

## AN APPROXIMATE POSITIVE PART OF A SELF-ADJOINT PSEUDO-DIFFERENTIAL OPERATOR II

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### 1. Introduction

Let  $P=P(x, D)$  be a self-adjoint pseudo-differential operator with the symbol  $p(x, \xi)$  in the class  $S_{1,0}^1$  of Hörmander. The positive part of  $P$  is defined by

$$P^+ = \int_0^\infty \lambda dE(\lambda),$$

where  $dE(\lambda)$  is the spectral measure of  $P$ . We shall be concerned with the following question: To what extent the correspondence;  $u \rightarrow P^+u$  can be localized? We shall prove a localization principle for the operator  $P^+$  which is analogous to Theorem 6.3 of Hörmander [5]. If we combine this with our previous discussions in [2], we can explicitly construct an operator  $B$  such that we have estimate

$$|((A^+ - B)u, v)| \leq C \|u\|_{1/6} \|v\|_{1/6},$$

where  $u$  and  $v$  are arbitrary functions in  $\mathcal{D}(\mathbf{R}^n)$  and  $C$  is a positive constant independent of  $u$  and  $v$ .

### 2. Localized operators

Let us repeat our notations.  $p(x, \xi)$  is a function in the class  $S_{1,0}^1$  which vanishes unless  $x$  lies in a compact set  $K$  in  $\mathbf{R}^n$ . We treat pseudo-differential operator  $P(x, D)$  defined as

$$(2.1) \quad P(x, D)u(x) = (2\pi)^{-n} \iint_{\mathbf{R}^{2n}} p(x, \xi) u(y) e^{i(x-y) \cdot \xi} dy d\xi.$$

We assume that  $P=P(x, D)$  is self-adjoint in Hilbert space  $L^2(\mathbf{R}^n)$ .

Now we make use of the partition of unity of Hörmander [5]. Let  $g_0=0, g_1, g_2, \dots$  be the unit lattice points in  $\mathbf{R}^n$ . Then  $\mathbf{R}^n$  is covered by open cubes of side 2 with center at these points. Let  $\Theta(x)$  be a non-negative  $C^\infty$  function which equals 1 on  $|x_j| \leq 1$  and 0 outside  $|x_j| \leq 3/2, j=1, 2, 3, \dots, n$ . We set

$$(2.2) \quad \varphi_k(x) = \Theta(x-g_k) / \left( \sum_{k=0}^{\infty} \Theta(x-g_k)^2 \right)^{\frac{1}{2}}$$

and

$$\hat{\varphi}_k(x) = \varphi_k \left( \frac{1}{2}(x-g_k) + g_k \right).$$

Note that  $\hat{\varphi}(x)=1$  on  $\text{supp } \varphi_k$ . We, by definition, have

$$(2.4) \quad \sum_k \varphi_k(x)^2 = 1$$

and

$$(2.5) \quad \sum_k |D^\alpha \varphi_k(x)|^2 \leq C_\alpha$$

for any multi-index  $\alpha=(\alpha_1, \alpha_2, \dots, \alpha_n)$ .

$$(2.6) \quad |x-y| \leq 3\sqrt{n} \quad \text{if } x \text{ and } y \text{ are in } \text{supp } \varphi_k.$$

We set

$$(2.7) \quad \psi_k(\xi) = \varphi_k(\xi/|\xi|^\rho) \quad \text{and}$$

$$(2.8) \quad \hat{\psi}_k(\xi) = \hat{\varphi}_k(\xi/|\xi|^\rho), \quad \frac{1}{2} \leq \rho \leq 1.$$

Then we have

$$(2.9) \quad \sum_k \psi_k(\xi)^2 = 1,$$

$$(2.10) \quad |\xi|^{2|\alpha|\rho} \sum_k |D^\alpha \psi_k(\xi)|^2 \leq C,$$

and

$$(2.11) \quad |\xi-\eta| \leq C|\xi|^\rho \quad \text{if } \xi \text{ and } \eta \text{ are in } \text{supp } \psi_k.$$

Here and hereafter  $C$  stands for positive constants which are different from time to time.

$$(2.12) \quad \sum_k |\psi_k(\xi) - \psi_k(\eta)|^2 \leq \frac{C(\xi-\eta)^2}{(1+|\xi|)^\rho(1+|\eta|)^\rho} \quad \text{for any } \xi, \eta \in \mathbf{R}^n.$$

Let  $\delta_j = |g_j|^{2/(1-\rho)}$ . Then  $g_j \delta_j \in \text{supp } \psi_j$ . We shall denote by  $\psi_j(D)$  the pseudo-differential operator corresponding to the symbol  $\psi_j(\xi)$ . Then we have

$$(2.13) \quad \sum_j \psi_j(D)^2 = I.$$

The Sobolef norm  $\|u\|_t$  of  $u$  is equivalent to  $(\sum_j \delta_j^{2t/\rho} \|\psi_j(D)u\|^2)^{\frac{1}{2}}$ .

We put  $\varphi_{jk}(x) = \varphi_j(\delta_k^\sigma x)$  and  $\phi_{jk}(x, \xi) = \varphi_{jk}(x) \psi_k(\xi)$ ,  $\hat{\phi}_{jk}(x, \xi) = \hat{\varphi}_{jk}(x) \hat{\psi}_k(\xi)$  where  $\sigma = (1-\rho)/\rho$ . It is obvious from definition that

$$(2.14) \quad \left| \left( \frac{\partial}{\partial x} \right)^\alpha \left( \frac{\partial}{\partial \xi} \right)^\beta \phi_{jk}(x, \xi) \right| \leq C \delta_k^{|\alpha|\sigma} \delta_k^{-|\beta|} \leq C |\xi|^{|\alpha|(1-\rho) - |\beta|\rho},$$

This means that the set  $\{\phi_{jk}\}_{jk}$  is bounded in the class  $S_{\rho,1-\rho}^0$ . We shall frequently use the inequality

$$(2.15) \quad C \|u\|_s^2 \leq \sum_{jk} \delta_k^{2s/\rho} \|\phi_{jk}(x, D)u\|_s^2 \leq C^{-1} \|u\|_s^2.$$

Choosing a point  $(x^{jk}, \xi^k)$  in  $\text{supp } \phi_{jk}$ , we set

$$(2.16) \quad Q_{jk}(x, D) = \sum_{|\alpha|+|\beta| < N} \frac{x^\alpha D^\beta}{\alpha! \beta!} p_{(\omega)}^{(\beta)}(x^{jk}, \xi^k), \quad N \geq \rho/(1-\rho),$$

and  $P_{jk}(x, D) = \frac{1}{2}(Q_{jk}(x, D) + Q_{jk}(x, D)^*)$ , where  $Q(x, D)^*$  is the formal adjoint of  $Q_{jk}(x, D)$ . We call these  $P_{jk}(x, D)$  localized operators.

### 3. Statement of results

**Theorem 1.** *For any given  $\gamma > \frac{1}{2}(1-\rho)$ , there exists a constant  $C_\gamma > 0$  such that inequality*

$$(3.1) \quad |(P^+u, u) - \sum_{jk} (P^+ \phi_{jk}(x, D)u, \phi_{jk}(x, D)u)| \leq C_\gamma \|u\|_\gamma \|u\|$$

holds for any  $u \in C_0^\infty(\mathbf{R}^n)$ .

**Theorem 2.** *Assume that the localized operators  $P_{jk}(x, D)$  are self-adjoint. Let  $P_{jk}^+$  denote the non-negative part of  $P_{jk}$ . Then, for any  $\gamma > \frac{1}{2}(1-\rho)$ , there exists a constant  $C_\gamma > 0$  such that we have estimate*

$$(3.2) \quad |(P^+u, u) - \sum_{jk} (P_{jk}^+ \phi_{jk}(x, D)u, \phi_{jk}(x, D)u)| \leq C_\gamma (\|u\|_\gamma \|u\| + \|u\|_{\frac{1}{2}(1-\rho)}^2)$$

for any  $u$  in  $C_0^\infty(\mathbf{R}^n)$ .

REMARK 3.1. When  $\rho=2/3$  and  $N=2$ , the assumption that  $P_{jk}(x, D)$  is self-adjoint is satisfied and  $P_{jk}^+$  is easily constructed. See [2] for the details. We can construct operator  $B$  for which the estimate  $|((P^+ - B)u, v)| \leq C \|u\|_{1/6} \|v\|_{1/6}$  holds for any  $u$  and  $v$  in  $C_0^\infty$ .

### 4. Proofs

We begin our proof by the following lemma.

**Lemma 4.1.** *Let  $A$  be a self-adjoint operator in a Hilbert space  $X$ . Let  $e^{isA}$  be the corresponding one-parameter group of unitary operators. Then the non-negative part  $A^+$  of  $A$  is given by the formula*

$$(4.1) \quad A^+ x = -(2\pi)^{-1} \int_{-\infty}^{\infty} \frac{e^{isA}}{(s-i0)^2} x ds$$

for any  $x$  in  $D(A^2)$ . Here  $(s-i0)^{-2}$  is the distribution  $\lim_{\epsilon \downarrow 0} (s-i\epsilon)^{-2}$ . (cf. Gelfand-Silov [3])

Proof. Let  $\lambda^+ = \max(\lambda, 0)$ . Then we have

$$(4.2) \quad \int_{-\infty}^{\infty} (s-i0)^{-2} e^{is\lambda} ds = -2\pi\lambda^+.$$

If  $\varphi$  is in  $\mathcal{B}(\mathbf{R}^n)$ , then

$$(4.3) \quad \langle (s-i0)^{-2}, \varphi(s) \rangle = \int_0^{\infty} (\varphi(s) + \varphi(-s) - 2\varphi(0)) / s^2 ds + i\pi\varphi'(0).$$

This and (4.2) mean that

$$(4.4) \quad -2\pi\lambda^+ = \int_0^{\infty} (e^{is\lambda} + e^{-is\lambda} - 2) / s^2 ds - \pi\lambda.$$

Now we need spectral representation  $A = \int_{-\infty}^{\infty} \lambda dE(\lambda)$  of  $A$ . Integrating (4.4) with respect to  $\lambda$  by measure  $d_\lambda E(\lambda)x$ , we have

$$-2\pi A^+ x = \int_0^{\infty} (e^{isA} + e^{-isA} - 2) / s^2 ds x - \pi Ax = \int_{-\infty}^{\infty} e^{isA} x / (s-i0)^2 ds.$$

Proof of Theorem 1. We have to deal with the difference

$$(4.5) \quad (P^+u, u) - \sum_{jk} (P^+ \phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \\ = \sum_{jk} ((P^+, \phi_{jk}^*(x, D)) \phi_{jk}(x, D)u, u).$$

Putting

$$(4.6) \quad \phi_{jk}(s; x, D) = e^{i\frac{1}{2}sP} \phi_{jk}(x, D) e^{-i\frac{1}{2}sP} \quad \text{and} \\ \phi_{jk}^*(s; x, D) = e^{i\frac{1}{2}sP} \phi_{jk}(x, D)^* e^{-i\frac{1}{2}sP},$$

we have

$$(4.7) \quad [e^{isP}, \phi_{jk}(x, D)^*] \phi_{jk}(x, D) \\ = e^{i\frac{1}{2}sP} (\phi_{jk}^*(s; x, D) - \phi_{jk}^*(-s; x, D)) \phi_{jk}(s; x, D) e^{i\frac{1}{2}sP}.$$

Therefore by lemma 4.1,

$$(4.8) \quad [P^+, \phi_{jk}(x, D)^*] \phi_{jk}(x, D) \\ = -(2\pi)^{-1} \int_{-\infty}^{\infty} (s-i0)^{-2} e^{i\frac{1}{2}sP} (\phi_{jk}^*(s; x, D) - \phi_{jk}^*(-s; x, D)) \phi_{jk}(s; x, D) e^{i\frac{1}{2}sP} ds.$$

The operator  $\phi_{jk}(s; x, D)$  is a pseudo-differential operator whose symbol is given in the following manner; Let  $(y(t; x, \xi), \eta(t; x, \xi))$  be the solution of the Hamilton-Jacobi equations

$$(4.9) \quad \frac{d\eta}{dt} = \frac{\partial p(y, \eta)}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial p(y, \eta)}{\partial \eta}$$

with initial conditions  $y(0; x, \xi) = x$ , and  $\eta(0; x, \xi) = \xi$ . The symbol of  $\phi_{jk}(s; x, D)$  is

$$(4.10) \quad \phi_{jk}(s; x, \xi) = \phi_{jk}(y(s; x, \xi), \eta(s, x, \xi)).$$

(cf. Egoroff [1], Hörmander [6] and Nirenberg-Trèves [7].). As a consequence, the sequence  $\phi_{jk}(s; x, \xi)$  is bounded in  $S_{\rho, 1-\rho}^0$  and the number of overlaps of  $\text{supp } \phi_{jk}(s; x, \xi)$  is bounded. Set

$$(4.11) \quad \Phi_{jk}(s; x, D) = (\phi_{jk}^*(s; x, D) - \phi_{jk}^*(-s; x, D))\phi_{jk}(s; x, D).$$

Then we have

**Lemma 4.2.**

$$(4.12) \quad 1^\circ \quad \Phi_{jk}(0; x, D) = 0,$$

$$(4.13) \quad 2^\circ \quad \frac{d}{ds} \Phi_{jk}(s; x, D) = \frac{1}{2}i\{[P, \Phi_{jk}^*(x, D)]_{jk} + [P, \phi_{jk}^*(x, D)]_{-cs}\}\phi_{jk}(s; x, D) + \frac{1}{2}i(\phi_{jk}^*(s; x, D) - \phi_{jk}(-s; x, D))[P, \phi_{jk}]_{cs}.$$

$$(4.15) \quad 3^\circ \quad |s|^{-\alpha} \left\{ \frac{d}{ds} \Phi_{jk}(s; x, D) - 2i[P, \phi_{jk}^*(x, D)]\phi_{jk}(x, D) \right\}, \quad j, k = 0, 1, 2, \dots,$$

is a bounded sequence in the space  $L_{\rho, 1-\rho}^{(1+\alpha)(1-\rho)}$ , if  $0 \leq \alpha < 1$ . Here we have used the notation  $[P, \phi_{jk}^*(x, D)]_{cs} = e^{i\frac{1}{2}sP}[P, \phi_{jk}^*(x, D)]e^{-i\frac{1}{2}sP}$ .

Proof.

$1^\circ$  is obvious.

$$2^\circ \quad \frac{d}{ds} \phi_{jk}^*(s; x, D) = \frac{1}{2}ie^{i\frac{1}{2}sP}[P, \phi_{jk}^*]e^{-i\frac{1}{2}sP} = \frac{1}{2}i[P, \phi_{jk}^*(x, D)]_{cs}.$$

$$3^\circ \quad \begin{aligned} \frac{d^2}{ds^2} \Phi_{jk}(s; x, D) &= \\ &= (i/2)^2\{[P, [P, \phi_{jk}^*]_{cs}] - [P, [P, \phi_{jk}^*(x, D)]_{-cs}]\}\phi_{jk}(s; x, D) \\ &\quad + 2(i/2)^2\{[P, \phi_{jk}(x, D)^*]_{cs} + [P, \phi_{jk}^*(x, D)]_{-cs}\}[P, \phi_{jk}]_{c-1s} \\ &\quad + (i/2)^2(\phi_{jk}^*(s; x, D) - \phi_{jk}(-s; x, D))[P, [P, \phi_{jk}]_{cs}]. \end{aligned}$$

This implies that the set  $\left\{ \frac{d^2}{ds^2} \Phi_{jk}(s; x, D) \right\}_{jk}$  is bounded in  $S_{\rho, 1-\rho}^{2(1-\rho)}$ . Applying convexity argument, we can prove that the set  $\left\{ \frac{d}{ds} \Phi_{jk}(s; x, D) - \frac{d}{ds} \Phi_{jk}(0; x, D) \right\} |s|^{-\alpha}$  is bounded in  $S_{\rho, 1-\rho}^{(1+\alpha)(1-\rho)}(\mathbf{R}^n)$ . This proves  $3^\circ$ .

Now we come back to the proof of Theorem 1. We divide integral (4.8) into two parts;

$$(4.16) \quad A_{jk} = \int_t^\infty s^{-2} (e^{i\frac{1}{2}sP} \Phi_{jk}(s; x, D) e^{i\frac{1}{2}sP} + e^{-i\frac{1}{2}sP} \Phi_{jk}(-s; x, D) e^{-i\frac{1}{2}sP}) ds$$

and

$$(4.17) \quad B_{jk} = -2\pi [P, \phi_{jk}^*(x, D)] \phi_{jk}(x, D) + \int_0^t s^{-2} (e^{i\frac{1}{2}sP} \Phi_{ij}(s; x, D) e^{i\frac{1}{2}sP} + e^{-i\frac{1}{2}sP} \Phi_{jk}(-s; x, D) e^{-i\frac{1}{2}sP}) ds.$$

We have to prove estimate

$$(4.18) \quad \left| \sum_{jk} (A_{jk}u, u) + \sum_{jk} (B_{jk}u, u) \right| \leq C_\gamma \|u\|_\gamma \|u\|.$$

Since  $\{\Phi_{jk}(s; x, \xi)\}_{jk}$  is bounded in  $S_{\rho, 1-\rho}^0$  and the number of overlaps of  $\text{supp } \Phi_{jk}$  is bounded, the series  $\sum_{jk} \Phi_{jk}(s; x, D)$  converges to an operator  $T(s; x, D)$  in  $L_{\rho, 1-\rho}^0$  of Hörmander [5]. Thus we have

$$(4.19) \quad \left| \sum_{jk} (A_{jk}u, u) \right| = \left| \int_t^\infty s^{-2} \{ (T(s; x, D) e^{i\frac{1}{2}sP} u, e^{-i\frac{1}{2}sP} u) + (T(-s; x, D) e^{-i\frac{1}{2}sP} u, e^{i\frac{1}{2}sP} u) \} ds \right| \leq C t^{-1} \|u\|^2.$$

We get estimate of  $\sum_{jk} (B_{jk}u, u)$  by virtue of lemma 4.2. The set  $\left\{ |s|^{-(1+\omega)} \left( \Phi_{jk}(s; x, D) - s \frac{d}{ds} \Phi_{jk}(0; x, D) \right) \right\}_{jk}$  is bounded in  $S_{\rho, 1-\rho}^{(1+\omega)(1-\rho)}$ . If we set  $\Lambda = (1 - \Delta)^{\frac{1}{2}}$  and

$$S_{jk}(s; x, D) = \Lambda^{-\frac{1}{2}(1+\omega)(1-\rho)} s^{-(1+\omega)} \left( \Phi_{jk}(s; x, D) - s \frac{d}{ds} \Phi_{jk}(0; x, D) \right) \Lambda^{-\frac{1}{2}(1+\omega)(1-\rho)},$$

the sequence of their symbols  $S_{jk}(s; x, D)$  is bounded in  $S_{\rho, 1-\rho}^0$  and the number of overlaps of supports of them is also bounded. The series  $\sum_{kj} S_{jk}(s; x, D)$  thus converges to an operator  $S(s; x, D)$  in the space  $L_{\rho, 1-\rho}^0$ . Hence we have

$$(4.20) \quad \begin{aligned} \sum_{jk} (B_{jk}u, u) &= \int_0^t s^{\omega-1} (S(s; x, D) e^{i\frac{1}{2}sP} \Lambda^{\frac{1}{2}(1+\omega)(1-\rho)}(s)u, e^{-i\frac{1}{2}sP} \Lambda^{\frac{1}{2}(1+\omega)(1-\rho)}(-s)u) ds \\ &\quad + \int_0^t s^{\omega-1} (S(-s; x, D) e^{-i\frac{1}{2}sP} \Lambda^{\frac{1}{2}(1+\omega)(1-\rho)}(-s)u, e^{-i\frac{1}{2}sP} \Lambda^{\frac{1}{2}(1+\omega)(1-\rho)}(-s)u) ds, \end{aligned}$$

where  $\Lambda(s) = e^{i\frac{1}{2}sP} \Lambda e^{-i\frac{1}{2}sP}$ .

Since  $\Lambda(s)$  and  $\Lambda(-s)$  are elliptic operators of order 1, we have

$$(4.21) \quad \begin{aligned} \left| \sum_{jk} (B_{jk}u, u) \right| &\leq C \int_0^t s^{\omega-1} ds \|u\|_{\frac{1}{2}(1+\omega)(1-\rho)}^2 \\ &= C t^\omega \|u\|_{\frac{1}{2}(1+\omega)(1-\rho)}^2 \end{aligned}$$

Setting  $\gamma = \frac{1}{2}(1 + \alpha)(1 - \rho)$  and adding (4.19) and (4.21), we obtain

$$|\sum_{jk} (A_{jk}u, u) + \sum_{jk} (B_{jk}u, u)| \leq C(t^\alpha \|u\|_Y^2 + t^{-1} \|u\|^2).$$

Since  $t$  was arbitrary positive number we take the minimum of the right side with respect to  $t$ . This completes proof of Theorem I.

Proof of Theorem II.

This time we have to deal with

$$(4.22) \quad |(P^+u, u) - \sum_{jk} (P_{jk}^+ \phi_{jk}(x, D)u, \phi_{jk}(x, D)u)| \\ \leq \sum_{jk} |((P^+ - P_{jk}^+) \phi_{jk}(x, D)u, \phi_{jk}(x, D)u)|.$$

Using Lemma 4.1 again, we have

$$(4.23) \quad ((P^+ - P_{jk}^+) \phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \\ = \int_{-\infty}^{\infty} (s - i0)^{-2} ((e^{isP} - e^{isP_{jk}}) \phi_{jk}(x, D)u, \phi_{jk}(x, D)u) ds.$$

We put

$$L(s) = ((e^{isP} - e^{isP_{jk}}) \phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \quad \text{and}$$

divide the integral in (4.23) into two parts;

$$(4.24) \quad M_{jk} = \int_0^{|\xi_k|^{\rho-1}} s^{-2} (L(s) + L(-s)) ds \quad \text{and}$$

$$(4.25) \quad N_{jk} = \pi i L'(0) + \int_{|\xi_k|^{\rho-1}}^{\infty} s^{-2} (L(s) + L(-s)) ds.$$

The latter is easily majorized. In fact, unitarity of operators  $e^{isP}$  and  $e^{isP_{jk}}$  imply that

$$(4.26) \quad \int_{|\xi_k|^{\rho-1}}^{\infty} s^{-2} |L(s) + L(-s)| ds \leq 2 \int_{|\xi_k|^{\rho-1}}^{\infty} s^{-2} \|\phi_{jk}(x, D)u\|^2 ds \\ \leq C |\xi_k|^{1-\rho} \|\phi_{jk}(x, D)u\|^2,$$

while

$$(4.27) \quad |L'(0)| = |((P - P_{jk}) \phi_{jk}(x, D)u, \phi_{jk}(x, D)u)| \\ \leq C |\xi_k|^{1-\rho} \|\phi_{jk}(x, D)u\|^2.$$

And we have

$$(4.28) \quad N_{jk} \leq C |\xi_k|^{1-\rho} \|\phi_{jk}(x, D)u\|^2.$$

$L(s)$  can be written in the form

$$(4.29) \quad \begin{aligned} L(s) &= \int_0^s \frac{d}{dt} ((e^{itP} e^{-i(s-t)P_{jk}}) \phi_{jk}(x, D)u, \phi_{jk}(x, D)u) dt \\ &= \int_0^s (e^{itP} (P - P_{jk}) e^{i(s-t)P_{jk}} \phi_{jk}(x, D)u, \phi_{jk}(x, D)u) dt. \end{aligned}$$

The integrand can be divided into two parts

$$(4.30) \quad J(t) = e^{itP} \dot{\phi}_{jk}^*(2t; x, D) (P - P_{jk}) e^{i(s-t)P_{jk}}$$

and

$$(4.31) \quad K(t) = e^{itP} (I - \dot{\phi}_{jk}^*(2t; x, D)) (P - P_{jk}) e^{i(s-t)P_{jk}}.$$

Here  $\dot{\phi}_{jk}^*(2t; x, D) = e^{-itP} \dot{\phi}_{jk}(x, D)^* e^{itP}$ . The symbol  $\dot{\phi}_{jk}(2t; x, \xi)^*$  of it is obtained from  $\dot{\phi}_{jk}(x, \xi)^*$  in exactly the same manner as  $\phi_{jk}(t; x, \xi)^*$  is obtained from  $\phi_{jk}^*(x, \xi)$ . A consequence of this is that there exists constant  $C > 0$  such that  $|x - x^{jk}| \leq C |\xi_k|^{p-1}$  and  $|\xi - \xi^{jk}| \leq C |\xi_k|^p$  hold if  $(x, \xi)$  is in  $\text{supp } \dot{\phi}_{jk}^*(2t; x, \xi)$  and  $|t| \leq |\xi_k|^{p-1}$ . This fact together with definition of  $P_{jk}$  imply that  $\{\dot{\phi}_{jk}^*(2t; x, \xi) (P - P_{jk})\}_{jk}$  is bounded in  $S_{\rho, 1-\rho}^{1, \rho}$  and at most bounded number of them have non-empty intersection.

**Lemma 4.3.** *We have the following estimates;*

$$(4.32) \quad (1) \quad |(J(t)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)| \leq C |\xi_k|^{1-\rho} \|\phi_{jk}(x, D)u\|^2,$$

$$(4.33) \quad (2) \quad |t|^{-\alpha} |(J(t)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) - (J(0)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)| \leq C |\xi_k|^{(1+\alpha)(1-\beta)} \|\phi_{jk}(x, D)u\|^2.$$

Proof.

(1) Since  $\{\dot{\phi}_{jk}^*(2t; x, D) (P - P_{jk})\}_{jk}$  is a bounded set in  $L_{\rho, 1-\rho}^{1, \rho}$ , we have

$$\begin{aligned} & |(J(t)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)| \\ &= |(e^{itP} \Lambda^{\rho-1} \dot{\phi}_{jk}^*(2t; x, D) (P - P_{jk}) e^{i(s-t)P_{jk}} \phi_{jk}(x, D)u, \Lambda^{1-\rho} (-2t)\phi_{jk}(x, D)u)| \\ &\leq C \|\phi_{jk}(x, D)u\| \|\Lambda^{1-\rho} (-2t)\phi_{jk}(x, D)u\| \\ &\leq C \|\phi_{jk}(x, D)u\|^2 |\xi_k|^{1-\rho}. \end{aligned}$$

(2) Differentiating (4.30), we have

$$\begin{aligned} \frac{d}{dt} J(t) &= e^{itP} \dot{\phi}_{jk}^*(2t; x, D) (P(P - P_{jk}) - (P - P_{jk})P_{jk}) e^{i(s-t)P_{jk}} \\ &= e^{itP} \dot{\phi}_{jk}^*(2t; x, D) \{(P - P_{jk})^2 + [P, P - P_{jk}]\} e^{i(s-t)P_{jk}}. \end{aligned}$$

We know, just as above, that

$$\dot{\phi}_{jk}^*(2t; x, D) \{(P - P_{jk})^2 + [P, P - P_{jk}]\} \Lambda^{-\alpha(1-\rho)}$$

is bounded. This fact implies that



$$\left| \left( \frac{d}{dt} J(t) \phi_{jk}(x, D)u, \phi_{jk}(x, D)u \right) \right| \leq C |\xi_h|^{2(1-\rho)} \|\phi_{jk}(x, D)u\|^2.$$

Convexity argument again proves

$$\begin{aligned} & \left| |t|^{-\alpha} \{ (J(t) \phi_{jk}(x, D)u, \phi_{jk}(x, D)u) - (J(0) \phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \} \right| \\ & \leq C |\xi_h|^{(1+\alpha)(1-\rho)} \|\phi_{jk}(x, D)u\|^2. \end{aligned}$$

**Lemma 4.4.**

$$(4.34) \quad |(K(t) \phi_{jk}(x, D)u, \phi_{jk}(x, D)u)| \leq C |\xi_h|^{-4n} \|\phi_{jk}(x, D)u\| \|u\|$$

and

$$(4.35) \quad \left| \left( \frac{d}{dt} K(t) \phi_{jk}(x, D)u, \phi_{jk}(x, D)u \right) \right| \leq C |\xi_h|^{-4n} \|\phi_{jk}(x, D)u\| \|u\|.$$

Proof. By definition (4.31) we have

$$\phi_{jk}^*(x, D)K(t) = e^{itP} \phi_{jk}^*(2t; x, D) (1 - \dot{\phi}_{jk}^*(2t; x, D)) (P - P_{jk}) e^{i(s-t)P_{jk}}.$$

Lemma 4.4 is a consequence of this and the fact that  $\phi_{jk}^*(2t; x, D) (1 - \dot{\phi}_{jk}^*(2t; x, D))$  belongs to  $L^{-\infty}$ .

Now we are able to manage (4.23).  $L(s)$  turns out to be

$$(4.36) \quad \begin{aligned} L(s) = & \int_0^s ((J(t) - J(0)) \phi_{jk}(x, D)u, \phi_{jk}(x, D)u) dt \\ & + s(J(0) \phi_{jk}(x, D)u, \phi_{jk}(x, D)u) \\ & + \int_0^s (s-t) \left( \frac{d}{dt} K(t) \phi_{jk}(x, D)u, \phi_{jk}(x, D)u \right) dt \\ & + s(K(0) \phi_{jk}(x, D)u, \phi_{jk}(x, D)u). \end{aligned}$$

The first term is estimated as a consequence of Lemma 4.3.

$$(4.37) \quad \begin{aligned} & \left| \int_0^s ((J(t) - J(0)) \phi_{jk}(x, D)u, \phi_{jk}(x, D)u) dt \right| \\ & = \left| \int_0^s t^\alpha t^{-\alpha} (J(t) - J(0)) (\phi_{jk}(x, D)u, \phi_{jk}(x, D)u) dt \right| \\ & \leq C s^{\alpha+1} |\xi_h|^{(1+\alpha)(1-\rho)} \|\phi_{jk}(x, D)u\|^2, \quad \alpha > 0. \end{aligned}$$

Estimate of the third term follows from Lemma 4.4;

$$(4.38) \quad \begin{aligned} & \left| \int_0^s (s-t) \left( \frac{d}{dt} K(t) \phi_{jk}(x, D)u, \phi_{jk}(x, D)u \right) dt \right| \\ & \leq C |\xi_h|^{-4ns^2} \|\phi_{jk}(x, D)u\| \|u\|. \end{aligned}$$

Thus we have proved that  $L(s) = sW(s) + R(s)$ , where

$$(4.39) \quad W(s) = ((P - P_{jk})e^{isP_{jk}}\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)$$

and

$$(4.40) \quad |R(s)| \leq C(s^{\alpha+1}|\xi_k|^{(1+\alpha)(1-\rho)}|\phi_{jk}(x, D)u|^2 + s^2|\xi_k|^{-4n}|\phi_{jk}(x, D)u|||u|).$$

Now we majorize  $M_{jk}$ . First we have

$$\begin{aligned} & \left| \int_0^{|\xi_k|^{\rho-1}} s^{-2}(R(s) + R(-s))ds \right| \\ & \leq C(|\xi_k|^{\alpha(\rho-1)}|\xi_k|^{(1+\alpha)(1-\rho)}|\phi_{jk}(x, D)u|^2 + |\xi_k|^{-4n+1-\rho}|\phi_{jk}(x, D)u|||u|). \end{aligned}$$

The remainder is

$$\int_0^{|\xi_k|^{\rho-1}} s^{-1}(\sin(sP_{jk})\phi_{jk}(x, D)u, (P - P_{jk})^*\phi_{jk}(x, D)u)ds.$$

Therefore we have proved estimate

$$(4.41) \quad |M_{jk}| \leq C(|\xi_k|^{1-\rho}|\phi_{jk}(x, D)u|^2 + |\xi_k|^{-4n+1-\rho}|\phi_{jk}(x, D)u|||u|)$$

if we admit the following lemma that will be proved later.

**Lemma 4.5.** *Let  $A$  be a self-adjoint operator in a Hilbert space  $X$ , then*

$$\left\| \int_0^K s^{-1} \sin(sA) ds \right\| \leq \pi.$$

It follows from (4.23), (4.24) and (4.26) that we must prove estimate

$$\left| \sum_{jk} M_{jk} + \sum_{jk} N_{jk} \right| \leq C(\|u\|_\gamma \|u\| + \|u\|_{(1-\rho)/2}^2)$$

This is proved in the following manner: Taking summation of (4.41) with respect to  $j$  and  $k$ , we have

$$\sum_{jk} |M_{jk}| \leq C \sum_{jk} |\xi_k|^{1-\rho} |\phi_{jk}(x, D)u|^2 \leq C \|u\|_{\frac{1}{2}(1-\rho)}^2.$$

On the other hand

$$\begin{aligned} \sum_{jk} |N_{jk}| & \leq C \left( \sum_{jk} |\xi_k|^{1-\rho} |\phi_{jk}(x, D)u|^2 + \xi_k^{-4n+1-\rho} |\phi_{jk}(x, D)u|||u| \right) \\ & \leq C \left( \sum_{jk} \|\phi_{jk}(x, D)u\|_{\frac{1}{2}(1-\rho)}^2 + \|u\|^2 \right) \\ & \leq C \|u\|_{\frac{1}{2}(1-\rho)}^2, \end{aligned}$$

This is because the number of those  $j$ 's for which  $\text{supp } \phi_{jk} \cap K \times R^n$ ,  $k$  being fixed, is of order  $|\xi_k|^{(1-\rho)n} \times (\text{the volume of the set } K)$ . Theorem II is now proved up to Lemma 4.5.

**Proof of Lemma 4.5.** Let  $A = \int_{-\infty}^{\infty} \lambda dE(\lambda)$  be the spectral representation of  $A$ . Then we have

$$\begin{aligned}
\int_0^K s^{-1}(\sin(sA)x, y) ds &= \int_0^K ds \int_{-\infty}^{\infty} s^{-1} \sin(\lambda s) d(E(\lambda)x, y) \\
&= \int_{-\infty}^{\infty} d(E(\lambda)x, y) \int_0^K s^{-1} \sin(\lambda s) ds \\
&= \int_{-\infty}^{\infty} d(E(\lambda)x, y) \int_0^{K\lambda} s^{-1} \sin s ds.
\end{aligned}$$

Therefore,

$$\left\| \int_0^K s^{-1} \sin s A ds \right\| \leq \text{Sup}_T \left\| \int_0^T s^{-1} \sin s ds \right\| \leq \pi.$$

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#### References

- [1] Yu. V. Egoroff: *On canonical transformations of pseudo-differential operators*, Uspehi Mat. Nauk. **24** (1969), 235–236.
- [2] D. Fujiwara: *Approximate positive part of a self-adjoint pseudo-differential operator*. I. Osaka J. Math. **11** (1974), 265–281.
- [3] Gelfand-Silov: *Generalised Functions*.
- [4] L. Hörmander: *Pseudo-differential operators and non elliptic boundary value problems*, Ann. of Math. **83** (1966), 129–209.
- [5] ———: *Pseudo-differential operators and hypoelliptic equations*, Proc. Symp in pure Math. A.M.S., Vol. X, 138–183.
- [6] ———: *Fourier integral operators I*, Acta Math. **127** (1971), 79–183.
- [7] L. Nirenberg and F. Trèves: *On local solvability of linear partial differential equations Part II, sufficient conditions*, Comm. Pure Appl. Math. **23** (1970), 459–510.

