for L

Now that a priori estimates for L have been established (Theorem 1.2), it can be shown by arguements quite similar to those used in §3 and §4 of [3] that the main results of [3] hold in our case, too. We sum up these in the following three theorems whose proof will be omitted.

Theorem 2.1 (*the properties of the set* $\sum$ *of the singular points of* L). *Let* (1.1) *and* (1.2) *be satisfied and let* $0<\varepsilon\leqq\max(1, \delta/2)$.

(i) *Then the set* $\sum=\sum(L, \varepsilon)$ *of the singular points of* L *is a bounded set of* $\boldsymbol{C}_+=\{\kappa\in\boldsymbol{C}/\kappa\neq 0, \operatorname{Im}\kappa\geqq 0\}$. $\sum_R=\sum\cap\boldsymbol{R}$ *is a bounded set with the Lebesgue measure* 0.

(ii) *For any* $a>0$ $\sum\cap\{\kappa\in\boldsymbol{C}_+/|\kappa|\geqq a\}$ *is a compact set of* $\boldsymbol{C}_+$, *Further,* $\sum-\sum_R$ *is an isolated, bounded set having no limit point in* $\{\kappa\in\boldsymbol{C}_+/\operatorname{Im}\kappa>0\}$.

(iii) *Let* $\kappa\in\boldsymbol{C}_+$ *and* $\operatorname{Im}\kappa>0$. *Then* $\nu\in\sum$ *if and only if* κ^2 *belongs to the point spectrum of* H, *where* H *is a densely defined, closed linear operator in* L_2 *given by*

$$\begin{cases} D(H) = H_2{}^{6)}, \\ Hu = Lu. \end{cases} \tag{2.1}$$

For $\kappa\in\boldsymbol{C}_+-\sum$ *with* $\operatorname{Im}\kappa>0$ *belongs to the resolvent set of* H.

Theorem 2.2. (the limiting absorption principle for L). *Let* (1.1) *and* (1.2) *be satisfied and let* ε *and* $\sum$ *be as above. Assume that* M *is an open set of* $\boldsymbol{C}$ *such that* $\bar{M}\cap\sum=\phi$ *and* $M\subset M_a$ *with some* $a>0$, $\bar{M}$ *being the closure of* M *and* M_a *being given as in Theorem* 1.2.

(i) *Then for any pair* $(\kappa, f)\in\bar{M}\times L_{2,(1+\varepsilon)/2}$ *there exists a unique solution* $u=u(\kappa, f)$ *of the equation*

$$\begin{cases} (L-\kappa^2)u = f, \qquad u\in H_{2,loc}\cap L_{2,-(1+\varepsilon)/2}, \\ \|\mathcal{D}u\|_{(-1+\varepsilon)/2,E_1}<\infty. \end{cases} \tag{2.2}$$

(ii) *The solution* $u=u(\kappa, f)$, $(\kappa, f)\in\bar{M}\times L_{2,(1+\varepsilon)/2}$, *satisfies the estimates*

$$\begin{cases} \|u\|_{-(1+\varepsilon)/2}\leqq\dfrac{C}{|\kappa|}\|f\|_{(1+\varepsilon)/2}, \\ \|\mathcal{D}u\|_{(-1+\varepsilon)/2,E_1}\leqq D\|f\|_{(1+\varepsilon)/2}, \\ \|u\|_{-(1+\varepsilon)/2,E_\rho}\leqq\dfrac{C}{|\kappa|}(1+\rho)^{-\varepsilon/2}\|f\|_{(1+\varepsilon)/2} \qquad (\rho\geqq 1) \end{cases} \tag{2.3}$$

with a positive constant $C=C(M, L, \varepsilon)$.

---

6) $D(T)$ is the domain of T.

(iii) *If we define an operator* $(L-\kappa^2)^{-1}$ *by*

$$(2.4)\qquad (L-\kappa^2)^{-1}f=u(\kappa, f) \qquad (f\in L_{2,(1+\varepsilon)/2})$$

for $\kappa\in\bar{M}$, *then* $(L-\kappa^2)^{-1}$ *is a* $\boldsymbol{B}(L_{2(1+\varepsilon)/2}, L_{2,-(1+\varepsilon)/2})$*-valued, continuous function on* $\bar{M}$[7], *and we have*

$$(2.5)\qquad \|(L-\kappa^2)^{-1}\|\leqq\frac{C}{|\kappa|} \qquad (\kappa\in\bar{M},\ C=C(M, L, \varepsilon)),$$

where $\|(L-\kappa^2)^{-1}\|$ *means the operator norm of* $\boldsymbol{B}(L_{2,(1+\varepsilon)/2}, L_{2,-(1+\varepsilon)/2})$.

(iv) $(L-\kappa^2)^{-1}\in\boldsymbol{C}(L_{2,(1+\varepsilon)/2}, L_{2,-(1+\varepsilon)/2})$[8]. *Moreover we have the following: let* $\{f_n\}$ *be any bounded sequence of* $L_{2,(1+\varepsilon)/2}$ *and let* $\{\kappa_n\}$ *be any sequence contained in* $\bar{M}$. *Then the sequence* $\{(L-\kappa_n^2)^{-1}f_n\}$ *is relatively compact in* $L_{2,-(1+\varepsilon)/2}$.

(v) $(L-\kappa^2)^{-1}$ *is a* $\boldsymbol{B}(L_{2,(1+\varepsilon)/2}, L_{2,(1+\varepsilon)/2})$*-valued, analytic function on* M.

Finally let us show some properties of the spectrum $\sigma(H)$ of H defined by (2.1). Its point spectrum, continuous spectrum and residual spectrum are denoted by $\sigma_p(H)$, $\sigma_c(H)$ and $\sigma_r(H)$, respectively. We define the essential spectrum $\sigma_e(H)$ of H as in [3][9].

Theorem 2.3 (the properties of $\sigma(H)$). *Let* (1.1) *and* (1.2) *be satisfied and let H be as defined in* (2.1). *Then we have the following* (i)~(iv):

(i) $\sigma_e(H)=[0, \infty)$.

(ii) $\sigma_r(H)=\phi$[10].

(iii) $\sigma(H)\cap(\boldsymbol{C}-[0, \infty))\subset\sigma_p(H)$ *and* $\sigma_p(H)\cap(0, \infty)=\phi$, *and hence* $\sigma_c(H)\supset(0, \infty)$.

(iv) *The eigenvalues in* $\boldsymbol{C}-[0, \infty)$, *if they exist, are of finite multiplicity and they form an isolated, bounded set having no limit point in* $\boldsymbol{C}-[0, \infty)$.

OSAKA CITY UNIVERSITY

References

[1] T. Ikebe and Y. Saitō: *Limiting absorption method and absolute continuity for the Schrödinger operator,* J. Math. Kyoto Univ. **12** (1972), 513–542.

---

7) $B(X, Y)$ denotes the set of all bounded linear operators on X into Y, X and Y being Banach spaces.

8) $\mathbf{C}(X, Y)$ is all compact operators on X into Y, where X and Y are Banach spaces.

9) See (1.20) of [3].

10) (iv) can be obtained from the fact that in our case the relation $Hu=0$ $(u\in D(H))$ is equivalent to $H^*\bar{u}=0$, where H^* is the adjoint of H and $\bar{u}(x)$ is the conjugate of $u(x)$. See Mochizuki [2], Remark 1.1 (p. 425). See also Remark 1.8 of [3].

[2] K. Mochizuki: *Eigenfunction expansions associated with the Schrödinger operator with a complex potential and the scattering theory,* Publ. RIMS Kyoto Univ. Ser. A **4** (1968), 419–466.

[3] Y. Saitō: *The principle of limiting absorption for the non-selfadjoint Schrödinger operator in* $R^N(N\neq 2)$, Publ. Res. Inst. Math. Sci. Kyoto Univ. **9** (1974), 397–428.