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CONTINUOUS MAPS OF MANIFOLDS WITH INVOLUTION II

MINORU NAKAOKA

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Introduction

Let N and M be closed manifolds on each of which an involution is given, and assume that the involution on N is free. In the previous paper [9], the author defined the equivariant Lefschetz class of a continuous map $f: N\to M$, and treated the class in the case when the involution on M is also free. The present paper is concerned with the equivariant Lefschetz class in the case when the involution on M is trivial. As applications, we show generalizations of the Borsuk-Ulam theorem and also theorem of group action on manifolds.

Compared with the previous case, the expression of the equivariant Lefschetz class in the present case is rather complicated, and the Wu classes of mainfolds and the operations of Breden [1] appear in it. Some properties of the semi-characteristic of manifolds are also needed in our applications. These are prepared in §1 and §2 (see also Appendix).

Throughout this paper, the homology and cohomology with coefficients in Z_2 are to be understood. For brevity, manifolds and actions on them are assumed to be differentiable.

1. Semicharacteristic of manifolds with involution

If M is a closed manifold such that the dimension of the vector space $H^*(M)$ is even, an integer mod 2 given by

$$\hat{\chi}(M)=\frac{1}{2}\dim H^*(M) \qquad \mathrm{mod}\ 2$$

is called the *semicharacteristic* of M. If N is a closed manifold with a free involution, it is known that dim $H^*(N)$ is even (see [1], [9]). In this section we shall consider the semicharacteristic of closed manifolds with involution.

(1.1) **Proposition.** *Let W be a compact $(n+1)$-dimensional manifold with boundary ∂W, and assume that W has a free involution T. Then we have $\hat{\chi}(\partial W)=0$.*

Proof. Consider the exact sequence

$$\cdots \to H^r(W, \partial W) \xrightarrow{j_r^*} H^r(W) \xrightarrow{i_r^*} H^r(\partial W) \xrightarrow{\delta_r^*} H^{r+1}(W, \partial W) \to \cdots .$$

Since $\dim H^r(W, \partial W) = \dim H^{n+1-r}(W)$ by the Lefschetz-Poincaré duality, it is easily seen theat

$$\dim \operatorname{Im} j_l^* \equiv \chi(W) + \sum_{r=0}^{l-1} \dim H^r(\partial W) \qquad \bmod 2$$

if $n=2l-1$, and

$$\dim \operatorname{Im} i_l^* \equiv \chi(W) + \sum_{r=0}^{l-1} \dim H^r(\partial W) \qquad \bmod 2$$

if $n=2l$, where $\chi(W)$ denotes the Euler characteristic of W.

Since W has a free involution, a triangulation of W can be taken in such a way that the number of r-simplices is even for each r. Therefore $\chi(W) \equiv 0 \bmod 2$. Since ∂W has a free involution, $\dim H^l(\partial W)$ is even if $n=2l$. By the Poincaré duality we have $\dim H^r(\partial W) = \dim H^{n-r}(\partial W)$.

Consequently the desired result follows from that

i) $\dim \operatorname{Im} j_l^*$ is even if $n=2l-1$,

ii) $\dim \operatorname{Im} i_l^* = \frac{1}{2} \dim H^l(\partial W)$ if $n=2l$.

Proof of i). It follows that a non-degenerate bilinear form

$$\varphi \colon \operatorname{Im} j_l^* \times \operatorname{Im} j_l^* \to Z_2$$

can be defined by

$$\varphi(\alpha, \beta) = \langle \alpha' \cup T^*\beta', [W] \rangle,$$
$$\alpha = j_l^*(\alpha'), \quad \beta = j_l^*(\beta'),$$

where $[W] \in H_{2l}(W, \partial W)$ is the fundamental homology class. If M is any closed $2l$-dimensional manifold with a free involution T, it is known that $\alpha \cup T^*\alpha = 0$ for any $\alpha \in H^l(M)$ (see [1], [9]). Therefore, by considering the double of W, it is easily seen that $\alpha' \cup T^*\alpha' = 0$ for any $\alpha' \in H^l(W, \partial W)$. This shows that the bilinear form φ is symplectic. Thus $\operatorname{Im} j_l^*$ has a symplectic, non-degenerate bilinear form, and hence $\dim \operatorname{Im} j_l^*$ is even.

Proof of ii). If follows that a bilinear form

$$\psi \colon H^l(\partial W)/\operatorname{Im} i_l^* \times \operatorname{Im} i_l^* \to Z_2$$

can be defined by

$$\psi(\bar{\alpha}, \beta) = \langle \alpha \cup \beta, [\partial W] \rangle,$$

where $\bar{\alpha}$ is the class represented by $\alpha \in H^l(\partial W)$, $\beta \in \operatorname{Im} i_l^*$, and $[\partial W] \in H_{2l}(\partial W)$ is the fundamental homology class. Consider the commutative diagram

$$\begin{array}{ccccc} H^l(W) & \xrightarrow{i_l^*} & H^l(\partial W) & \xrightarrow{\delta_l^*} & H^{l+1}(W, \partial W) \\ \downarrow \cap [W] & & \downarrow \cap [\partial W] & & \downarrow \cap [W] \\ H_{l+1}(W, \partial W) & \xrightarrow{\partial_*} & H_l(\partial W) & \xrightarrow{i_*} & H_l(W), \end{array}$$

then it is easily proved that the bilinear form ψ is non-degenerate. Therefore it holds that

$$\dim \operatorname{Im} i_l^* = \dim H^l(\partial W) - \dim \operatorname{Im} i_l^*,$$

and the proof of ii) completes.

REMARK. R. Lee [6] proves (1.1) in a more general form when n is odd.

Let $\mathfrak{N}_*$ denote the unoriented Thom bordism ring, and let $\mathfrak{N}_*(Z_2)$ denote the unoriented bordism group of free involutions. As is shown in [2], $\mathfrak{N}_*(Z_2)$ may be regarded as an $\mathfrak{N}_*$-module by defining

$$[N, T]\cdot[M] = [N\times M, T\times 1]$$

for $[M]\in\mathfrak{N}_*$ and $[N, T]\in\mathfrak{N}_*(Z_2)$, and it is a free $\mathfrak{N}_*$-module with basis $\{[S^n, A]; n=0, 1, 2, \cdots\}$, where A denotes the antipodal map on the n-sphere S^n.

If N_1 and N_2 are closed manifold with free involution,

$$\hat{\chi}(N_1 \cup N_2) = \hat{\chi}(N_1) + \hat{\chi}(N_2)$$

is obvious. Therefore, in virtue of (1.1) a group-homomorphism

$$\hat{\chi}\colon \mathfrak{N}_*(Z_2) \to Z_2$$

can be defined by seding $[N, T]$ to $\hat{\chi}(N)$. If N is a closed manifold with free involution and M is a closed manifold, it follows that

$$\hat{\chi}(N\times M) \equiv \hat{\chi}(N)\chi(M) \qquad \bmod 2\,.$$

Consequently, if we regard Z_2 as an $\mathfrak{N}_*$-module by defining $r\cdot[M]=r\chi(M)$ mod 2 ($r\in Z$), it turns out that $\hat{\chi}$ is a homomorphism of $\mathfrak{N}_*$-modules.

Let X be a paracompact space with a free involution T. Denote by X_T the orbit space, and consider the principal Z_2-bundle $\pi\colon X\to X_T$ defined by the projection. The 1-st Stiefel-Whitney class

$$c = c(X, T)\in H^1(X_T)$$

of the bundle π is called the *involution class* of (X, T). For an equivariant map $f\colon X\to Y$, we have

$$f^*c(Y, T) = c(X, T)\,. \tag{1.2}$$

We denote by $w_k(M)\in H^k(M)$ the k-th Stiefel-Whitney class of a manifold M. The following is due to F. Uchida.

(1.3) **Theorem.** *For a closed n-dimensional manifold N with a free involution T, it holds that*

$$\hat{\chi}(N)=\langle\sum_{k=0}^{n}c^{n-k}w_k,\ [N_T]\rangle\,,$$

where $c=c(N,\ T)$, $w_k=w_k(N_T)$.

Proof. As is shown above, $\hat{\chi}$: $\mathfrak{N}_*(Z_2)\rightarrow Z_2$ is a homomorphism of $\mathfrak{N}_*$-modules. We have also a homomorphism $\mathfrak{N}_*(Z_2)\rightarrow Z_2$ of $\mathfrak{N}_*$-modules by sending $[N,\ T]$ to $\langle\sum_{k=0}^{n}c^{n-k}w_k,\ [N_T]\rangle$. This is easily shown if we recall that $c^{n-k}w_k$ depends on the class $[N,\ T]\in\mathfrak{N}_*(Z_2)$ ([2]) and if we note $c(N\times M,\ T\times 1)=c(N,\ T)\times 1$, $w_k(N_T\times M)=\sum_{i+j=k}w_i(N_T)\times w_j(M)$ and $\langle w_m(M),\ [M]\rangle\equiv\chi(M)$ mod 2. Thus it suffices to prove the theorem in the special case when N is S^n and T is the anitpodal map.

In this case, N_T is the real projective space RP^n, and it follows that

$$\begin{aligned}&\langle\sum_{k=0}^{n}c^{n-k}w_k,\ [RP^n]\rangle\\&=\langle\sum_{k=0}^{n}\binom{n+1}{k}c^n,\ [RP^n]\rangle\\&=\sum_{k=0}^{n}\binom{n+1}{k}\equiv 1\qquad\bmod 2\,.\end{aligned}$$

Since $\hat{\chi}(S^n)=1$, we have the desired result.

Remark. If N is a closed even-dimensional manifold with a free involution T, it is easily seen that

$$\chi(N_T)=\chi(N)/2\equiv\hat{\chi}(N)\qquad\bmod 2\,.$$

2. The Bredon operation

Let S^∞ denote the infinite dimensional sphere, and let X be a paracompact space. We shall regard as a space with involution S^∞ by the antipodal map T, and $X^2=X\times X$ by the map T such that $T(x_1,\ x_2)=(x_2,\ x_1)$. We consider the diagonal action on $S^\infty\times X^2$, and denote the orbit space by $S^\infty\underset{T}{\times}X^2$.

A continuous map $f\colon X\rightarrow Y$ defines a continuous map $1\underset{T}{\times}f^2\colon S^\infty\underset{T}{\times}X^2\rightarrow S^\infty\underset{T}{\times}Y^2$, and the diagonal map $d\colon X\rightarrow X^2$ defines a continuous map $1\underset{T}{\times}d\colon S^\infty_T\times X\rightarrow S^\infty\underset{T}{\times}X^2$. The projection $p\colon S^\infty\times X^2\rightarrow S^\infty$ defines a bundle p_T:

$S^\infty \underset{T}{\times} X^2 \to S^\infty_T$ with fiber X^2, and there is the inclusion i: $X^2 \to S^\infty \underset{T}{\times} X^2$ of a typical fiber.

According to Steenrod [9] (see also [1]), there is the operation

$$P: H^r(X) \to H^{2r}(S^\infty \underset{T}{\times} X^2)$$

satisfying the following properties:

$$P(f^*\alpha) = (1 \underset{T}{\times} f^2)^* P(\alpha)\,, \tag{2.1}$$

$$i^* P(\alpha) = \alpha \times \alpha\,, \tag{2.2}$$

$$P(\alpha\beta) = P(\alpha)P(\beta)\,, \tag{2.3}$$

$$(1 \underset{T}{\times} d)^* P(\alpha) = \sum_{i+j=r} w^i \times Sq^j \alpha\,, \tag{2.4}$$

where $w = w(S^\infty, T)$ is the generator of $H^1(S^\infty_T)$.

Suppose now that X has an involution T, and consider the diagonal action on $S^\infty \times X$. Then an equivariant map Δ: $X \to X^2$ is defined by

$$\Delta(x) = (x, Tx)\,,$$

and it defines a continuous map $1 \underset{T}{\times} \Delta$: $S^\infty \underset{T}{\times} X \to S^\infty \underset{T}{\times} X^2$. Bredon [1] defines an operation

$$Q: H^r(X) \to H^{2r}(S^\infty \underset{T}{\times} X)$$

by $Q = (1 \underset{T}{\times} \Delta)^* \circ P$.

We suppose now that the involution T on X is free. Then the projection q: $S^\infty \times X \to X$ defines a bundle q_T: $S^\infty \underset{T}{\times} X \to X_T$ with fiber S^∞, and hence we have the isomorphism

$$q_T^*: H^r(X_T) \cong H^r(S^\infty \underset{T}{\times} X)\,.$$

Thus, if X has a free involution T, there is the operation

$$Q: H^r(X) \to H^{2r}(X_T)$$

by regarding q_T^* as the identification.

Corresponding to (2.1)-(2.3), the following (2.5)-(2.7) hold:

(2.5) *If f is equivariant, then*

$$Qf^*(\alpha) = f_T^* Q(\alpha)\,.$$

(2.6) *For the projection π: $X \to X_T$, we have*

$$\pi^* Q(\alpha) = \alpha \cup T^*\alpha\,.$$

$$Q(\alpha\beta) = Q(\alpha)Q(\beta)\,. \tag{2.7}$$

We shall prove

(2.8) $$Q(\pi^*\alpha)=\sum_{i+j=r} c^i\, Sq^j\alpha\,,$$

where $\alpha\in H^r(X_T)$ *and* $c=c(X,\ T)$.

Proof. Consider the diagram

$$\begin{tikzcd}
H^*(S^\infty\underset{T}{\times}X^2) \arrow[r, "(1\underset{T}{\times}\Delta)^*"] & H^*(S^\infty\underset{T}{\times}X) & H^*(Y) \arrow[l, "q_T^*"'] \arrow[dl, "q^*"] \\
H^*(S^\infty\underset{T}{\times}Y^2) \arrow[u, "(1\underset{T}{\times}\pi^2)^*"] \arrow[r, "(1\underset{T}{\times}d)^*"] & H^*(S_T^\infty\times Y) \arrow[u, "(1\underset{T}{\times}\pi)^*"'] &
\end{tikzcd}$$

where $Y=X_T$ and d is the diagonal map. Since the diagram is commutative, it follows from (2.1) and (2.4) that

$$\begin{aligned}
&Q(\pi^*\alpha)\\
&=q_T^{*-1}(1\underset{T}{\times}\Delta)^*P(\pi^*\alpha)\\
&=q_T^{*-1}(1\underset{T}{\times}\Delta)^*(1\underset{T}{\times}\pi^2)^*P(\alpha)\\
&=q_T^{*-1}(1\underset{T}{\times}\pi)^*(1\underset{T}{\times}d)^*P(\alpha)\\
&=q_T^{*-1}(1\underset{T}{\times}\pi)^*\Big(\sum_{i+j=r}w^i\times Sq^j\alpha\Big)\,.
\end{aligned}$$

Let $h\colon X\to S^\infty$ be an equivariant map, and consider the diagram

$$\begin{tikzcd}
H^*(S_T^\infty) \arrow[rr, "h_T^*"] \arrow[d, "p^*"] \arrow[drr, "p_T^*"] & & H^*(Y) \arrow[d, "q_T^*"] \\
H^*(S_T^\infty\times Y) \arrow[rr, "(1\underset{T}{\times}\pi)^*"] & & H^*(S^\infty\underset{T}{\times}X)\,.
\end{tikzcd}$$

Since the maps $h\circ q,\ p\colon S^\infty\times X\to S^\infty$ are equivariant, we have $p_T^*=q_T^*\circ h_T^*$. It is obvious that $(1\underset{T}{\times}\pi)^*\circ p^*=p_T^*$. Therefore it follows that

$$\begin{aligned}
&Q(\pi^*\alpha)\\
&=\sum_{i+j=r}q_T^{*-1}(1\underset{T}{\times}\pi)^*(p^*w^i\cup q^*Sq^j\alpha)\\
&=\sum_{i+j=r}q_T^{*-1}(1\underset{T}{\times}\pi)^*p^*w^i\cup q_T^{*-1}(1\underset{T}{\times}\pi)^*q^*Sq^j\alpha\\
&=\sum_{i+j=r}h_T^*w^i\cup Sq^j\alpha\\
&=\sum_{i+j=r}c^i Sq^j\alpha\,.
\end{aligned}$$

This completes the proof of (2.8).

For a closed manifold M, let $v(M)=\sum_i v_i(M)\in H^*(M)$ denote the total Wu class of M, *i.e.* the class such that

$$\langle Sq\alpha,\ [M]\rangle = \langle \alpha \cup v(M),\ [M]\rangle$$

for any $\alpha\in H^*(M)$, where $Sq=1+Sq^1+Sq^2+\cdots$. For the total Stiefel-Whitney class $w(M)=\sum_i w_i(M)$ we have

$$w(M) = Sqv(M)\,,$$

and it holds that

$$v_i(M) = 0 \quad \text{for} \quad i>[\dim M/2]$$

(see [5]).

We shall prove

(2.9) *If $g\colon M\to M$ is a continuous map such that $g^*\colon H^*(M)\to H^*(M)$ is onto, then*

$$g^*v_i(M) = v_i(M)\,.$$

Proof. It follows that

$$\begin{aligned} &\langle Sqg^*\alpha,\ [M]\rangle = \langle g^*Sq\alpha,\ [M]\rangle \\ &= \langle Sq\alpha,\ g_*[M]\rangle = \langle Sq\alpha,\ [M]\rangle \\ &= \langle \alpha\cup v(M),\ [M]\rangle = \langle \alpha\cup v(M),\ g_*[M]\rangle \\ &= \langle g^*\alpha\cup g^*v(M),\ [M]\rangle \end{aligned}$$

and any element of $H^*(M)$ has the form $g^*\alpha$. Therefore we have $g^*v(M)=v(M)$.

(2.10) *For a closed manifold N with a free involution T, we have*

$$v(N) = \pi^*v(N_T)\,.$$

Proof. Since the tangent bundle to N is induced from the tangent bundle to N_T by the projection π, we have

$$w(N) = \pi^*w(N_T)\,.$$

Therefore it follows that

$$\begin{aligned} &Sq\pi^*v(N_T) = \pi^*Sqv(N_T) \\ &= \pi^*w(N_T) = w(N) \\ &= Sqv(N)\,. \end{aligned}$$

Since Sq is invertible, this proves (2.10).

As for the semicharacteristic, we have

(2.11) **Theorem.** *For a closed n-dimensional manifold N with a free involution T, it holds that*

$$\hat{\chi}(N) = \langle \sum_i c^{n-2i} Q(v_i),\ [N_T] \rangle ,$$

where $c=c(N, T)$, $v_i=v_i(N)$.

Proof. It follows from (2.10), (2.8) that

$$\begin{aligned}
&\sum_i c^{n-2i} Q(v_i(N)) \\
&= \sum_i c^{n-2i} Q(\pi^* v_i(N_T)) \\
&= \sum_i c^{n-2i} \sum_{j+k=i} c^k Sq^j v_i(N_T) \\
&= \sum_{i+j+k=n} c^k Sq^j v_i(N_T) \\
&= \sum_k c^k w_{n-k}(N_T) .
\end{aligned}$$

Therefore, by Theorem (1.3) we have

$$\begin{aligned}
&\langle \sum_i c^{n-2i} Q(v_i(N)),\ [N_T] \rangle \\
&= \langle \sum_k c^k w_{n-k}(N_T),\ [N_T] \rangle \\
&= \hat{\chi}(N) ,
\end{aligned}$$

which completes the proof.

3. The equivariant Lefschetz class

Let N be a paracompact space with a free involution, and M be a closed manifold with an involution. In [9], for a continuous map $f: N \to M$, we defined the equivariant Lefschetz class $\hat{f}_T^*(\Delta_N) \in H^*(N_T)$, and considered the class in the case when the involution on M is free. In this section we shall consider the class in the case when the involution on M is trivial. To distinguish Δ_N in the two cases, we shall write θ_N for Δ_N in the present case.

Let M be a closed m-dimensional manifolds. We denote by ν the normal bundle of the diagonal imbedding $d: M \to M^2$. We regard M^2 as a manifold with involution by the map T interchanging factors. Then the total space of ν may be regarded as an equivariant tubular neighborhood U of $d(M)$ in M^2. Therefore it turns out that ν is a vector bundle with involution.

Let N be a paracompact space with a free involution T. As in §2, we regard $N \times M^2$ as a space with involution by the diagonal action, and consider the orbit space $N \underset{T}{\times} M^2$. Then we have the real n-dimensional vector bundle $1 \underset{T}{\times} \nu: N \underset{T}{\times} U \to N_T \times M$. We regard the Thom class $t(1 \underset{T}{\times} \nu) \in H^m(N \underset{T}{\times} (U, U - dM))$ as an element of $H^m(N \underset{T}{\times} (M^2, M^2 - dM))$ by the excision. Then

$$\theta_N \in H^m(N \underset{T}{\times} M^2)$$

is defined to be the restriction of $t(1 \underset{T}{\times} \nu)$.

We have

(3.1) *If h: $N \to N'$ is an equivariant map of paracompact spaces with free involution, then* $(h \underset{T}{\times} 1)^* \theta_{N'} = \theta_N$.

(3.2) *If N is a closed manifold with a free involution T, then*

$$(1 \times d)_*[N_T \times M] = \theta_N \cap [N \underset{T}{\times} M^2] .$$

Let f: $N \to M$ be a continuous map. By definition, the *equivariant Lefschetz class* of f is

$$\hat{f}_T^*(\theta_N) \in H^m(N_T) ,$$

where $\hat{f}$: $N \to N \times M^2$ is an equivariant map given by $\hat{f}(y) = (y, f(y), fT(y))$.

As is shown in [9], we have

(3.3) *Let N be a closed n-dimensional manifold with a free involution T. If $\hat{f}_T^*(\theta_N) \neq 0$, then the covering dimension of*

$$A(f) = \{y \in N; fT(y) = f(y)\}$$

is at least $n - m$.

To study $\hat{f}_T^*(\theta_N)$ we shall proceed in parallel with [9].

We denote by θ_∞ the element θ_N for $N = S^\infty$ and $T =$ antipodal map.

(3.4) **Proposition.** *Let $\{\alpha_1, \alpha_2, \cdots, \alpha_s\}$ be a basis for the vector space $H^*(M)$, and put $a_i = \alpha_i \cap [M]$. Let*

$$d_*([M]) = \sum_{j,k} \eta_{jk} a_j \times a_k$$

for d_*: $H_*(M) \to H_*(M^2)$. Then we have

$$\theta_\infty = \sum_{i=0}^{[m/2]} c^{m-2i} P(v_i) + \sum_{j<k} (\eta_{jk} + \eta_{jj} \eta_{kk}) \phi^*(1 \times \alpha_j \times \alpha_k) ,$$

where $v_i = v_i(M)$, $c = c(S^\infty \times M^2, T)$ and ϕ^*: $H^*(S^\infty \times M^2) \to H^*(S^\infty \underset{T}{\times} M^2)$ is the transfer homomorphism.

Proof. It is known that

$$\theta_\infty = \sum_{i=0}^{[m/2]} c^{m-2i} P(v_i) + \sum_{j<k} \varepsilon_{jk} \phi^*(1 \times \alpha_j \times \alpha_k)$$

with $\varepsilon_{jk} \in Z_2$ (see [3], [8]). Therefore it suffices to prove

$$\varepsilon_{jk} = \eta_{jk} + \eta_{jj} \eta_{kk} \qquad (j<k) .$$

Consider the inclusion i: $M^2 \to S^\infty \underset{T}{\times} M^2$ of a typical fibre. It follows that $i^*(\theta_\infty) \in H^m(M^2)$ is the restriction of the Thom class $t(\nu) \in H^m(M^2, M^2 - dM)$ of the normal bundle ν. Therefore it follows that

$$d^*[M] = i^*(\theta_\infty) \cap [M \times M] .$$

Since $i^*(c)=0$, $i^*P(\alpha)=\alpha\times\alpha$ and $i^*\phi^*(1\times\alpha\times\beta)=\alpha\times\beta+\beta\times\alpha$, we have

$$\begin{aligned}&\sum_{j,k}\eta_{jk}a_j\times a_k\\ &=\begin{cases}\sum_{j<k}\varepsilon_{jk}(a_j\times a_k+a_k\times a_j) & \text{if } m \text{ is odd,}\\ \sum_{j,k}\varepsilon_j\varepsilon_k a_j\times a_k+\sum_{j<k}\varepsilon_{jk}(a_j\times a_k+a_k\times a_j) & \text{if } m \text{ is even,}\end{cases}\end{aligned}$$

where we put $v_{[m/2]}=\sum_j \varepsilon_j\alpha_j$ if m is even. Consequently it holds that

$$\varepsilon_{jk} = \eta_{jk}(j<k), \qquad \eta_{jj} = 0$$

if m is odd, and

$$\eta_{jk} = \varepsilon_{jk}+\varepsilon_j\varepsilon_k\ (j<k), \qquad \eta_{jj} = \varepsilon_j$$

if m is even. Thus $\varepsilon_{jk}=\eta_{jk}+\eta_{jj}\eta_{kk}$ $(j<k)$ holds, and the proof completes.

As for the equivariant Lefschetz class $\hat{f}_T^*(\theta_N)$ we have

(3.5) **Theorem.** *Let N be a paracompact space with a free involution T, and M be a closed manifold. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_s\}$ be a basis for the vector space $H^*(M)$, and put*

$$d_*[M] = \sum_{j,k}\eta_{jk}a_j\times a_k, \qquad a_i = \alpha_i\cap[M] .$$

Then, for any continuous map $f\colon N\to M$, it holds that

$$\hat{f}_T^*(\theta_N) = \sum_{i=0}^{[m/2]} c^{m-2i}Q(f^*v_i)+\sum_{j<k}(\eta_{jk}+\eta_{jj}\eta_{kk})\phi^*(f^*\alpha_j\cup T^*f^*\alpha_k) ,$$

where $c=c(N, T)$, $v_i=v_i(M)$ and $\phi^*\colon H^*(N)\to H^*(N_T)$ is the transfer homomorphism.

Proof. There exists an equivariant map $h\colon N\to S^\infty$, and we have

$$\theta_N = (h\underset{T}{\times}1)^*\theta_\infty$$

by (3.1). The diagram

$$\begin{array}{ccccc} H^m(S^\infty\underset{T}{\times}M^2) & \xrightarrow{(h\underset{T}{\times}1)^*} & H^m(N\underset{T}{\times}M^2) & \xrightarrow{\hat{f}_T^*} & H^m(N_T)\\ \Big\downarrow{\scriptstyle (1\underset{T}{\times}f^2)^*} & & & & \Big\uparrow{\scriptstyle d_T^*}\\ H^m(S^\infty\underset{T}{\times}N^2) & \xrightarrow{(1\underset{T}{\times}\Delta)^*} & H^m(S^\infty\underset{T}{\times}N) & \xrightarrow{(h\underset{T}{\times}1)^*} & H^m(N\underset{T}{\times}N)\end{array}$$

is commutative, where $\Delta\colon N\to N\times N$ is given by $\Delta(y)=(y, Ty)$, and $d\colon N\to N\times N$ is the diagonal map. It is obvious that

$$d_T^* \circ (h \underset{T}{\times} 1)^* \circ q_T^* = id$$

for the isomorphism q_T^*: $H^*(N_T) \cong H^*(S^\infty \underset{T}{\times} N)$. Therefore it follows that

$$\begin{aligned} \hat{f}_T^*(\theta_N) &= \hat{f}_T^*(h \underset{T}{\times} 1)^* \theta_\infty \\ &= d_T^*(h \underset{T}{\times} 1)^*(1 \underset{T}{\times} \Delta)^*(1 \underset{T}{\times} f^2)^* \theta_\infty \\ &= q_T^{*-1}(1 \underset{T}{\times} \Delta)^*(1 \underset{T}{\times} f^2)^* \theta_\infty . \end{aligned}$$

We have

$$q_T^{*-1}(1 \underset{T}{\times} \Delta)^* \, (1 \underset{T}{\times} f^2)^* c(S^\infty \times M^2, T) = c(N, T)$$

by (1.2), and

$$q_T^{*-1}(1 \underset{T}{\times} \Delta)^*(1 \underset{T}{\times} f^2)^* P(\alpha) = Q(f^* \alpha)$$

by (2.1) and the definition of Q. It follows that

$$\begin{aligned} & q_T^{*-1}(1 \underset{T}{\times} \Delta)^*(1 \underset{T}{\times} f^2)^* \phi^*(1 \times \alpha \times \beta) \\ &= q_T^{*-1} \phi^*(1 \times d^*)(1 \times 1 \times T^*)(1 \times f^* \times f^*)(1 \times \alpha \times \beta) \\ &= q_T^{*-1} \phi^*(1 \times (f^* \alpha \cup T^* f^* \beta)) \\ &= \phi^*(f^* \alpha \cup T^* f^* \beta) . \end{aligned}$$

Therefore, in virtue of Proposition (3.4), we have the desired result.

REMARK 1. With the notations in (3.5), consider the matrices $X = (\xi_{jk})$, $Y = (\eta_{jk})$ over Z_2, where $\xi_{jk} = \langle \alpha_j \cup \alpha_k, [M] \rangle$. Then we have $Y = X^{-1}$. In fact, it follows that

$$\begin{aligned} \xi_{jk} &= \langle \alpha_j \times \alpha_k, d_*[M] \rangle \\ &= \sum_{s,t} \eta_{st} \langle \alpha_j \times \alpha_k, a_s \times a_t \rangle \\ &= \sum_{s,t} \eta_{st} \langle \alpha_j, \alpha_s \cap [M] \rangle \langle \alpha_k, \alpha_t \cap [M] \rangle \\ &= \sum_{s,t} \xi_{js} \eta_{st} \xi_{tk} , \end{aligned}$$

and hence we have $X = XYX$, *i.e.* $Y = X^{-1}$.

REMARK 2. If there exists a basis $\{\alpha_1, \cdots, \alpha_r, \beta_1, \cdots, \beta_r\}$ for the vector space $H^*(M)$ such that $\langle \alpha_j \cup \alpha_k, [M] \rangle = 0$, $\langle \beta_j \cup \beta_k, [M] \rangle = 0$ and $\langle \alpha_j \cup \beta_k, [M] \rangle = \delta_{jk}$, then the conclusion of Theorem (3.5) is written as follows:

$$\hat{f}_T^*(\theta_N) = \sum_{i=0}^{[m/2]} c^{m-2i} Q(f^* v_i) + \sum_{j=1}^{r} \phi^*(f^* \alpha_j \cup T^* f^* \beta_j) .$$

Such a basis exists when dim $M = 2l - 1$ or when dim $M = 2l$ and $\alpha^2 = 0$ for any $\alpha \in H^l(M)$.

4. Applications

1) Borsuk-Ulam type theorem

Combining Theorem (3.5) with (3.3), we can obtain generalizations of the Borsuk-Ulam theorem. As a typical example we have

(4.1) **Theorem.** *Let N be a closed n-dimensional manifold with a free involution T, and let $f: N \to M$ be a continuous map to an m-dimensional manifold M. Assume that $c^m \neq 0$ for $c=c(N, T)$ and $f_*: \tilde{H}_*(N) \to \tilde{H}_*(M)$ is trivial. Then the covering dimension of $A(f)=\{y \in N; f(y)=f(Ty)\}$ is at least $n-m$.*

Proof. Without loss of generality, we may assume that M is closed (see p. 88 in [2]). Since $f^*: \tilde{H}^*(M) \to \tilde{H}^*(N)$ is trivial, we have

$$\hat{f}_T^*(\theta_N) = c^m \neq 0$$

by Theorem (3.5). Therefore we get the desired result by (3.3).

If N is a mod 2 homology sphere, then $c^m \neq 0$ for $m \leq n$. Therefore (4.1) has as a corollary the following result which is known in [8] (see also [2], [7]).

(4.2) **Corollary.** *Let N be a mod 2 homology n-sphere with a free involution, and let $f: N \to M$ be a continuous map to an n-dimensional manifold M. Then it holds that*

i) *If $n>m$,* $\dim A(f) \geq n-m$,

ii) *If $n=m$ and the degree of f is even, $A(f)$ is not empty.*

2) Equivariant map

As a direct consequence of (4.1), we have

(4.3) **Theorem.** *Let N and M be closed manifolds on each of which a free involution T is given, and let $f: N \to M$ be an equivariant map. Assume that $c^m \neq 0$ for $c=c(N, T)$ and $m=\dim M$. Then $f_*: \tilde{H}_*(N) \to \tilde{H}_*(M)$ is not trivial.*

In some case, the converse of (4.3) is also true.

(4.4) **Theorem.** *Let N be a closed m-dimensional manifold with a free involution T, and M be a mod 2 homology m-sphere with a free involution. Let $f: N \to M$ be an equivariant map. Then if the degree of f is odd, we have $c^m \neq 0$ for $c=c(N, T)$.*

Proof. By (3.5) we have

$$\begin{aligned} \hat{f}_T^*(\theta_N) &= c^m + \phi^* T^* f^*(\alpha) \\ &= c^m + (\deg f)\beta \\ &= c^m + \beta , \end{aligned}$$

where $\alpha \in H^m(M)$ and $\beta \in H^m(N_T)$ are the generators. Suppose $c^m=0$. Then we have $\hat{f}_T^*(\theta_N)\neq 0$, and by (3.3) there exists $y\in N$ such that $f(y)=f(Ty)$. Since f is equivariant and the involution on M is free, this is a contradiction. Therefore we have the desired result.

REMARK. Assuming that N is orientable and M is S^n with the antipodal involution, (4.4) is proved in Theorem 5.1 of Holm-Spanier [4].

3) Group action on manifolds

As is shown in (5.4) and (5.5) of [9], the following theorem holds under weaker conditions. But it is good enough for applications to group action on manifolds (see §6 of [9]). We shall derive it from (3.5) by using the results in §2.

(4.5) **Theorem.** *Let N be a closed manifold with a free involution T, and let $g: N\to N$ be a homeomorphism. Assume $\hat{\chi}(N)\neq 0$ and also assume the following* i) *or* ii):

i) $g^*=id: H^*(N)\to H^*(N)$,

ii) $T^*=id: H^*(N)\to H^*(N)$.

Then there exists $y\in N$ such that $gT(y)=Tg(y)$.

Proof. Put $f=\pi\circ g: N\to N_T$. Then, for α, $\beta\in H^*(N_T)$ we have

$$\begin{aligned}
&\langle\phi^*(f^*\alpha\cup T^*f^*\beta), [N_T]\rangle\\
&=\langle f^*\alpha\cup T^*f^*\beta, [N]\rangle\\
&=\langle g^*\pi^*\alpha\cup T^*g^*\pi^*\beta, [N]\rangle\\
&=\begin{cases}\langle\pi^*(\alpha\cup\beta), [N]\rangle & \text{if } g^*=\mathrm{id},\\ \langle g^*\pi^*(\alpha\cup\beta), [N]\rangle & \text{if } T^*=\mathrm{id}\end{cases}\\
&=\langle\alpha\cup\beta, \pi_*[N]\rangle\\
&=0\,.
\end{aligned}$$

Therefore it follows from Theorem (3.5) that

$$\hat{f}_T^*(\theta_N)=\sum_{i=0}^{[n/2]}c^{n-2i}Q(g^*\pi^*v_i)\,,$$

where $c=c(N, T)$ and $v_i=v_i(N_T)$. In virtue of (2.10), (2.9) and Theorem (2.11), it follows that

$$\begin{aligned}
&\langle\hat{f}_T^*(\theta_N), [N_T]\rangle\\
&=\langle\sum_{i=0}^{[n/2]}c^{n-2i}Q(v_i(N)), [N_T]\rangle\\
&=\hat{\chi}(N)\,.
\end{aligned}$$

By the assumption this shows $\hat{f}_T^*(\theta_N)\neq 0$. Thus,by (3.3) there exists $y\in N$ such that $\pi gT(y)=\pi g(y)$. This means that $gT(y)=g(y)$ or $gT(y)=Tg(y)$. Since g

is monic, $gT(y)=g(y)$ implies $T(y)=y$ which contradicts to that T is free. Therefore there exists $y\in N$ such that $gT(y)=Tg(y)$.

Appendix

In the proof of Theorem (1.3), the bordism theory was used. In this appendix, we shall give another proof of (1.3) which makes no use of bordism theory and which is an application of Theorem (5.2) in [9]. First of all we shall prove

(5.1) **Theorem.** *Let N be a closed n-dimensional manifold with a free involution T. Denote by $\tau(N_T)$ the tangent bundle of N_T, and ρ the line bundle associated to the $O(1)$-bundle π: $N\to N_T$. Then it holds that*

$$\hat{\chi}(N)=\langle w_n(\rho\otimes\tau(N_T)),\ [N_T]\rangle\,.$$

Proof. Regard N as submanifold of $N\times N$ through an imbedding Δ: $N\to N\times N$ given by $\Delta(y)=(y,\ Ty)$, and let ν denote the normal bundle of N in $N\times N$. Regard $N\times N$ as a manifold with involution by the map T interchanging factors. This makes ν a bundle with involution. Consider the bundle $1\underset{T}{\times}\nu$ over $N\underset{T}{\times}N$, and define $\Delta_N\in H^n(N\underset{T}{\times}N^2)$ to be the restriction of the Thom class $t(1\underset{T}{\times}\nu)\in H^n(N\underset{T}{\times}(N^2,\ N^2-N))$. Let d: $N\to N\times N$ be the diagonal map. Then the composition

$$N_T\xrightarrow{d_T}N\underset{T}{\times}N\xrightarrow{1\underset{T}{\times}\Delta}N\underset{T}{\times}N^2$$

is induced from the map sending $y\in N$ to $(y,\ y,\ Ty)\in N\times N^2$. Therefore, by applying Theorem (5.2) in [8] to the identity map of N, we see

$$\hat{\chi}(N)=\langle d_T^*(1\underset{T}{\times}\Delta)^*(\Delta_N),\ [N_T]\rangle\,.$$

By the well known relation between the Thom class and the Euler class, we have

$$w_n(1\times\nu)=(1\underset{T}{\times}\Delta)^*\Delta_N\,.$$

Therefore it holds that

$$\hat{\chi}(N)=\langle d_T^*w_n(1\underset{T}{\times}\nu),\ [N_T]\rangle\,.$$

Thus it suffices to prove that there is an isomorphism

$$\rho\otimes\tau(N_T)\cong d_T^*(1\underset{T}{\times}\nu)\,. \tag{5.2}$$

To prove this, we consider on the tangent bundle $\tau(N)$ an unusual free involution given by sending a tangent vector v_y of N at y to $-dT(v_y)$, where dT

is the differential of T. The bunldle $\tau(N)$ together with this involution will be denoted by $\tilde{\tau}(N)$

For the bundles $\tau(N)$, $\tau(N\times N)$ together with the usual involution, we have the following exact sequence of bundles with involution:

$$0 \to \tau(N) \to \tau(N\times N)\,|\,N \to \nu \to 0\,.$$

It is easily seen that an equivariant bundle map of ν to $\tilde{\tau}(N)$ can be defined by sending $(v_y, v'_{Ty})\in\tau(N\times N)\,|\,N$ to $v_y - dT(v'_{Ty})\in\tau(N)$. Therefore there is an isomorphism $\nu\cong\tilde{\tau}(N)$ of bundles with involution, and hence we have an isomorphism

$$(5.3) \qquad 1\underset{T}{\times}\nu\cong 1\underset{T}{\times}\tilde{\tau}(N)$$

of bundles over $N\underset{T}{\times}N$.

Denote by $\tilde{\tau}(N)_T$ the bundle over N_T obtained from $\tilde{\tau}(N)$ by taking the orbit spaces. If we consider the involution on $\boldsymbol{R}$ given by $T(t)=-t$, we have $\rho=\boldsymbol{R}\underset{T}{\times}\pi$. It follows that a bundle map of $\tilde{\tau}(N)_T$ to $\rho\otimes\tau(N_T)$ can be defined by sending v_y to $(1\underset{T}{\times}y)\otimes d\pi(v_y)$, and that a bundle map of $\tilde{\tau}(N)_T$ to $d_T^*(1\underset{T}{\times}\tilde{\tau}(N))$ can be defined by sending v_y to $(\pi(y), y\underset{T}{\times}v_y)$. Thus we have isomorphisms

$$(5.4) \qquad \rho\otimes\tau(N_T)\cong\tilde{\tau}(N)_T\cong d_T^*(1\underset{T}{\times}\tilde{\tau}(N))$$

of bundles over N_T.

From (5.3) and (5.4) we obtain (5.2), and the proof is complete.

The following is well known, and is easily derived from the elementary properties of Stiefel-Whitney classes by using the splitting principle.

(5.5) *Let ξ be an n-dimensional bundle over X, and ρ be a line bundle over X. Put $c=w_1(\rho)$. Then we have*

$$w_n(\rho\otimes\xi)=\sum_{k=0}^{n}c^{n-k}w_k(\xi)\,.$$

Now (1.3) is a consequence of (5.1) and (5.5).

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References

[1] G.E. Bredon: *Cohomological aspects of transformation groups*. Proc. Conf. Transformation Groups, New Orleans, 1967, Springer-Verlag, 245–280.

[2] P.E. Conner and E.E. Floyd: Differentiable Periodic Maps. Springer-Verlag, 1964.

[3] A. Haefliger: *Points multiples d'une application et produit cyclique reduit*, Amer. J. Math. **83** (1961), 57–70.
[4] P. Holm and E.H. Spanier: *Involutions and Fredholm maps*, Topology **10** (1971), 203–218,
[5] D. Husemoller: Fibre Bundles, McGraw-Hill, 1966.
[6] R. Lee: *Semicharacteristic classes*, Topology **12** (1973), 183–199.
[7] H.J. Munkholm: *Borsuk-Ulam type theorems for proper Z_p-actions on (mod p homology) n-spheres*, Math. Scand. **24** (1969), 167–185.
[8] M. Nakaoka: *Generalizations of Borsuk-Ulam theorem*, Osaka J. Math. **7** (1970) 433–441.
[9] M. Nakaoka: *Continuous maps of manifolds with involution* I, Osaka J. Math. **11** (1974), 129–145.
[10] N.E. Steenrod: Cohomology Operations, Ann. of Math. Studies, 50. Princeton Univ., 1962.