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CONTINUOUS MAPS OF MANIFOLDS WITH INVOLUTION I

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Introduction

To study the question of which finite groups can act freely on a sphere, J. Milnor proved in [5] that if M is a mod 2 homology sphere with a free involution T, then for any continuous map $f: M \to M$ of odd degree there exists a point $x \in M$ such that $fT(x)=Tf(x)$. In the present paper we generalize this theorem, and apply it to the problem of group action on spheres.

Let M be a closed manifold with a free involution T. Then a non-degenerate symplectic pairing $\circ: H^*(M;Z_2)\times H^*(M;Z_2)\to Z_2$ can be defined by $\alpha\circ\beta=\langle\alpha\cup T^*\beta, [M]\rangle$, where $[M]$ is the mod 2 fundamental class of M. Therefore there exists a symplectic basis $\{\mu_1, \cdots, \mu_r, \mu_1', \cdots, \mu_r'\}$ for the vector space $H^*(M;Z_2)$. Let N be also a closed manifold with a free involution T, and $f: N\to M$ be a continuous map. Then it is seen that

$$\hat{\chi}(f)=\sum_{i=1}^{r} f^*\mu_i \circ f^*\mu_i' \in Z_2$$

is independent of the choice of $\{\mu_1, \cdots, \mu_r, \mu_1', \cdots, \mu_r'\}$. Now the Milnor theorem is generalized as follows: If $\hat{\chi}(f)\neq 0$ then there exists a point $y\in N$ such that $Tf(y)=fT(y)$.

This theorem is paraphrased that if the *equivariant Lefschetz number* $\hat{\chi}(f)$ is not zero then there exists an *equivariant* point $y\in N$, and may be regarded as an analogue of the classical Lefschetz fixed point theorem. We shall prove it after the cohomological proof of the Lefschetz fixed point theorem (see e.g. [9]). As is well known, the Lefschetz theorem asserts that the fixed point index is equal to the Lefschetz number. Correspondingly, we define the *equivariant point index* $\hat{I}(f)\in Z_2$ which has a property that $\hat{I}(f)\neq 0$ implies the existence of equivariant points of f, and we prove that the equivariant point index $\hat{I}(f)$ is equal to the equivariant Lefschetz number $\hat{\chi}(f)$.

Our theorem is applicable well for the problem of group action on manifolds as the Lefschetz fixed point theorem is. The theorem is effective to show the non-existence of free action of dihedral group on a given manifold.

Let $Q(8n, k, l)$ denote the group with generators X, Y, A and relations

$$X^2=(XY)^2=Y^{2n}, \quad A^{kl}=1,$$
$$XAX^{-1}=A^r, \quad YAY^{-1}=A^{-1},$$

where $8n$, k, l are pariwise relatively prime positive integers, $r\equiv-1 \pmod{k}$ and $r\equiv+1 \pmod{l}$. Milnor asks in [5] if $Q(8n, k, l)$ can act freely on a 3-sphere. Recently, R. Lee introduced a group homomorphism $\chi_{1/2}$ from the bordism group $\mathfrak{N}_{2m+1}(G)$ to a certain Grothendieck group $\hat{R}_{GL, ev}(G)$ for any finite group G, and applied it to prove that if n is even and $l>1$ then $Q(8n, k, l)$ can not act freely on any mod 2 homology sphere whose dimension is 3 mod 8 (see [4]). We shall give another proof of this result as an application of our theorem.

Milnor asks also if the group $P''(48r)$ (see §6) can act freely on a 3-sphere, and R. Lee answers that $P''(48r)$ ($O(48, k, l)$ in his notation) can not act freely on any mod 2 homology sphere whose dimension is 3 mod 8. We also prove this fact in the case when r is not a power of 3, but our method gives no information when r is a power of 3. It seems to me that the proof of Corollary 4.17 in [4] is incorrect and his method also gives no information for $P''(48\cdot 3^k)$ $(k\geqq 1)$.

Throughout this paper, the homology and cohomology with coefficients in Z_2 are to be understood. For brevity, manifolds and actions on them are assumed to be differentiable.

1. The equivariant Lefschetz class

Let M be a closed m-dimensional manifold with an involution T. We regard the product $M^2=M\times M$ as a manifold with involution by defining $T(x_1, x_2)=(x_2, x_1)$. Then we have an equivariant imbedding $\Delta: M\rightarrow M^2$ given by $\Delta(x)=(x, Tx)$. We shall identify M with its image under Δ. Let ν denote the normal bundle of the imbedding Δ. As usual we shall regard the total space of ν as an equivariant tubular neighborhood U of M in M^2. Then $\nu: U\rightarrow M$ is a vector bundle with involution.

Let N be a paracompact space with a free involution T. Consider $N\underset{T}{\times}M$ and $N\underset{T}{\times}M^2$, the orbit spaces under the diagonal action of T on $N\times M$ and $N\times M^2$. Then we have the vector bundle $\nu_N=1\underset{T}{\times}\nu: N\underset{T}{\times}U\rightarrow N\underset{T}{\times}M$. Regard the Thom class $t(\nu_N)\in H^m(N\underset{T}{\times}(U, U-M))$ as an element of $H^m(N\underset{T}{\times}(M^2, M^2-M))$ by the excision, and define

$$\Delta_N\in H^m(N\underset{T}{\times}M^2)$$

to be the restriction of $t(\nu_N)$.

Obviously we have

(1.1) *If* $h:N\to N'$ *is an equivariant map, then* $(h\underset{T}{\times}1)^*:H^m(N'\underset{T}{\times}M^2)\to H^m(N\underset{T}{\times}M^2)$ *sends* $\Delta_{N'}$ *to* Δ_N.

For a closed manifold W, we denote by $[W]$ the mod 2 fundamental homology class of W. As is easily seen we have

(1.2) *If* N *is a closed m-dimensional manifold, then the Poincaré duality takes* Δ_N *to* $(1\underset{T}{\times}\Delta)_*[N\underset{T}{\times}M]$, *i.e.*

$$(1\underset{T}{\times}\Delta)_*[N\underset{T}{\times}M]=\Delta_N\cap[N\underset{T}{\times}M^2],$$

where $(1\underset{T}{\times}\Delta)_*:H_{n+m}(N\underset{T}{\times}M]\to H_{n+m}(N\underset{T}{\times}M^2)$.

Given a continuous map $f:N\to M$, we define an equivariant map $\hat{f}:N\to N\times M^2$ by $\hat{f}(y)=(y,f(y),fT(y))$. Denote by N_T the orbit space of N under the action T. We have the homomorphism $\hat{f}^*_T:H^*(N\underset{T}{\times}M^2)\to H^*(N_T)$. We call the element

$$\hat{f}^*_T(\Delta_N)\in H^m(N_T)$$

the *equivariant Lefschetz class* of f.

If N is a closed manifold and $\dim M=\dim N$, an integer mod 2 given by the Kronecker product

$$\hat{I}(f)=\langle\hat{f}^*_T(\Delta_N),[N_T]\rangle$$

is called the *equivariant point index* of f.

(1.3) **Proposition.** *Let* N *be a closed manifold, and let* $f:N\to M$ *be a continuous map. If the equivariant Lefschetz class* $\hat{f}^*_T(\Delta_N)$ *is not zero, the covering dimension of*

$$A(f)=\{y\in N;fT(y)=Tf(y)\}$$

is at least $n-m$.

Proof. Denote by $A(f)_T$ the image of $A(f)$ under the projection $\pi:N\to N_T$. Then we have the following commutative diagram:

$$\begin{array}{ccc}H^m(N\underset{T}{\times}(M^2,M^2-M)) & \xrightarrow{j^*} & H^m(N\underset{T}{\times}M^2)\\ \downarrow{\hat{f}^*_T} & & \downarrow{\hat{f}^*_T}\\ H^m(N_T,N_T-A(f)_T) & \xrightarrow{j^*} & H^m(N_T),\end{array}$$

where j are the inclusions. Therefore we have $j^*\hat{f}^*_Tt(\nu_N)=\hat{f}^*_T\Delta_N\neq0$. In particular $H^m(N_T,N_T-A(f)_T)\neq0$. Since this shows $H_m(N_T,N_T-A(f)_T)\neq0$, it follows that the Čech cohomology group $\check{H}^{n-m}(A(f)_T)$ is not zero (see [8]). Therefore dim $A(f)_T \geqq n-m$, and hence we have dim $A(f) \geqq n-m$.

(1.4) **Corollary.** *Let N be a closed manifold, and let $f: N \to M$ be a continuous map. If $\hat{I}(f) \neq 0$ there exists $y \in N$ such that $fT(y) = Tf(y)$.*

2. Preliminaries

Regard the standard n-sphere S^n as a space with involution by the antipodal map, where $n = 1, 2, \cdots, \infty$. The corresponding Δ_N will be denoted by $\Delta_n \in H^m(S^n \underset{T}{\times} M^2)$. Since for any paracompact space N with involution there exists an equivariant map of N to S^∞, the element Δ_∞ is universal among $\{\Delta_N\}$. In the next section we shall consider Δ_∞ in the case when the involution T on M is free. For this purpose, we shall recall in this section some facts from [6] and [7].

We have the following theorem due to N. Steenrod (see §3 of [6]).

(2.1) *$H_*(S^\infty \underset{T}{\times} M^2)$ is naturally isomorphic with $H_*(Z_2; H_*(M)^{(2)})$, the homology group of the group Z_2 with coefficients in $H_*(M)^{(2)} = H_*(M) \otimes H_*(M)$ on which Z_2 acts by permutation of factors. Similarly $H^*(S^\infty \underset{T}{\times} M^2)$ is naturally isomorphic with $H^*(Z_2; H^*(M)^{(2)})$. These isomorphisms preserve the cup product and the cap product.*

We shall regard these isomorphisms as the identifications.

Let W be a Z_2-free acyclic complex which has one cell e_i and its transform Te_i in each dimension $i \geqq 0$ and has the boundary ∂ given by $\partial(e_{2i+1}) = e_{2i} - Te_{2i}$, $\partial(e_{2i+2}) = e_{2i+1} + Te_{2i+1}$. For $a, b \in H_*(M)$, let $P_i(a)$, $P(a, b) \in H_*(Z_2; H_*(M)^{(2)}) = H_*(S^\infty \underset{T}{\times} M^2)$ denote the homology classes represented by the cycles $e_i \otimes a \otimes a$, $e_0 \otimes a \otimes b \in W \underset{Z_2}{\otimes} H_*(M)^{(2)}$ respectively. Similarly, for $\alpha, \beta \in H^*(M)$, let $P_i(\alpha)$, $P(\alpha, \beta) \in H^*(Z_2; H^*(M)^{(2)}) = H^*(S^\infty \underset{T}{\times} M^2)$ denote the cohomology classes represented by the cocylces $u_i(\alpha)$, $u(\alpha, \beta) \in \mathrm{Hom}_{Z_2}(W, H^*(M)^{(2)})$ respectively, where $\langle u_i(\alpha), e_i \rangle = \alpha \otimes \alpha$, $\langle u_i(\alpha), e_j \rangle = 0$ $(i \neq j)$, $\langle u(\alpha, \beta), e_0 \rangle = \alpha \otimes \beta + \beta \otimes \alpha$, $\langle u(\alpha, \beta), e_j \rangle = 0$ $(j \neq 0)$.

As is easily seen we have

(2.2) *If $\{a_1, a_2, \cdots, a_s\}$ is a basis for the vector space $H_*(M)$, then $\{P_i(a_j), P(a_j, a_k); i \geqq 0, j < k\}$ is a basis for the vector space $H_*(S^\infty \underset{T}{\times} M^2)$. Similarly, if $\{\alpha_1, \alpha_2, \cdots, \alpha_s\}$ is a basis for the vector space $H_*(M)$, then $\{P_i(\alpha_j), P(\alpha_j, \alpha_k); i \geqq 0, j < k\}$ is a basis for the vector space $H^*(S^\infty \underset{T}{\times} M^2)$.*

Since a diagnoal approximation $d_\#: W \to W \otimes W$ is given by

$$d_\sharp(e_i)=\sum_{j=0}^{[i/2]} e_{2j}\otimes e_{i-2j}+e_{2j+1}\otimes Te_{i-2j-1}\,,$$

it follows that

$$\text{(2.3)}\qquad \begin{aligned} &P(\alpha,\beta)\cap P_i(a)=0\,,\\ &P_j(\alpha)\cap P_i(a)=\begin{cases} P_{i-j}(\alpha\cap a) & \text{if } j\leqq i\,,\\ 0 & \text{if } j>i\,.\end{cases}\end{aligned}$$

We see

(2.4) *For the homomorphism* $(i\underset{T}{\times}1)_*: H_*(S^n\underset{T}{\times}M^2)\to H_*(S^\infty\underset{T}{\times}M^2)$ *induced by the inclusion, we have*

$$(i\underset{T}{\times}1)_*[S^n\underset{T}{\times}M^2]=P_n([M])\,.$$

Let X be a Hausdorff space with a free involution T. Consider the induced chain map $T_\sharp: S(X)\to S(X)$, $\pi_\sharp: S(X)\to S(X_T)$ of singular complexes, where $\pi: X\to X_T$ is the projection. Then a chain map $\phi: S(X_T)\to S(X)$ can be defined by

$$\phi(c)=\tilde{c}+T_\sharp(\tilde{c}),\quad \pi_\sharp(\tilde{c})=c$$

($c\in S(X_T)$, $\tilde{c}\in S(X)$), and we have 'transfer homomorphisms'

$$\phi_*: H_*(X_T)\to H_*(X),\quad \phi^*: H^*(X)\to H^*(X_T)\,.$$

These are obviously functorial with respect to equivariant maps.

We have the following (2.5) and (2.6) (see §2 of [7]).

(2.5) *For any* $a\in H_*(X_T)$, *the diagram*

$$\begin{array}{ccc} H^*(X_T) & \xrightarrow{\ \cap a\ } & H_*(X_T)\\ \big\downarrow{\scriptstyle \pi^*} & & \big\downarrow{\scriptstyle \phi_*}\\ H^*(X) & \xrightarrow{\ \cap \phi_*(a)\ } & H_*(X)\end{array}$$

is commutative.

(2.6) *If* X *is a closed manifold, then* $\phi_*[X_T]=[X]$.

The following is easily seen.

(2.7) *For* $\phi^*: H^*(S^\infty\times M^2)\to H^*(S^\infty\underset{T}{\times}M^2)$, *we have*

$$\phi^*(1\times\alpha\times\beta)=P(\alpha,\beta)\,.$$

3. Expression of Δ_∞

Throughout this section, we assume that the involution T on M is *free*.

We shall consider the element $\Delta_\infty \in H^m(S^\infty \underset{T}{\times} M^2)$.

(3.1) **Lemma.** *For $n \geqq 1$ we have*

$$\Delta_\infty \cap P_n([M]) = 0 .$$

Proof. In the commutative diagram

$$\begin{array}{ccc} H_{n+m}(S^n \underset{T}{\times} M) & \xrightarrow{(1 \underset{T}{\times} \Delta)_*} & H_{n+m}(S^n \underset{T}{\times} M^2) \\ \downarrow (i \underset{T}{\times} 1)_* & & \downarrow (i \underset{T}{\times} 1)_* \\ H_{n+m}(S^\infty \underset{T}{\times} M) & \xrightarrow{(1 \underset{T}{\times} \Delta)_*} & H_{n+m}(S^\infty \underset{T}{\times} M^2) , \end{array}$$

we have $H_{n+m}(S^\infty \underset{T}{\times} M) \cong H_{n+m}(M_T) = 0$ $(n \geqq 1)$, for the involution T on M is free. Therefore by (2.4), (1.1) and (1.2) we see

$$\begin{aligned} \Delta_\infty \cap P_n([M]) &= \Delta_\infty \cap (i \underset{T}{\times} 1)_*[S^n \underset{T}{\times} M^2] \\ &= (i \underset{T}{\times} 1)_*((i \underset{T}{\times} 1)^* \Delta_\infty \cap [S^n \underset{T}{\times} M^2]) \\ &= (i \underset{T}{\times} 1)_*(\Delta_n \cap [S^n \underset{T}{\times} M^2]) \\ &= (i \underset{T}{\times} 1)_*(1 \underset{T}{\times} \Delta)_*[S^n \underset{T}{\times} M] \\ &= (1 \underset{T}{\times} \Delta)_*(i \underset{T}{\times} 1)_*[S^n \underset{T}{\times} M] \\ &= 0 . \end{aligned}$$

(3.2) **Proposition.** *Let $\{\alpha_1, \alpha_2, \cdots, \alpha_s\}$ be a basis for the vector space $H^*(M)$, and put $a_i = \alpha_i \cap [M]$, $i = 1, 2, \cdots, s$. For $\Delta_*: H_*(M) \to H_*(M^2)$, let*

$$\Delta_*([M]) = \sum_{j,k} \varepsilon_{jk} a_j \times a_k \qquad (\varepsilon_{jk} \in Z_2) .$$

Then we have

$$\varepsilon_{jj} = 0 , \qquad \varepsilon_{jk} = \varepsilon_{kj}$$

for each j, k, and

$$\Delta_\infty = \sum_{j<k} \varepsilon_{jk} \phi^*(1 \times \alpha_j \times \alpha_k) ,$$

where $\phi^*: H^*(S^\infty \times M^2) \to H^*(S^\infty \underset{T}{\times} M^2)$ is the transfer homomorphism.

Proof. In virtue of (2.2) we can put

$$\Delta_\infty = \sum_{i,j} g_{ij} P_i(\alpha_j) + \sum_{j<k} h_{jk} P(\alpha_j, \alpha_k)$$

$(g_{ij}, h_{jk} \in Z_2)$. Then it follows from (3.1) and (2.3) that

$$0=\Delta_\infty\cap P_n([M])=\sum_{i=0}^{n}\sum_{j=1}^{s}g_{ij}P_{n-i}(a_j)$$

for any $n\geqq 1$. Therefore by (2.2) it holds $g_{ij}=0$ for any i, j, and hence by (2.7)

$$\Delta_\infty=\sum_{j<k}h_{jk}\phi^*(1\times\alpha_j\times\alpha_k).$$

By (2.5) and (2.6) the diagram

$$\begin{array}{ccccc} H_*(S^n\underset{T}{\times}M) & \xrightarrow[T]{(1\times\Delta)_*} & H_*(S^n\underset{T}{\times}M^2) & \xleftarrow[T]{\cap[S^n\times M^2]} & H^*(S^n\underset{T}{\times}M^2) \\ \downarrow\phi_* & & \downarrow\phi_* & & \downarrow\pi^* \\ H_*(S^n\times M) & \xrightarrow{(1\times\Delta)_*} & H_*(S^n\times M^2) & \xleftarrow{\cap[S^n\times M^2]} & H^*(S^n\times M^2) \end{array}$$

is commutative. Therefore by (2.6) and (1.2) we have

$$\begin{aligned}&(1\times\Delta)_*[S^n\times M]\\ &=(1\times\Delta)_*\phi_*[S^n\underset{T}{\times}M]=\phi_*(1\underset{T}{\times}\Delta)_*[S^n\underset{T}{\times}M]\\ &=\phi_*(\Delta_n\cap[S^n\underset{T}{\times}M^2])=\pi^*(\Delta_n)\cap[S^n\times M^2].\end{aligned}$$

Since the diagram

$$\begin{array}{ccc} H^*(S^\infty\underset{T}{\times}M^2) & \xrightarrow[T]{(i\times 1)^*} & H^*(S^n\underset{T}{\times}M^2) \\ \downarrow\pi^* & & \downarrow\pi^* \\ H^*(S^\infty\times M^2) & \xrightarrow{(i\times 1)^*} & H^*(S^n\times M^2) \end{array}$$

is commutative, we have

$$\begin{aligned}&(1\times\Delta)_*[S^n\times M]\\ &=\pi^*(i\underset{T}{\times}1)^*(\Delta_\infty)\cap[S^n\times M^2]\\ &=(i\times 1)^*\pi^*(\Delta_\infty)\cap[S^n\times M^2]\\ &=(i\times 1)^*\pi^*(\sum_{j<k}h_{jk}\phi^*(1\times\alpha_j\times\alpha_k))\cap[S^n\times M^2]\\ &=\sum_{j<k}h_{jk}(i\times 1)^*(1\times\alpha_j\times\alpha_k+1\times\alpha_k\times\alpha_j)\cap[S^n\times M^2]\\ &=\sum_{j<k}h_{jk}(1\times\alpha_j\times\alpha_k+1\times\alpha_k\times\alpha_j)\cap([S^n]\times[M]\times[M])\\ &=[S^n]\times\sum_{j<k}h_{jk}(a_j\times a_k+a_k\times a_j).\end{aligned}$$

On the other hand, by the assumption we have

$$\begin{aligned}&(1\times\Delta)_*([S^n]\times[M])\\ &=[S^n]\times\sum_{j,k}\varepsilon_{jk}a_j\times a_k.\end{aligned}$$

Thus we see that $\varepsilon_{jj}=0$, $\varepsilon_{jk}=\varepsilon_{kj}=h_{jk}$ $(j<k)$ and $\Delta_\infty=\sum_{j<k}\varepsilon_{jk}\phi^*(1\times\alpha_j\times\alpha_k)$. This completes the proof.

Define a bilinear form

$$\circ : H^*(M)\times H^*(M)\to Z_2$$

by

$$\alpha\circ\beta=\langle\alpha\cup T^*\beta,\ [M]\rangle\,.$$

By Poincaré duality this is non-singular. We have also

(3.3) **Proposition.** *The bilinear form* $\circ$ *is symplectic, i.e.* $\alpha\circ\alpha=0$ *for any* $\alpha\in H^*(M)$.

Proof. Note first that $\circ$ is symmetric. In fact,

$$\begin{aligned}\alpha\circ\beta &= \langle\alpha\cup T^*\beta,\ [M]\rangle\\ &= \langle T^*(T^*\alpha\cup\beta),\ [M]\rangle = \langle T^*\alpha\cup\beta,\ T_*[M]\rangle\\ &= \langle T^*\alpha\cup\beta,\ [M]\rangle = \langle\beta\cup T^*\alpha,\ [M]\rangle\\ &= \beta\circ\alpha\,.\end{aligned}$$

Therefore we have

$$(\alpha+\beta)\circ(\alpha+\beta)=\alpha\circ\alpha+\beta\circ\beta\,.$$

Thus it suffices to prove that $\alpha\circ\alpha=0$ for each element α of a basis for $H^*(M)$. To do this, take the basis $\{a_1^*, \ldots, a_s^*\}$ dual to a basis $\{a_1, \cdots, a_s\}$ for $H_*(M)$. Then it follows from (3.2) that

$$\begin{aligned}a_i^*\circ a_i^* &= \langle a_i^*\cup T^*a_i^*,\ [M]\rangle\\ &= \langle a_i^*\times a_i^*,\ \Delta_*[M]\rangle\\ &= \langle a_i^*\times a_i^*,\ \sum_{j\neq k}\varepsilon_{jk}a_j\times a_k\rangle\\ &= \sum_{j\neq k}\varepsilon_{jk}\langle a_i^*,\ a_j\rangle\langle a_i^*,\ a_k\rangle\\ &= 0\,.\end{aligned}$$

REMARK. (3.3) is known by G. Bredon (see Corollary 1.11 of [2]).

Let V be a finite dimensional vector space over Z_2, on which a non-singular symplectic bilinear form

$$\circ : V\times V\to Z_2$$

is given. Such V is called a *non-singular symplectic vector space* over Z_2. It is known that for such V we can take a *symplectic basis*, *i.e.* a basis $\{v_1, \cdots, v_r, v_1', \cdots, v_r'\}$ such that

$$v_i\circ v_j=0,\quad v_i'\circ v_j'=0,\quad v_i\circ v_j'=\delta_{ij}$$

(see [1]).

As is shown above, if M is a closed manifold with a free involution, then $H^*(M)$ is a non-singular symplectic vector space over Z_2 with respect to the bilinear form $\circ$ defined above.

(3.4) **Theorem.** *Let $\{\mu_1, \cdots, \mu_r, \mu_1', \cdots, \mu_r'\}$ be a symplectic basis for $H^*(M)$, then we have*

$$\Delta_\infty = \sum_{i=1}^{r} \phi^*(1\times \mu_i \times \mu_i'),$$

where $\phi^*: H^*(S^\infty \times M^2) \to H^*(S^\infty \underset{T}{\times} M^2)$ is the transfer homomorphism.

Proof. Put $a_i=\mu_i \cap [M]$, $a_i'=\mu_i' \cap [M]$ $(i=1, \cdots, r)$. Then $\{a_1, \cdots, a_r, a_1', \cdots, a_r'\}$ is a basis for $H_*(M)$. We have

$$\begin{aligned}\langle T^*\mu_i', a_j\rangle &= \langle T^*\mu_i', \mu_j \cap [M]\rangle \\ &= \langle \mu_j \cup T^*\mu_i', [M]\rangle = \mu_j \circ \mu_i' = \delta_{ij},\end{aligned}$$

and similarly $\langle T^*\mu_i, a_j'\rangle=\delta_{ij}$, $\langle T^*\mu_i, a_j\rangle=0$, $\langle T^*\mu_i', a_j'\rangle=0$. Therefore if $\{a_1^*, \cdots, a_r^*, a_1'^*, \cdots, a_r'^*\}$ denote the basis dual to $\{a_1, \cdots, a_r, a_1', \cdots, a_r'\}$, we have

$$a_i^* = T^*\mu_i', \quad a_i'^* = T^*\mu_i .$$

Consequently it follows that

$$\begin{aligned}&\langle a_i^* \times a_j'^*, \Delta_*[M]\rangle \\ &= \langle T^*\mu_i' \times T^*\mu_j, \Delta_*[M]\rangle \\ &= \mu_j \circ \mu_i' = \delta_{ij},\end{aligned}$$

and similarly

$$\langle a_i'^* \times a_j^*, \Delta_*[M]\rangle = \delta_{ij} .$$

This shows that

$$\Delta_*[M] = \sum_{i=1}^{r} a_i \times a_i' + a_i' \times a_i .$$

Thus, by (3.2) we get the desired result.

4. The number $\hat{\chi}(\psi)$

Let V and W be non-singular symplectic vector spaces over Z_2, and $\psi: V \to W$ be a linear map of vector spaces. Then we define a number

$$\hat{\chi}(\psi) = \sum_{i=1}^{r} \psi(v_i) \circ \psi(v_i') \in Z_2$$

by making use of a symplectic basis $\{v_1, \cdots, v_r, v_1', \cdots, v_r'\}$ for V.

If $\{w_1, \cdots, w_t, w_1', \cdots, w_t'\}$ is a symplectic basis for W and if

$$\psi(v_j) = \sum_i a_{ij} w_i + \sum_i c_{ij} w_i' ,$$
$$\psi(v_j') = \sum_i b_{ij} w_i + \sum_i d_{ij} w_i' ,$$

then it can be easily seen that

$$\hat{\chi}(\psi) = \text{trace}\ ({}^tAD + {}^tBC)$$

for the matrices $A=(a_{ij}), \cdots$, where ${}^tA$ denotes the transposed matrix of A.

(4.1) **Lemma** *$\hat{\chi}(\psi)$ is independent of the choice of symplectic bases for V.*

Proof. Let $\{u_1, \cdots, u_r, u_1', \cdots, u_r'\}$ be another symplectic basis for V, and put

$$\psi(u_j) = \sum_i a'_{ij} w_i + \sum_i c'_{ij} w_i' ,$$
$$\psi(u_j') = \sum_i b'_{ij} w_i + \sum_i d'_{ij} w_i' .$$

We shall show

$$\text{trace}\ ({}^tA'D' + {}^tB'C') = \text{trace}\ ({}^tAD + {}^tBC) .$$

Let

$$u_j = \sum_i p_{ij} v_i + \sum_i r_{ij} v_i' ,$$
$$u_j' = \sum_i q_{ij} v_i + \sum_i s_{ij} v_i' .$$

Then the symplectic conditions imply

$$ {}^tPR + {}^tRP = 0 , \quad {}^tQS + {}^tSQ = 0 ,$$
$$ {}^tPS + {}^tRQ = E ,$$

where E is the identity matrix. This shows that

$$\begin{pmatrix} {}^tP & {}^tR \\ {}^tQ & {}^tS \end{pmatrix} \begin{pmatrix} S & R \\ Q & P \end{pmatrix} = E .$$

Therefore we have

$$(*) \qquad S{}^tR + R{}^tS = 0 , \quad Q{}^tP + P{}^tQ = 0 ,$$
$$S{}^tP + R{}^tQ = E .$$

On the other hand, since

$$\begin{pmatrix} A', & B' \\ C', & D' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} ,$$

we have

$$\begin{aligned}
&\operatorname{trace}({}^tA'D'+{}^tB'C')\\
&=\operatorname{trace}({}^t(AP+BR)(CQ+DS)+{}^t(AQ+BS)(CP+DR))\\
&=\operatorname{trace}({}^tP{}^tACQ+{}^tP{}^tADS+{}^tR{}^tBCQ+{}^tR{}^tBDS\\
&\quad+{}^tQ{}^tACP+{}^tQ{}^tADR+{}^tS{}^tBCP+{}^tS{}^tBDR)\\
&=\operatorname{trace}(Q{}^tP{}^tAC+S{}^tP{}^tAD+Q{}^tR{}^tBC+S{}^tR{}^tBD\\
&\quad+P{}^tQ{}^tAC+R{}^tQ{}^tAD+P{}^tS{}^tBC+R{}^tS{}^tBD)\,.
\end{aligned}$$

By (*) this is equal to trace $({}^tAD+{}^tBC)$, and the proof is complete.

The following is obvious.

(4.2) **Lemma.** *Let V be a non-singular symplectic vector space over Z_2. Then* dim V *is even, and for the identity map $id: V\to V$ we have*

$$\hat{\chi}(id)=\frac{1}{2}\dim V \bmod 2\,.$$

5. Main theorem

We assume that N is a closed manifold and the involution on M is *free*, and consider the element $\Delta_N\in H^m(N\underset{T}{\times}M^2)$. Since there exists an equivariant map $h:N\to S^\infty$, by (1.1) and (3.4) we have immediately

(5.1) **Lemma.** *For any symplectic basis $\{\mu_1,\cdots,\mu_r,\mu_1',\cdots,\mu_r'\}$ for $H^*(M)$, it holds*

$$\Delta_N=\sum_{i=1}^r\phi^*(1\times\mu_i\times\mu_i')\,,$$

where $\phi^*: H^*(N\times M^2)\to H^*(N\underset{T}{\times}M^2)$ is the transfer homomorphism.

Let $f:N\to M$ be a continuous map. Then $f^*:H^*(M)\to H^*(N)$ is a linear map of non-singular symplectic vector spaces over Z_2, and hence we have the number $\hat{\chi}(f^*)$ which will be denoted by $\hat{\chi}(f)$. We call $\hat{\chi}(f)$ the *equivariant Lefschetz number* of f:

$$\hat{\chi}(f)=\sum_{i=1}^r\langle f^*\mu_i\cup T^*f^*\mu_i',\,[N]\rangle\,.$$

Analogously to the Lefschetz fixed point theorem which asserts that the fixed point index coincides with the Lefschetz number, we have

(5.2) **Theorem.** *If dim $M=$ dim N, then the equivariant point index $\hat{I}(f)$ coincides with the equivariant Lefschetz number $\hat{\chi}(f)$.*

Proof. Consider an equivariant map $k:N\to N\times N^2$ given by $k(y)=(y, y, T(y))$. Since the diagram

$$\begin{array}{ccc} H^m(N \underset{T}{\times} M^2) & \xrightarrow{\hat{f}_T^*} & H^m(N_T) \\ & \searrow (1 \underset{T}{\times} f^2)^* \quad \nearrow k_T^* & \\ & H^m(N \underset{T}{\times} N^2) & \end{array}$$

is commutative, it follows from (5.1) that

$$\begin{aligned} \hat{f}_T^*(\Delta_N) &= k_T^*(1 \underset{T}{\times} f^2)^*(\Delta_N) \\ &= \sum_{i=1}^{r} k_T^*(1 \underset{T}{\times} f^2)^* \phi^*(1 \times \mu_i \times \mu_i') \\ &= \sum_{i=1}^{r} k_T^* \phi^*(1 \times f^2)^*(1 \times \mu_i \times \mu_i') \\ &= \sum_{i=1}^{r} k_T^* \phi^*(1 \times f^* \mu_i \times f^* \mu_i') \,. \end{aligned}$$

Let $d: N \to N^3$ be the diagonal map, then the diagram

$$\begin{array}{ccc} & H^*(N^3) & \\ (1 \times 1 \times T)^* \nearrow & & \searrow d^* \\ H^*(N^3) & \xrightarrow{k^*} & H^*(N) \\ \downarrow \phi^* & & \downarrow \phi^* \\ H^*(N \underset{T}{\times} N^2) & \xrightarrow{k_T^*} & H^*(N_T) \end{array}$$

is commutative. Consequently we have

$$\begin{aligned} \hat{f}_T^*(\Delta_N) &= \sum_{i=1}^{r} \phi^* d^*(1 \times 1 \times T)^*(1 \times f^* \mu_i \times f^* \mu_i') \\ &= \sum_{i=1}^{r} \phi^*(f^* \mu_i \cup T^* f^* \mu_i') \,, \end{aligned}$$

and hence

$$\langle \hat{f}_T^*(\Delta_N), [N_T] \rangle = \sum_{i=1}^{r} \langle f^* \mu_i \cup T^* f^* \mu_i', \phi_*[N_T] \rangle$$

$$= \sum_{i=1}^{r} \langle f^* \mu_i \cup T^* f^* \mu_i', [N] \rangle = \sum_{i=1}^{r} f^* \mu_i \circ f^* \mu_i' \,.$$

This completes the proof.

Now the following main theorem is a consequence of (1.3) and (5.2).

(5.3) **Main theorem.** *Let M and N be closed manifolds on each of which a free involution T is given. Let $f: N \to M$ be a continuous map such that $\hat{\chi}(f) \neq 0$. Then there exists a point $y \in N$ such that $fT(y) = Tf(y)$.*

For a closed manifold M such that the dimension of the vector space $H_*(M)$ is even, an integer mod 2 given by

$$\hat{\chi}(M) = \frac{1}{2} \dim H_*(M) \bmod 2$$

is called the *semicharacteristic* of M.

By (5.2) we have

(5.4) **Corollary.** *Let T, T' be free involutions on a closed manifold M with $\hat{\chi}(M) \not\equiv 0$. Let f: $M \to M$ be a continuous map of degree odd such that $f_* \circ T'_* = T_* \circ f_*$: $H_*(M) \to H_*(M)$. Then there exists a point $x \in M$ such that $fT'(x) = Tf(x)$. In particular, if $T_* = T_*'$: $H(M) \to H_*(M)$ then T and T' have a coincidence.*

We have also the following corollary of (5.3).

(5.5) **Corollary.** *Let M be a closed manifold with a free involution T, and assume $\hat{\chi}(M) \not\equiv 0 \bmod 2$. Then, for a continuous map $f: M \to M$ such that $f_*: H_*(M) \to H_*(M)$ is the identity, there exists a point $x \in M$ such that $fT(x) = Tf(x)$.*

REMARK. If we take in (5.5) a mod 2 homology sphere as M, we get Theorem 1 in Milnor [5].

6. Applications

(6.1) **Theorem.** *Let M be a closed manifold such that $\dim H^*(M) \equiv 2 \bmod 4$, and G be a group acting freely on M. Then*

i) *G can have at most one element T of order 2 such that $T_*: H_*(M) \to H_*(M)$ is a given isomosphism.*

ii) *If $T \in G$ is an element of order 2 such that $T_*: H_*(M) \to H_*(M)$ is the identity, T lies in the center of G.*

iii) *If $T \in G$ is an element of order 2, T lies in the centralizer of $G_0 = \{S \in G; S_* = id: H_*(M) \to H_*(M)\}$.*

Proof. Let T, T', $S \in G$, and let T, T' have order 2. It follows from (5.4) that if $T_* = T'_*$ then $T(x_1) = T'(x_1)$ for some $x_1 \in M$, and that if $T_* = T'_* = id$ then $ST(x_2) = TS(x_2)$ for some $x_2 \in M$. It follows from (5.5) that if $S \in G_0$ then $ST(x_3) = TS(x_3)$ for some $x_3 \in M$. Since G acts freely on M, we have the desired results.

Let $D(2l)$ denote the dihedral group with presentation $(X, Y; X^2 = (XY)^2 = Y^l = 1)$.

(6.2) **Theorem.** *Let M be a closed manifold on which $D(2l)$ acts freely. Assume that $\hat{\chi}(M) \not\equiv 0$ and l is an odd > 1. Then the representation of $D(2l)$ on $H_*(M)$ given by sending $S \in D(2l)$ to $S_*: H_*(M) \to H_*(M)$ is faithful.*

Proof. Any element of $D(2l)$ has a form $X^\varepsilon Y^i (\varepsilon = 0, 1, 0 \leqq i < l)$. We shall show that $X_* \neq \mathrm{id}$ and $(X^\varepsilon Y^i)_* \neq \mathrm{id}$ ($\varepsilon=0, 1, 1 \leqq i < l$).

i) Assume $X_* = \mathrm{id}$. Then we have $XY=YX$ by ii) of (6.1). Since $X=YXY$, this implies $Y^2=1$. Since the order of Y is l, this is a contradiction. Thus $X_* \neq \mathrm{id}$.

ii) Assume $(X^\varepsilon Y^i)_* = \mathrm{id}$ with $\varepsilon=0, 1, 1 \leqq i < l$. Then we have $X^{\varepsilon+1} Y^i = X^\varepsilon Y^i X$, *i.e.* $XY^i = Y^i X$ by iii) of (6.1). This implies $Y^{2i}=1$ which shows $i=0$. Thus $(X^\varepsilon Y^i)_* \neq \mathrm{id}$ for $\varepsilon=0, 1$ and $1 \leqq i < l$.

Consider the group $Q(8n, k, l)$ stated in Introduction.

(6.3) **Theorem.** *If n is even and $l>1$, the group $Q(8n, k, l)$ can not act freely on any mod 2 homology sphere whose dimension is 3 mod 8.*

Proof. Put $\bar{A}=A^k$, then we have

$$X^2 = (XY)^2 = Y^{2n}, \ \bar{A}^l = 1 ,$$
$$X\bar{A}X^{-1} = \bar{A} , \quad Y\bar{A}Y^{-1} = \bar{A}^{-1} .$$

Therefore the subgroup in $Q(8n, k, l)$ generated by $\{X, Y, \bar{A}\}$ is isomorphic to $Q(8n, 1, l)$. Thus it suffices to prove (6.3) in the special case when $k=r=1$.

Put $\bar{Y}=Y^2$, then we have in $Q(8n, 1, l)$

$$X^2 = (X\bar{Y})^2 = \bar{Y}^n ,$$
$$YXY^{-1} = \bar{Y}X , \qquad Y\bar{Y}Y^{-1} = \bar{Y} ,$$
$$AXA^{-1} = X , \qquad A\bar{Y}A^{-1} = \bar{Y} .$$

Therefore the subgroup in $Q(8n, 1, l)$ generated by $\{X, \bar{Y}\}$ is a normal subgroup isomorphic to the binary dihedral group $Q(4n)$. The quotient group $Q(8n, 1, l)/Q(4n)$ is generated by the classes $T=[Y]$ and $S=[A]$ with relations $T^2=(TS)^2=S^l=1$, and so is isomorphic to $D(2l)$.

Suppose now that we have a free action of $Q(8n, 1, l)$ on a mod 2 homology sphere L of dimension $8t+3$. Let $M=L/Q(4n)$ be the quotient manifold of L under the action of the normal subgroup $Q(4n)$. Then there is a natural free action of $D(2l)$ on M.

Since $\tilde{H}_i(L)=0$ for $i<8t+3$, it follows that

$$H_i(M) \cong H_i(Q(4n)) \qquad (i<8t+3) .$$

Since n is even, we have

$$H_i(Q(4n)) = \begin{cases} Z_2 & i \equiv 0 \bmod 4 , \\ Z_2 \oplus Z_2 & i \equiv 1 \bmod 4 , \\ Z_2 \oplus Z_2 & i \equiv 2 \bmod 4 , \\ Z_2 & i \equiv 3 \bmod 4 \end{cases}$$

(see [3], p. 254). Therefore it holds

$$\hat{\chi}(M)=\sum_{i=0}^{4t+1}\dim H_i(M)\not\equiv 0 \bmod 2\,.$$

Under the isomorphism of $H_i(M)$ to $H_i(Q(4n))$ $(i<8t+3)$, the induced homomorphism $S_*:H_i(M)\to H_i(M)$ corresponds to the homomorphism $\sigma_*:H_i(Q(4n))\to H_i(Q(4n))$ induced by the homomorphism $\sigma:Q(4n)\to Q(4n)$ sending each element U to AUA^{-1}. Since $AXA^{-1}=X$, $A\bar{Y}A^{-1}=\bar{Y}$, we see that S_* is the identity for $i<8t+3$. This is obvious for $i\geqq 8t+3$. Since T is of order 2, it follows from (6.1) that $ST=TS$. Since l is odd >1, this is a contradiction, and the proof completes.

Let $P''(48r)$ denote the group with generators X, Y, Z, A and relations

$$\begin{aligned}&X^2=Y^2=Z^2=(XY)^2\,,\quad A^{3r}=1\,,\\&ZXZ^{-1}=YX\,,\quad ZYZ^{-1}=Y^{-1}\,,\quad AXA^{-1}=Y\,,\\&AYA^{-1}=XY\,,\quad ZAZ^{-1}=A^{-1}\,,\end{aligned}$$

where r is an odd positive integer. Milnor proves in [5] that if r is not a power of 3 then $P''(48r)$ can not act freely on any homotopy 3-sphere. More generally we have

(6.4) **Theorem.** *If r is not a power of 3, the group $P''(48r)$ can not act freely on any mod 2 homology sphere whose dimension is 3 mod 8.*

Proof. Let $r=3^{k-1}l$ with $(l,6)=1$, $l\geqq 5$. Then it follows that the subgroup in $P''(48r)$ generated by $\{X, Y, A^l\}$ is a normal subgroup isomorphic to $P'(8\cdot 3^k)$ and its quotient group is isomorphic to $D(2l)$, where $P'(8\cdot 3^k)$ denotes the group with presentation $(X, Y, A; X^2=Y^2=(XY)^2, A^{3^k}=1, AXA^{-1}=Y, AYA^{-1}=XY)$.

Suppose now that we have a free action of $P''(48r)$ on a mod 2 homology sphere L of dimension $8t+3$. If we put $M=L/P'(8\cdot 3^k)$, there is a natural free action of $D(2l)$ on M. We have $H_i(M)\cong H_i(P'(8\cdot 3^k))$ for $i<8t+3$. The subgroup in $P'(8\cdot 3^k)$ generated by $\{X, Y\}$ is isomorphic to the quaternion group $Q(8)$, and its quotient group is isomorphic to Z_{3^k}. Therefore it is easily seen that

$$H_i(P'(8\cdot 3^k))=\begin{cases}Z_2 & i\equiv 0 \bmod 4\,,\\ 0 & i\equiv 1 \bmod 4\,,\\ 0 & i\equiv 2 \bmod 4\,,\\ Z_2 & i\equiv 3 \bmod 4\,.\end{cases}$$

Thus $\hat{\chi}(M)\not\equiv 0$ and the action of $D(2l)$ on $H_*(M)$ is trivial. By (6.1) this is a contradiction, and the proof completes.

(Added Nov. 27, 1973). R.E. Stong [10] proves the following theorem. As an application of Theorem (5.2) we shall prove this theorem.

(6.5) **Theorem.** *If a closed mainfold N admits a free action of* $Z_2\times Z_2$, *then* $\hat{\chi}(N)=0$.

Proof. Taking generators T and S of $Z_2\times Z_2$, regard N as a manifold with involution by T, and S a continuous map of N to itself. Then it follows from (5.2) that $\hat{I}(S)=\hat{\chi}(S)$.

Define Δ, $\Delta':N\rightarrow N\times N$ by $\Delta(y)=(y, Ty)$, $\Delta'(y)=(y, Sy)$. Then the map $\hat{S}_T:N_T\rightarrow N\underset{T}{\times}N^2$ is the composition of $\Delta_T':N_T\rightarrow N\underset{T}{\times}N$ and $1\underset{T}{\times}\Delta:N\underset{T}{\times}N\rightarrow N\underset{T}{\times}N^2$. Therefore it holds that

$$\begin{aligned}\hat{I}(S) &= \langle\hat{S}_T^*(\Delta_N)\,,\,[N_T]\rangle \\ &= \langle\Delta_T'^*(1\underset{T}{\times}\Delta)^*(\Delta_N),\,[N_T]\rangle\,.\end{aligned}$$

Let ν_N denote the normal bundle of the imbedding $1\underset{T}{\times}\Delta:N\underset{T}{\times}N\rightarrow N\underset{T}{\times}N^2$. Then it is obvious that $(1\underset{T}{\times}\Delta)^*(\Delta_N)$ is the n-th Stiefel-Whitney class $w_n(\nu_N)$, where $n=\dim N=\dim \nu_N$. The involution T on N gives rise to a free involution T on the orbit manifold N_S. If ν_N' denotes the normal bundle of the imbedding $1\underset{T}{\times}\Delta:N_S\underset{T}{\times}N_S\rightarrow N_S\underset{T}{\times}N_S^2$, we have $\nu_N=(p\underset{T}{\times}p)^*\nu_N'$, where $p:N\rightarrow N_S$ is the projection. Therefore it follows that

$$\begin{aligned}&\Delta_T'^*(1\underset{T}{\times}\Delta)^*(\Delta_N)=\Delta_T'^*w_n(\nu_N) \\ &=\Delta_T'^*(p\underset{T}{\times}p)^*w_n(\nu_N')=p_T^*d_T^*w_n(\nu_N')\,,\end{aligned}$$

where $d:N_S\rightarrow N_S\times N_S$ is the diagonal map. Hence

$$\hat{I}(S)=\langle d_T^*w_n(\nu_N'),\,p_{T*}[N_T]\rangle=0\,.$$

On the other hand, we have

$$\begin{aligned}\hat{\chi}(S) &= \sum_{i=1}^{r}\langle S^*\mu_i\cup T^*S^*\mu_i',\,[N]\rangle \\ &= \sum_{i=1}^{r}\langle S^*(\mu_i\cup T^*\mu_i'),\,[N]\rangle \\ &= \sum_{i=1}^{r}\langle \mu_i\cup T^*\mu_i',\,[N]\rangle \\ &= \hat{\chi}(N)\,,\end{aligned}$$

where $\{\mu_1, \cdots, \mu_r, \mu_i', \cdots, \mu_r'\}$ is a symplectic basis for $H^*(N)$. Thus $\hat{\chi}(N)=0$.

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