Vick, J.W.
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SOME APPLICATIONS OF THE ROTHENBERG-STEENROD SPECTRAL SEQUENCE

JAMES W. VICK

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1. Introduction. In the study of principal actions of a group G, a fundamental role is played by the classifying space B_G. Thus it is natural to seek algebraic invariants which describe the geometrical properties of these spaces. For the purpose of studying their homology and cohomology, Rothenberg and Steenrod [15] introduced a variation of the Eilenberg-Moore spectral sequence and gave several applications. Hodgkin [11] and Anderson and Hodgkin [2] recast the cohomological form of this spectral sequence into K^*-theory and used it to study the K^*-theory of Lie groups and Eilenberg-MacLane spaces.

It is our purpose here to extend the homological form of the spectral sequence to arbitrary multiplicative generalized homology theories and give some brief applications. Since the constructions require a Künneth isomorphism, we must introduce cyclic groups of coefficients and investigate the existence of associated multiplicative structures. This is done in §2 and follows the corresponding constructions of Araki and Toda [3] for cohomology. In §3 the spectral sequence is described and the E^2-stage and edge homomorphism are identified.

The applications are given in §4. These include the computation of the K_*-groups of certain Eilenberg-MacLane spaces, using results of Anderson and Hodgkin [2]. The implications of these computations in complex bordism are noted briefly. Finally we give the following generalization of a theorem of Borel [5]: If h_* is a multiplicative homology theory, p is a prime, $h_*(pt.; Z_p)=R$ is zero in odd dimensions and G is a group having $h_*(G; Z_p)$ an exterior algebra over R on a finite number of odd dimensional generators, then $h_*(B_G; Z_p)$ is a modified polynomial algebra over R on corresponding generators of one dimension higher.

We assume throughout that spaces are in the category A of spaces having the homotopy type of a CW complex with finite skeleta and that all homology theories are additive. It is a pleasure to acknowledge recent conversations with Gary Hamrick on this and related subjects.

2. Multiplicative homology theories. Let h_* be a generalized homology theory [18], [9]. A *multiplicative structure* in h_* is an associative bilinear pairing

$$\mu:\quad h_*(X, A)\otimes h_*(Y, B) \to h_*((X, Z)\times(Y, B))$$

for all pairs of spaces (X, A) and (Y, B) subject to the following requirements:

(i) μ is natural with respect to maps of pairs;

(ii) there is a two-sided unit $1\in h_0(pt.)$;

(iii) $\mu(\sigma_*(x)\otimes y)=\sigma_*(\mu(x\otimes y))$ and
$T_*\mu(x\otimes\sigma_*(y))=(-1)^n\sigma_*(\mu(x\otimes y))$

where $x\in h_n(X, A)$, $y\in h_m(Y, B)$, σ_* is the suspension isomorphism and T: $(X, A)\times(I, \partial I)\times(Y, B)\to(I, \partial I)\times(X, A)\times(Y, B)$ reverses the first two coordinates. For representable homology theories such products arise naturally from pairings of the associated spectra [18].

Let q be a positive integer and denote by T_q an homology Moore space of type $(Z_q, 1)$. Then define

$$h_n(X, A; Z_q) = \tilde{h}_{n+1}((X/A)\wedge T_q)$$

where it is understood that if $A=\phi$, then $X/A=X^+=X\cup pt.$ As is well known, this defines an homology theory, the h_*-homology with Z_q coefficients. Following techniques of Anderson [1] and Araki and Toda [3] we want to show that under certain conditions there is an associated multiplicative structure for $h_*(\ \ ; Z_q)$.

From the cofibration sequence

$$S^1 \xrightarrow{\times q} S^1 \xrightarrow{r} T_q \xrightarrow{b} S^2 \longrightarrow \cdots$$

there follows the corresponding sequence for any pair (X, A),

$$(X/A)\wedge S^1 \xrightarrow{id\wedge\times q} (X/A)\wedge S^1 \xrightarrow{id\wedge r} (X/A)\wedge T_q \xrightarrow{id\wedge b} \cdots$$

Its exact homology sequence may be abrieviated to give the *universal coefficient sequence*

$$\text{(2.1)}\quad 0\longrightarrow h_n(X, A)\otimes Z_q\longrightarrow h_n(X,A; Z_q)\longrightarrow Tor\,(h_{n-1}(X,A), Z_q)\longrightarrow 0.$$

The homomorphism

$$\rho_q = (id\wedge r)_*: h_n(X, A) \longrightarrow h_n(X, A; Z_q) \text{ is reduction mod } q$$

and

$$\beta_q = (id\wedge b)_*: h_n(X,A; Z_q) \longrightarrow h_{n-1}(X,A) \text{ is the mod } q \text{ integral Bockstein.}$$

(2.2) *If h_* is a multiplicative homology theory and $q\not\equiv 2$ (mod 4) then there is a multiplicative structure μ_q in $h_*(\ \ ; Z_q)$ with the property that $\mu_q(\rho_q(x)\otimes\rho_q(y)) =\rho_q(\mu(x\otimes y))$. Furthermore if 1 is a unit for μ, $\rho_q(1)$ is a unit for μ_q.*

Proof. Let i: $pt.\to S^1$ be the inclusion of the base point and consider the two cofibration sequences

$$\begin{array}{ccccccccc}
S^1\wedge T_q & \xrightarrow{\times q\wedge id} & S^1\wedge T_q & \xrightarrow{r\wedge id} & T_q\wedge T_q & \xrightarrow{b\wedge id} & S^2\wedge T_q & \xrightarrow{\Sigma(\times q\wedge id)} & S^2\wedge T_q \\
\vdots\, g & & \downarrow\, id & & \vdots\, f & & \vdots\, \Sigma g & & \downarrow\, id \\
pt.\wedge T_q & \xrightarrow{i\wedge id} & S^1\wedge T_q & \xrightarrow{id} & S^1\wedge T_q & \longrightarrow & pt.\wedge T_q & \longrightarrow & S^2\wedge T_q
\end{array}$$

A result of Barratt [4] states that for $q\not\equiv 2 \pmod 4$ the map $\times q\wedge id$: $S^1\wedge T_q\to S^1\wedge T_q$ is null homotopic. Thus by taking g to be the constant map, the first rectangle commutes up to homotopy. According to Puppe [14] this implies the existence of a map f so that the second rectangle is homotopy commutative, i.e. $f\circ(r\wedge id)$ is homotopic to the identity on $S^1\wedge T_q$. Note that since the homotopy class of f is not necessarily uniquely determined, the same may hold true for the resulting multiplication.

Now define μ_q to be the composition.

$$\begin{array}{ccc}
h_r(X;Z_q)\otimes h_s(Y;Z_q) & = & \tilde{h}_{r+1}(X^+\wedge T_q)\otimes\tilde{h}_{s+1}(Y^+\wedge T_q) \\
\vdots & & \downarrow\, \mu \\
\vdots & & \tilde{h}_{r+s+2}((X\times Y)^+\wedge T_q\wedge T_q) \\
\vdots\, \mu_q & & \downarrow\, (id\wedge f)_* \\
\vdots & & \tilde{h}_{r+s+2}((X\times Y)^+\wedge S^1\wedge T_q) \\
\vdots & & \downarrow\, {\sigma_*}^{-1} \\
h_{r+s}(X\times Y;Z_q) & = & \tilde{h}_{r+s+1}((X\times Y)^+\wedge T_q)
\end{array}$$

The basic properties of the multiplication follow.

For the case $q\equiv 2 \pmod 4$ there are considerable difficulties in defining a mod q multiplication with the desired properties. Following techniques of Anderson [1], we now give some partial results for this case. A multiplicative homology theory h_* is said to be *pre-associative* if the ring $\Lambda=h_*(pt.)$ is associative and for all X, $h_*(X)$ is both a left and right Λ-module. We immediately conclude the following:

(2.3) *If h_* is a pre-associative multiplicative homology theory, then for each n there is an isomorphism*

$$\mu:\ h_*(S^n)\otimes_\Lambda h_*(X)\xrightarrow{\approx} h_*(S^n\times X)\,.$$

From now on we assume that our multiplicative theory h_* is pre-associative.

(2.4) *Let*

$$S^k \xrightarrow{\eta} P \xrightarrow{\phi} Q \xrightarrow{j} S^{k+1}$$

be a cofibration sequence in which $\sigma_*{}^k(1) \in \tilde{h}_k(S^k)$ lies in the kernel of η_*. Then there is a natural short exact sequence for any X

$$0 \longrightarrow h_n(X) \xrightarrow{\bar{j}} h_{n+k+1}(Q \wedge X) \xrightarrow{\bar{\phi}} h_{n+k+1}(P \wedge X) \longrightarrow 0$$

such that $\sigma_*{}^{-(k+1)} \circ (j \wedge 1)_* \circ \bar{j}$ is the identity on $h_n(X)$ and $\bar{\phi} \circ \phi_*$ is the identity on $h_{n+k+1}(P \wedge X)$.

Let $\beta \in h_{k+1}(Q)$ with $j_*(\beta) = \sigma_*{}^{k+1}(1)$, and define $\bar{j}(x) = \mu(\beta \otimes x)$. Then

$$\begin{aligned} \sigma_*{}^{-(k+1)}(j \wedge 1)_* \bar{j}(x) &= \sigma_*{}^{-(k+1)}(j \wedge 1)_*(\mu(\beta \otimes x)) \\ &= \sigma_*{}^{-(k+1)}(\mu(\sigma_*{}^{k+1}(1) \otimes x)) \\ &= x \,. \end{aligned}$$

To see that this is natural, let $g: X \to Y$ and $x \in h_n(X)$. Then

$$\begin{aligned} (1 \wedge g)_* \bar{j}(x) &= (1 \wedge g)_*(\mu(\beta \otimes x)) = \mu(\beta \otimes g_*(x)) \\ &= \bar{j}(g_*(x)) \,. \end{aligned}$$

Now suppose that $P = S^n$ and

$$S^k \xrightarrow{\eta} S^n \xrightarrow{\phi} Q_n \xrightarrow{j} S^{k+1}$$

is the corresponding cofibration sequence. Suppose further that $\phi \wedge 1: S^n \wedge T_q \to Q_\eta \wedge T_q$ stably has order q. Then in the exact sequence

$$\begin{aligned} [S^n \wedge T_q \wedge T_q, Q_\eta \wedge S^1 \wedge T_q] &\xrightarrow{(1 \wedge r \wedge 1)^*} [S^n \wedge S^1 \wedge T_q, Q_\eta \wedge S^1 \wedge T_q] \\ &\xrightarrow{(\times q)^*} [S^n \wedge S^1 \wedge T_q, Q_\eta \wedge S^1 \wedge T_q] \end{aligned}$$

we have $(\times q)^*(\phi \wedge 1) = 0$, so there exists stably a map $f: S^n \wedge T_q \wedge T_q \to Q_\eta \wedge S^1 \wedge T_q$ such that the following diagram is homotopy commutative

$$\begin{array}{ccc} S^n \wedge T_q \wedge T_q & \xrightarrow{f} & Q_\eta \wedge S^1 \wedge T_q \\ & {}_{1 \wedge r \wedge 1} \nwarrow \quad \nearrow {}_{\phi \wedge 1 \wedge 1} & \\ & S^n \wedge S^1 \wedge T_q \,. & \end{array}$$

We may now define a mod q multiplication in h_* corresponding to the map η. The product μ_q is given by the following composition:

$$\begin{array}{ccc}
\tilde{h}_{r+1}(X^+\wedge T_q)\otimes\tilde{h}_{s+1}(Y^+\wedge T_q) & \xrightarrow{\mu} & \tilde{h}_{r+s+2}((X\times Y)^+\wedge T_q\wedge T_q) \\
\vdots & & \downarrow \sigma_*{}^n \\
\vdots & & \tilde{h}_{r+s+n+2}((X\times Y)^+\wedge S^n\wedge T_q\wedge T_q) \\
\vdots\ \mu_q & & \downarrow (1\wedge f\wedge 1)_* \\
\vdots & & \tilde{h}_{r+s+n+2}((X\times Y)^+\wedge Q_\eta\wedge S^1\wedge T_q) \\
\vdots & & \downarrow \overline{\phi} \\
\tilde{h}_{r+s+1}((X\times Y)^+\wedge T_q) & \xleftarrow{\sigma_*{}^{-(n+1)}} & \tilde{h}_{r+s+n+2}((X\times Y)^+\wedge S^n\wedge S^1\wedge T_q)
\end{array}$$

Note that this is analogous to the previous construction in that for $q\not\equiv 2\,(\text{mod } 4)$, every element of $[S^n\wedge T_q, Q\wedge T_q]$ has order q and we may take η to be null homotopic. Then ϕ becomes the identity and we have the situation in (2.2)

To check that this has the desired properties of a mod q multiplication, we will show that $\mu_q(\rho(x)\otimes\rho(y))=\rho(\mu(x\otimes y))$.

$$\begin{aligned}
\mu_q(\rho(x)\otimes\rho(y)) &= \mu_q((1\wedge r)_*(x)\otimes(1\wedge r)_*(y)) \\
&= \sigma_*{}^{-(n+1)}\overline{\phi}(1\wedge f\wedge 1)_*\sigma_*{}^n\mu((1\wedge r)_*(x)\otimes(1\wedge r)_*(y)) \\
&= \sigma_*{}^{-(n+1)}\overline{\phi}(1\wedge f\wedge 1)_*\sigma_*{}^n(1\wedge r\wedge 1)_*(1\wedge 1\wedge r)_*\mu(x\otimes y) \\
&= \sigma_*{}^{-(n+1)}\overline{\phi}(1\wedge f\wedge 1)_*(1\wedge(1\wedge r)\wedge 1)_*\sigma_*{}^n(1\wedge 1\wedge r)_*\mu(x\otimes y) \\
&= \sigma_*{}^{-(n+1)}\overline{\phi}(1\wedge\phi\wedge 1\wedge 1)_*\sigma^n(1\wedge 1\wedge r)_*\mu(x\otimes y) \\
&= \sigma_*{}^{-(n+1)}(1\wedge 1\wedge r)_*\mu(x\otimes y) = \rho(\mu(x\otimes y))\,.
\end{aligned}$$

The fact that $\rho(1)$ is a unit for μ_q follows similarly. Using the notation above, we have established the result:

(2.5) *If* $q\equiv 2$ (mod 4) *and* $\eta\colon S^k\to S^n$ *is a map such that* $\eta_*(\sigma_*{}^k(1))=0$ *and* $\phi\wedge 1\colon S^n\wedge T_q\to Q_\eta\wedge T_q$ *stably has order q, then there exists a mod q multiplication for* h_* *as in* (2.2).

As a specific case, let $\eta\colon S^3\to S^2$ be the Hopf map. We will compute the order of $\phi\wedge 1\in[S^2\wedge T_q, Q_\eta\wedge T_q]$ where $q\equiv 2$ (mod 4). From the cofibration sequence

$$S^3\wedge T_q\longrightarrow S^2\wedge T_q\longrightarrow Q_\eta\wedge T_q\longrightarrow S^4\wedge T_q$$

we obtain the exact sequence

$$\begin{aligned}
&[S^2\wedge T_q, S^3\wedge T_q]\xrightarrow{(\eta\wedge 1)_*}[S^2\wedge T_q, S^2\wedge T_q]\xrightarrow{(\phi\wedge 1)_*}[S^2\wedge T_q, Q_\eta\wedge T_q] \\
&\qquad\xrightarrow{(j\wedge 1)_*}[S^2\wedge T_q, S^4\wedge T_q]\,.
\end{aligned}$$

According to [3, Theorem 4.1], we have

(i) $[S^2\wedge T_q, S^3\wedge T_q]\approx Z_q$ generated by $(1\wedge r)\circ b$,

(ii) $[S^2\wedge T_q, S^2\wedge T_q]\approx Z_{2q}$ generated by 1,

(iii) $[S^2\wedge T_q, S^4\wedge T_q]=0$.

By using [3, Theorem 1.1] we get the relation

$$(\eta\wedge 1)_*((1\wedge r)\circ b) = q\cdot 1\in[S^2\wedge T_q, S^2\wedge T_q].$$

Hence we can easily see that

$$\phi\wedge 1\in[S^2\wedge T_q, Q_\eta\wedge T_q]_* \text{ has order } q.$$

Combining this result with (2.5) gives a more specific result.

(2.6) *If the homology theory h_* in (2.2) has $\eta_*(\sigma_*^3(1))=0$ where $\eta: S^3\to S^2$ is the Hopf map, then the hypothesis that $q\not\equiv 2 \pmod 4$ may be dropped.*

The following weak form of the Künneth formula may be established by the technique in [6, §44]:

(2.7) *If X and Y have the homotopy type of CW complexes of finite type and $h_*(X; Z_q)$ is a free $R=h_*(pt.; Z_q)$-module, then the mod q multiplication defines an isomorphism*

$$\mu_q\colon h_*(X; Z_q)\otimes_R h_*(Y; Z_q)\longrightarrow h_*(X\times Y; Z_q).$$

3. The spectral sequence. The results of this section are due to Rothenberg and Steenrod [15] [16]. Since the details of their proofs have at present only appeared in the form of mimeographed notes [16], some of their arguments are reproduced here.

Let G be a group in A with identity e. A *right action* of G on a space X is a continuous function $\phi: X\times G\to X$ such that $\phi(x, e)=x$ and $\phi(\phi(x, g_1), g_2)=\phi(x, g_1g_2)$ for all $g_1, g_2\in G$, $x\in X$. A space X together with such an action is a G-space. X is a *filtered* G-space if there is a sequence $X_0\subseteq X_1\subseteq\cdots\subseteq X_n\subseteq\cdots$ of closed G-invariant subspaces such that $X_0\neq\phi, X=\bigcup_{i=0}^{\infty}X_i$ and X has the topology of the union. A filtered G-space X is *acyclic* if X_n is contractible to a point in X_{n+1}; it is *free* if for each n there exists a closed subspace D_n, $X_{n-1}\subseteq D_n\subseteq X_n$, such that the restriction of the action mapping gives a relative homeomorphism

$$\phi_n\colon (D_n, X_{n-1})\times G\longrightarrow (X_n, X_{n-1}).$$

A filtered G-space which is both free and acyclic is a *G-resolution.*

Let $\mathcal{E}$ be the Milnor resolution for G [12]. Then $\mathcal{E}$ is a universal G-space and the decomposition space $B=\mathcal{E}/G$ is a *classifying space* for G. If $p: \mathcal{E}\to B$ is the identification map, then B is filtered by setting $B_n=p(\mathcal{E}_n)$. Then B_{n-1} is a deformation retract of a neighborhood in B_n and p restricts to a relative homeomorphism p: $(D_n, \mathcal{E}_{n-1})\to(B_n, B_{n-1})$ for $n\geq 0$.

Let h_* be a multiplicative homology theory as in (2.5) and q an integer for which $h_*(G, Z_q)$ is a free R-module where $R=h_*(pt.; Z_q)$. Then the Künneth isomorphism of (2.7) gives $h_*(G; Z_q)$ the structure of an R-algebra and $h_*(X; Z_q)$ the structure of a right $h_*(G; Z_q)$-module.

The space $X_G=X\times_G\mathcal{E}$ may be filtered by setting $(X_G)_n=X\times_G\mathcal{E}_n$. Note that if X is a point, $X_G=B$ is the corresponding classifying space. This filtration of X_G produces a natural filtration of $h_*(X_G; Z_q)$ which yields a convergent spectral sequence $E^*(X_G)$ in the usual way. We now want to establish some of the basic properties of this spectral sequence as in [16]. These results dualize those of Anderson and Hodgkin [2] [11].

Since $\mathcal{E}_{n-1}$ is contractible to a point in $\mathcal{E}_n$ we have a long exact sequence

$$(3.1)\quad \cdots\xrightarrow{d} h_*(\mathcal{E}_{n+1}, \mathcal{E}_n: Z_q)\xrightarrow{d} h_*(\mathcal{E}_n, \mathcal{E}_{n-1}; Z_q)\xrightarrow{d}\cdots$$
$$\cdots\xrightarrow{d} h_*(\mathcal{E}_1, \mathcal{E}_0; Z_q)\xrightarrow{d} h_*(\mathcal{E}_0; Z_q)\xrightarrow{\mathcal{E}} R\longrightarrow 0$$

where each d is the boundary operator of the respective triple and $\mathcal{E}$ is induced by the map onto a point.

(3.2) *If $h_*(G; Z_q)$ is a free finitely generated R-module then the sequence given in* (3.1) *is a free $h_*(G; Z_q)$ resolution of R.*

To prove this consider the pair $(D_n, \mathcal{E}_{n-1})$ where $D_n=\{e\}\circ\mathcal{E}_{n-1}\approx C\mathcal{E}_{n-1}$ and $e\in G$ is the identity. The map

$$\phi_n:\ (D_n, \mathcal{E}_{n-1})\times G\longrightarrow(\mathcal{E}_n, \mathcal{E}_{n-1})$$

induced by the action of G is a relative homeomorphism. Using the right action of G on the second factor of $(D_n, \mathcal{E}_{n-1})\times G$, we see that

$$\phi_{n*}:\ h_*((D_n, \mathcal{E}_{n-1})\times G; Z_q)\longrightarrow h_*(\mathcal{E}_n, \mathcal{E}_{n-1}; Z_q)$$

is an isomorphism of $h_*(G; Z_q)$-modules. The Künneth formula (2.7) gives an isomorphism

$$\mu_q:\ h_*(D_n, \mathcal{E}_{n-1}; Z_q)\otimes_R h_*(G; Z_q)\xrightarrow{\approx} h_*((D_n, \mathcal{E}_{n-1})\times G; Z_q)$$

of $h_*(G; Z_q)$-modules, where the action on the left side is given by acting on the second factor.

Thus to show that $h_*(\mathcal{E}_n, \mathcal{E}_{n-1}; Z_q)$ is a free $h_*(G; Z_q)$-module it is sufficient to show that $h_*(D_n, \mathcal{E}_{n-1}; Z_q)\approx h_*(C\mathcal{E}_{n-1}, \mathcal{E}_{n-1}; Z_q)\approx\tilde{h}_*(\sum\mathcal{E}_{n-1}; Z_q)$ is a free R-module. Since $\mathcal{E}_{n-1}$ is a join of copies G and $h_*(G; Z_q)$ is a finitely generated free R-module, it is sufficient to show that if $\tilde{h}_*(X;Z_q)$ and $\tilde{h}_*(Y;Z_q)$ are finitely generated free R-modules, then so is $\tilde{h}_*(X\circ Y;Z_q)$.

Here the argument of Milnor [12, lemma 2.1], together with the mod q Künneth formula (2.7), establishes an isomorphism

$$\tilde{h}_*(X\circ Y;Z_q)\approx\tilde{h}_*(X;Z_q)\otimes_R\tilde{h}_*(Y;Z_q)$$

of degree -1. We conclude that $h_*(\mathcal{E}_n,\mathcal{E}_{n-1};Z_q)$ is a free $h_*(G;Z_q)$-module.

Finally since the boundary operator on a cross product $\partial(x\times y)=\partial(x)\times y$ for x a relative class and y an absolute class, the homomorphisms d of (3.1) are $h_*(G;Z_q)$-module homomorphisms. This completes the proof of (3.2).

It is now possible to determine the E^2-stage of the spectral sequence for X_G. Consider the diagram

$$\begin{array}{lcl}
h_*(X;Z_q)\otimes_R h_*(D_n,\mathcal{E}_{n-1};Z_q) & & \\
\quad\Big\downarrow h & & \\
h_*(X;Z_q)\otimes_R h_*(D_n,\mathcal{E}_{n-1};Z_q)\otimes_R h_*(G;Z_q) & & \\
\approx\Big\downarrow id\otimes(\phi_n{}^*\circ\mu_q) & & \\
h_*(X;Z_q)\otimes_R h_*(\mathcal{E}_n,\mathcal{E}_{n-1};Z_q) & \searrow\eta & \\
\approx\Big\downarrow \mu_q & & h_*(X;Z_q)\otimes_{h_*(G;Z_q)} h_*(\mathcal{E}_n,\mathcal{E}_{n-1};Z_q) \\
h_*(X\times\mathcal{E}_n,X\times\mathcal{E}_{n-1};Z_q) & & \\
\quad\Big\downarrow p_* & & \\
h_*(X\times_G\mathcal{E}_n,X\times_G\mathcal{E}_{n-1};Z_q) & &
\end{array}$$

where $h(x\otimes y)=x\otimes y\otimes 1$ and η is the natural map onto the quotient. It is a standard fact that since $id\otimes(\phi_n{}^*\circ\mu_q)$ is an isomorphism, the composition $\eta\circ(id\otimes(\phi_n{}^*\circ\mu_q))\circ h$ is also an isomorphism. On the other hand, a diagram chasing argument shows that $p_*\circ\mu_q\circ(id\otimes(\phi_{n*}\circ\mu_q))\circ h$ is just the composition

$$\begin{array}{ccc}
h_*(X;Z_q)\otimes_R h_*(D_n,\mathcal{E}_{n-1};Z_q) & \xrightarrow[\approx]{\mu_q} & h_*(X\times(D_n,\mathcal{E}_{n-1};Z_q) \\
 & & p_*\Big\downarrow\approx \\
 & & h_*(X\times_G\mathcal{E}_n,X\times_G\mathcal{E}_{n-1});Z_q)
\end{array}$$

where p_* is an isomorphism since the restriction of p is a relative homeomorphism.

This allows us to conclude:

(3.3) *The homomorphism* $\eta\circ\mu_q^{-1}\circ p_*^{-1}$ *defines a natural isomorphism from*

$$h_*(X\times_G\mathcal{E}_n,\ X\times_G\mathcal{E}_{n-1};\ Z_q)\ \textit{to}\ h_*(X;\ Z_q)\otimes_{h_*(G;\,Z_q)}h_*(\mathcal{E}_n,\ \mathcal{E}_{n-1};\ Z_q)\,.$$

As an immediate consequence we have:

(3.4) *Denoting the R-algebra* $h_*(G;\ Z_q)$ *by* Λ *there is a natural isomorphism*

$$E^2_{*,*}(X_G)\approx \operatorname{Tor}^{\Lambda}_{**}(h_*(X;\ Z_q),\ R)\,.$$

We make the following observation concerning the edge homomorphism in $E^*(X_G)$:

(3.5) *The edge homomorphism given by the composition*

$$\begin{array}{ccccc} E^1_{0,*}(X_G) & \longrightarrow & E^2_{0,*}(X_G) & \longrightarrow & E^\infty_{0,*}(X_G) \\ \wr\wr & & \wr\wr & & \wr\wr \\ h_*(X\times_G G;Z_q) & & h_*(X;Z_q)\otimes_\Lambda R & & J_{0,*}\longrightarrow h_*(X_G;Z_q) \\ \wr\wr & & & & \\ h_*(X;Z_q) & & & & \end{array}$$

is induced by the inclusion

$$X=X\times_G G=(X\times_G\mathcal{E})_0\longrightarrow X\times_G\mathcal{E}\,.$$

4. Applications. Our first application is to dualize the results of Anderson and Hodgkin [2] by applying the spectral sequence of §3 to the problem of computing the K_* groups of Eilenberg-MacLane spaces. For the moment let p be an odd prime.

Denote by K_* the Z_2-graded homology theory associated with the unitary spectrum [10] [1]. Let π be a finite abelian group and denote by B Milnor's realization of $B_\pi=K(\pi,\ 1)$. The multiplication in the group π defines a product m: $B\times B\rightarrow B$ so that for any prime power $q=p^r$, $K_*(B;\ Z_q)$ becomes an algebra over $R=K_*(pt.;\ Z_q)\approx Z_q$.

(4.1) *The algebra structure in* $K_*(B;Z_q)$ *is dual to the coalgebra structure in* $K^*(B;\ Z_q)$.

It appears that this need not be true for H-spaces in general; the proof given here depends on certain properties of the space B. In view of the determination in [2] of the coalgebra structure in $K^*(B;\ Z_q)$, this relationship will enable us to make the desired computations.

Note that from the results of [17] and the universal coefficient theorem, $K_*(B;\ Z_q)$ is additively isomorphic to $K^*(B;\ Z_q)$. If we denote by ∇ and Δ the product and coproduct respectively, the problem may be stated as follows: if $\gamma\in K^*(B;Z_q)$ and $\alpha,\beta\in K_*(B;Z_q)$ then show that $\langle\Delta(\gamma),\alpha\otimes\beta\rangle=\langle\gamma,\ \nabla(\alpha\otimes\beta)\rangle$, where the brackets denote the mod q Kronecker index.

We recall briefly the definition of the coproduct Δ. First the mod q multiplication in cohomology is defined [1] [3]. This may be done by first finding a map

f: $S^2\wedge T\to T\wedge T$, as in §2, such that the composition $(b\wedge 1)\circ f$ is homotopic to the identity. Then from the ordinary cohomology multiplication μ, the mod q multiplication μ^q is defined by the composition

$$\begin{array}{ccc} K^n(X;Z_q)\otimes K^m(Y;Z_q) & \longrightarrow & \tilde{K}^{n+2}(X^+\wedge T)\otimes\tilde{K}^{m+2}(Y^+\wedge T) \\ \vdots & & \downarrow \mu \\ \vdots\ \mu^q & & \tilde{K}^{n+m+4}((X\times Y)^+\wedge T\wedge T) \\ \vdots & & \downarrow (id\wedge f)^* \\ K^{n+m}(X\times Y;Z_q) & \xleftarrow{\sigma^{-2}} & \tilde{K}^{n+m+4}((X\times Y)^+\wedge T\wedge S^2)\,. \end{array}$$

Now since $K^*(B;Z_q)$ is a finitely generated free Z_q-module [2] we have Δ defined by the composition

$$K^*(B;Z_q)\xrightarrow{m^*}K^*(B\times B;Z_q)\approx K^*(B;Z_q)\otimes K^*(B;Z_q)$$

where the second homomorphism is the inverse of the Künneth isomorphism.

The product ∇ is given by the composition

$$K_*(B;Z_q)\otimes K_*(B;Z_q)\underset{\approx}{\xrightarrow{\mu_q}}K_*(B\times B;Z_q)\xrightarrow{m^*}K_*(B;Z_q)\,.$$

So in order to prove (4.1) it will be sufficient to show that if $\alpha\in K_r(B;Z_q)$, $\beta\in K_s(B;Z_q)$, $\gamma\in K^r(B;Z_q)$ and $\delta\in K^s(B;Z_q)$ then

(4.2) $$\langle\gamma\otimes\delta,\ \alpha\otimes\beta\rangle=\langle\mu^q(\gamma\otimes\delta),\ \mu_q(\alpha\otimes\beta)\rangle.$$

First note that since $K^1(B)=0$ and $K^0(B)$ is torsin free, the mod q reduction homomorphism

$$(1\wedge b)^*:\ K^*(B)\longrightarrow K^*(B;Z_q)\text{ is an epimorphism.}$$

So we choose elements γ' and δ' with $(1\wedge b)^*(\gamma')=\gamma$ and $(1\wedge b)^*(\delta')=\delta$.

Following the notation for the unitary spectrum used in [10], we assume that the elements α, β, γ' and δ' are represented by the maps

$$\begin{aligned} h&:\ S^{n+r+1}\longrightarrow B\wedge T\wedge U_n \\ g&:\ S^{m+s+1}\longrightarrow B\wedge T\wedge U_m \\ k&:\ B\longrightarrow U_r \\ l&:\ B\longrightarrow U_s \end{aligned}$$

respectively.

The Kronecker index $\langle(1\wedge b)^*(\gamma'),\ \alpha\rangle$ is represented by the composition

$$
(4.3)\quad \begin{array}{ccccc}
S^2\wedge S^{n+r+1} & \xrightarrow{1\wedge h} & S^2\wedge B\wedge T\wedge U_n & \xrightarrow{1\wedge \bar{f}\wedge 1} & B\wedge T\wedge T\wedge U_n \\
 & & & & \Big\downarrow {\scriptstyle (1\wedge b)\wedge 1\wedge 1} \\
S^2\wedge T\wedge U_{r+n} & \longleftarrow & S^2\wedge U_r\wedge T\wedge U_n & \xrightarrow{k\wedge 1\wedge 1} & B\wedge S^2\wedge T\wedge U_n
\end{array}
$$

Note that the composition of the second and third maps is homotopic to the identity. Similarly the Kronecker index $\langle(1\wedge b)^*(\delta'),\beta\rangle$ may be represented by the composition

$$
(4.4)\quad \begin{array}{ccccc}
S^2\wedge S^{m+s+1} & \xrightarrow{1\wedge g} & S^2\wedge B\wedge T\wedge U_m & \xrightarrow{1\wedge l\wedge 1} & S^2\wedge T\wedge U_s\wedge U_m \\
 & & & & \Big\downarrow \\
 & & & & S^2\wedge T\wedge U_{s+m}
\end{array}
$$

On the other hand, the Kronecker index $\langle\mu^q((1\wedge b)^*(\gamma')\otimes(1\wedge b)^*(\delta')),\ \mu_q(\alpha\otimes\beta)\rangle$ is represented by the composition

$$
(4.5)\quad \begin{array}{ccc}
S^4\wedge S^{r+n+1}\wedge S^{m+s+1} & & \\
\Big\downarrow {\scriptstyle 1\wedge h\wedge g} & & \\
S^4\wedge B\wedge T\wedge U_n\wedge B\wedge T\wedge U_m & & \\
\Big\downarrow {\scriptstyle 1\wedge f\wedge 1} & & \\
S^2\wedge S^2\wedge B\wedge B\wedge T\wedge S^1\wedge U_{n+m} & \xrightarrow{1\wedge \bar{f}\wedge 1} & B\wedge B\wedge S^2\wedge T\wedge T\wedge S^1\wedge U_{n+m} \\
\vdots {\scriptstyle \text{identity}} & & \Big\downarrow {\scriptstyle 1\wedge \bar{f}\wedge 1} \\
B\wedge B\wedge S^2\wedge S^2\wedge T\wedge S^1\wedge U_{n+m} & \xrightarrow{(1\wedge b)\wedge(1\wedge b)\wedge 1} & B\wedge B\wedge T\wedge T\wedge S^1\wedge U_{n+m} \\
\Big\downarrow {\scriptstyle (k\wedge 1)\wedge(l\wedge 1)\wedge 1} & & \\
U_r\wedge U_s\wedge S^2\wedge S^2\wedge T\wedge S^1\wedge U_{n+m} & & \\
\Big\downarrow & & \\
U_{n+m+r+s+5}\wedge T\,. & &
\end{array}
$$

It can be checked that the rectangle commutes up to homotopy where the dotted map is the identity. However, this shorter composition represents the product in $K_0(pt.;Z_q)$ of the previously defined Kronecker indices (4.3) and (4.4). This establishes (4.2), and (4.1) follows.

Denote by π_p the p-component of π. Then Anderson and Hodgkin [2] have shown that $K^*(B_\pi;Z_q)\approx K^*(B_{\pi_p};Z_q)\approx Z_q[\pi_p^*]$ for any $q=p^r$. Here $Z_q[\pi_p^*]$ is the group ring of the character group π_p^* of π_p. So assume that π is a p-group and let $1=u_0, u_1, \cdots, u_k$ be the the characters in π^*. The coproduct [2] is given by

$$\Delta(u_i) = u_i \otimes u_i$$

for each i.

Define elements $w_0, w_1, \cdots, w_k$ in $K_*(B; Z_q)$ by requiring that $\langle u_i, w_j\rangle = \delta_{ij} \in Z_q$. Thus $K_*(B; Z_q)$ is the free Z_q-module with basis $w_0, w_1, \cdots, w_k$. From (4.1) it is then a simple matter to determine the product in $K_*(B; Z_q)$.

$$\langle u_n, \nabla(w_i \otimes w_j)\rangle = \langle \Delta u_n, w_i \otimes w_j\rangle$$
$$= \langle u_n, w_i\rangle \cdot \langle u_n, w_j\rangle = \begin{cases} 1 \text{ if } i = j = n \\ 0 \text{ otherwise}. \end{cases}$$

Thus $\nabla(\sum n_i w_i \otimes \sum m_j w_j) = \sum_{i=0}^{k} (n_i m_i) \cdot w_i$ for any $n_i, m_j \in Z_q$. The unit ε: $Z_q \to K_*(B; Z_q)$ is the dual of the augmentation of the coalgebra and is given by $\varepsilon(1) = w_0 + w_1 + \cdots + w_k$. The augmentation η: $K_*(B; Z_q) \to Z_q$ given by

$$\eta(w_i) = \begin{cases} 1 \text{ if } i = 0 \\ 0 \text{ otherwise} \end{cases}$$

gives Z_q the structure of a $K_*(B; Z_q)$-algebra and it is a simple exercise to verify that

(4.6) *For q a power of the prime p, Z_q is a projective $K_*(B; Z_q)$-module.*
As an immediate corollary we have

(4.7) *For any $K_*(B; Z_q)$-module M and integer $n > 0$,*

$$\mathrm{Tor}_{n,*}^{K_*(B;Z_q)}(M, Z_q) = 0.$$

This fact may now be applied to the spectral sequence in §3 to make the desired computations as in [2, §4]. As the group G in §3 we take the CW group B. Its classifying space will then be a $K(\pi, 2)$. So in the spectral sequence we take X to be a point so that X_G is a $K(\pi, 2)$.

The spectral sequence of §3 collapses since by (3.4) and (4.7)

$$E^2_{*,*}(X_G) \approx \mathrm{Tor}_{*,*}^{K_*(B;Z_q)}(K_*(pt.; Z_q), Z_q)$$
$$\approx K_*(pt.; Z_q) \otimes_{K_*(B;Z_q)} Z_q \approx Z_q.$$

Hence the edge homomorphism (3.5)

$$K_*(pt., Z_q) \longrightarrow K_*(K(\pi, 2); Z_q)$$

is an isomorphism. This completes the proof that

(4.8) *For any finite abelian group π and any prime power q,*

$$\tilde{K}_*(K(\pi, 2); Z_q) = 0 .$$

Note that this result and the following one are stated for all primes. Special arguments for the case $p=2$ may be made using the universal coefficient theorem and results of Anderson and Hodgkin [2]; however, the preceding approach cannot be employed due to the absence of the maps f and $\bar{f}$.

The same result for $K(\pi, n)$ may be proved inductively, for if $G=K(\pi,n-1)$ has $\tilde{K}_*(G; Z_q)=0$ then $K_*(G; Z_q)\approx Z_q$ is a finitely generated free Z_q-module and trivial as a Z_q-algebra. Thus by the same argument

$$K_*(pt.; Z_q)\otimes_{K_*(G;Z_q)} Z_q \longrightarrow K_*(K(\pi, n); Z_q)$$

is an isomorphism. Thus

(4.9) *For any finite abelian group* π, *integer* $n\geq 2$ *and prime power* q, $\tilde{K}_*(K(\pi, n); Z_q)=0$.

As in [2] we may extend (4.9) to the case of π a countable torsion group.

Note that it is an immediate consequence of the universal coefficient theorem that $\tilde{K}_*(K(\pi, n); Z_q)=0$ for all prime powers q implies $\tilde{K}_*(K(\pi, n))$ is both torsion free and divisible. If π is a finite abelian group, $\tilde{H}_i(K(\pi, n))$ is finite for all i, so that from the Atiyah-Dold-Hirzebruch spectral sequence for $K_*(K(\pi, n))$ we have $\tilde{K}_i(K(\pi, n))$ is a direct limit of finite groups. Thus every element must be a torsion element and we conclude

(4.10) *For* π *a countable torsion group*, $n\geq 2$, $\tilde{K}_*(K(\pi, n))=0$.

These results may be extended following [2] to show that if π is a countable torsion group and $n\geq 3$, then

$$K_*(K(\pi, n))\approx K_*(K(\pi\otimes Q, n)) .$$

We now briefly interpret these results in terms of the complex bordism of Eilenberg-MacLane spaces. Let $U_*(X)$ denote the weakly complex bordism of the space X [8]. There is a natural transformation

$$\tau: \quad U_*(\ \ ) \longrightarrow K_*(\ \ )$$

which when restricted to the coefficient groups becomes the Todd genus T: $U_* \to Z$. Giving a U_*-module structure to Z via T, we state the theorem of Conner and Floyd (see [8] for a proof):

(4.11) τ *induces a natural isomorphism for each* X

$$U_*(X)\otimes_{U_*} Z \xrightarrow{\approx} K_*(X)$$

Denote by $I\subseteq U_*$ the kernel of the homomorphism T and note that I contains no non-zero homogeneous elements of degree zero. As an immediate corollary of (4.10) we have

(4.12) *If π is a countable torsion group, $n\geq 2$, then every element $\alpha\in\tilde{U}_*(K(\pi, n))$ may be written in the form $\alpha=\beta\cdot\gamma$ where $\beta\in I$ and $\gamma\in\tilde{U}_*(K(\pi, n))$. Hence $I^m\cdot\tilde{U}_*(K(\pi, n))=\tilde{U}_*(K(\pi, n))$ for all positive integers m.*

It would be natural to ask if this spectral sequence could not be used to determine the homology of $K(\pi, 1)$. Indeed, if π is a finite group, then $h_*(\pi; Z_q)$ is isomorphic to $R[\pi]$ the group ring of π, a finitely generated free R-module, where $R=h_*(pt.; Z_q)$. In this case the E^2 stage becomes the homology of the group π with coefficients in R. If the spectral sequence collapses, this gives the standard relationship between the homology of the group π with coefficients in the π-module R and the singular homology with coefficients in R of the classifying space B_π. However for the interesting cases the spectral sequence does not collapse and the desired results require a deeper analysis.

For the final application we require the following definition. Let R be a graded commutative ring with unit. A *modified polynomial algebra* A over R on generators $x_1, x_2, \cdots, x_n$ of degrees $m_1, m_2, \cdots, m_n$ is the free R-module generated by elements $x_1^{(r_1)}x_2^{(r_2)}\cdots x_n^{(r_n)}$, for all collections of non-negative integers $r_1, \cdots, r_n$, of degree $r_1m_1+r_2m_2+\cdots+r_nm_n$, in which the multiplication is given by

$$(x_1^{(r_1)}x_2^{(r_2)}\cdots x_n^{(r_n)})\cdot(x_1^{(s_1)}x_2^{(s_2)}\cdots x_n^{(s_n)}) = \binom{r_1+s_1}{r_1}\binom{r_2+s_2}{r_2}\cdots\binom{r_n+s_n}{r_n}x_1^{(r_1+s_1)}x_2^{(r_2+s_2)}\cdots x_n^{(r+s_n)}.$$

Let h_* be a multiplicative theory, p a prime and G a group in A. If $p=2$ assume h_* is as in (2.5). We may now prove a generalization of a theorem of Borel [5], [13].

(4.13) *If $h_*(G; Z_p)=E$ is an exterior algebra over $R=h_*(pt.; Z_p)$ on a finite number of odd dimensional generators and R is zero in odd dimensions, then $h_*(B_G; Z_p)$ is a modified polynomial algebra over R on corresponding generators of one dimension higher.*

Aside from complications of notation; the proof in general is the same as the proof for two generators. So suppose that E is an exterior algebra over R on generators x and y of dimension $2n-1$ and $2m-1$, respectively.

Define a free E resolution of R as follows

$$(4.14) \quad \cdots\xrightarrow{\beta_4} E_3\xrightarrow{\beta_3} E_2\xrightarrow{\beta_2} E_1\xrightarrow{\beta_1} E_0\xrightarrow{\varepsilon} R\longrightarrow 0\,.$$

Let E_m be the free E-module on $(m+1)$ generators denoted by $a^{(m)}b^{(0)}$, $a^{(m-1)}b^{(1)}$, $\cdots$, $a^{(0)}b^{(m)}$. Define the homomorphism

$$\beta_m\colon E_m\longrightarrow E_{m-1}$$

by setting

$$\beta_m(a^{(k)}b^{(m-k)}) = x\cdot a^{(k-1)}b^{(m-k)} + y\cdot a^{(k)}b^{(m-k-1)}$$

where $a^{(j)}b^{(l)}$ is understood to be zero if either j or l is less than zero. Using the fact that $xy=-yx$ it is not difficult to check that this is a free E-resolution of R. Note that the bidegree of $a^{(k)}b^{(m-k)}$ is $(m, k(2n-1)+(m-k)(2m-1))$, so that the total degree is always even.

Tensoring the resolution (4.14) throughout with R over E results in a complex having each differential identically zero. It follows immediately that $\mathrm{Tor}^E_{k,*}(R,R)$ is a free R-module of rank $(k+1)$ whose basis we identify with $a^{(k)}b^{(0)}, \cdots, a^{(0)}b^{(k)}$.

We now determine the multiplicative structure inductively. Consider the product complex

$$\begin{array}{ccccccccc}
\longrightarrow & \sum_{i+j=2} E_i\otimes E_j & \xrightarrow{\partial_2} & \sum_{i+j=1} E_i\otimes E_j & \xrightarrow{\partial_1} & E_0\otimes_E E_0 & \longrightarrow & R\otimes R \\
 & \downarrow \mu_2 & & \downarrow \mu_1 & & \downarrow \mu_0 & & \downarrow m \\
\longrightarrow & E_2 & \xrightarrow{\beta_2} & E_1 & \xrightarrow{\beta_1} & E_0 & \longrightarrow & R
\end{array}$$

The multiplication m can be lifted to a sequence of homomorphisms $\{\mu_i\}$, unique up to chain homotopy, so that each rectangle commutes.

It is apparent then that

$$\mu_0(a^{(0)}b^{(0)}\otimes a^{(0)}b^{(0)}) = a^{(0)}b^{(0)} = \binom{0}{0}\binom{0}{0}a^{(0)}b^{(0)}.$$

Now let $r>0$ and suppose that for any $k<r$ and any integers $i\le l\le k$ and $j\le k-l$,

$$\mu_k(a^{(i)}b^{(l-i)}\otimes a^{(j)}b^{(k-l-j)}) = \binom{i+j}{i}\binom{k-i-j}{l-i}a^{(i+j)}b^{(k-i-j)}.$$

Consider then $z=a^{(i)}b^{(l-i)}\otimes a^{(j)}b^{(r-l-j)}$ where $i\le l\le r$ and $j\le r-l$. We have

$$\begin{aligned}\partial_r(z) &= x\cdot(a^{(i-1)}b^{(l-i)}\otimes a^{(j)}b^{(r-l-j)} + a^{(i)}b^{(l-i)}\otimes a^{(j-1)}b^{(r-l-j)}) \\ &\quad + y(a^{(i)}b^{(l-i-1)}\otimes a^{(j)}b^{(r-l-j)} + a^{(i)}b^{(l-i)}\otimes a^{(j)}b^{(r-l-j-1)}).\end{aligned}$$

So by the inductive hypothesis,

$$\begin{aligned}&\mu_{r-1}\partial_r(z) \\ &= x\cdot\left[\binom{i+j-1}{i-1}\binom{r-i-j}{l-i}a^{(i+j-1)}b^{(r-i-j)} + \binom{i+j-1}{i}\binom{r-i-j}{l-i}a^{(i+j-1)}b^{(r-i-j)}\right] \\ &+ y\cdot\left[\binom{i+j}{i}\binom{r-i-j-1}{l-i-1}a^{(i+j)}b^{(r-i-j-1)} + \binom{i+j}{i}\binom{r-i-j-1}{l-i}a^{(i+j)}b^{(r-i-j-1)}\right]\end{aligned}$$

$$=x\cdot\left[\binom{i+j}{i}\binom{r-i-j}{l-i}a^{(i+j-1)}b^{(r-i-j)}\right]+y\cdot\left[\binom{i+j}{i}\binom{r-i-j}{l-i}a^{(i+j)}b^{(r-i-j-1)}\right]$$

$$=\beta_r\left[\binom{i+j}{i}\binom{r-i-j}{l-i}a^{(i+j)}b^{(r-i-j)}\right].$$

Hence it is natural to choose inductively

$$\mu_r(z)=\binom{i+j}{i}\binom{r-i-j}{l-i}a^{(i+j)}b^{(r-i-j)}.$$

This establishes the multiplication in $\mathrm{Tor}^E_{*,*}(R, R)$.

Note that since all non-zero elements have even total degree the spectral sequence must collapse. Finally it may be checked that this product corresponds to the product in $h_*(B_G; Z_p)$ so that $h_*(B_G; Z_p)$ is the desired modified polynomial algebra on generators corresponding to $a^{(1)}b^{(0)}$ and $a^{(0)}b^{(1)}$ of dimension $2n$ and $2m$ respectively.

THE UNIVERSITY OF TEXAS

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