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KORTEWEG-DE VRIES EQUATION: CONSTRUCTION OF SOLUTIONS IN TERMS OF SCATTERING DATA

SHUNICHI TANAKA

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In this paper we show that the solution of the initial value problem for the Korteweg-de Vries (KdV) equation

$$(1) \qquad u_t - 6uu_x + u_{xxx} = 0 \qquad u = u(t) = u(x, t) \quad (-\infty < x, t < \infty)$$

can be constructued by a method suggested in Gardner, Greene, Kruskal and Miura (GGKM) [2].

GGKM have associated one dimensional Schrödinger operator

$$L_{u(t)} = -(d/dx)^2 + u(x, t)$$

to a solution of (1). They have found remarkable facts concerning the time dependence of scattering data of $L_{u(t)}$: eigen-values are invariants and reflection coefficient $r(\xi, t)$ of $L_{u(t)}$ is given by

$$r(\xi, t) = r(\xi, 0) \exp(8i\,\xi^3 t) \qquad -\infty < \xi < \infty$$

etc. In [2], the authors also have pointed out that application of inverse scattering theory leads to certain explicit realization of the solution. It is not unexpected that use of inverse scattering theory may even lead to the construction of general solutions of the initial value problem. However the existence of solutions of nice properties has been a priori assumed in [2] and this possibility has not been explored.

The results of the present paper have been announced in [7]. GGKM's method has been also formulated in Faddeev and Zakharov [8] in a different form.

In §1 we describe results from scattering theory of one dimensional Schrödinger operator. The materials which connect the KdV equation and scattering theory are described in §2. Then in §3 analytical properties of the reflection coefficients are studied. We prove the existence of potential $u(x, t)$ whose scattering data depend on t according to GGKM's formulas. Finally in §4 we show that this function satisfies the KdV equation.

Throughout the paper subscripts with independent variables denote partial differentiations. Integrations are taken over the whole real axis unless explicitly indicated otherwise. For a complex number c, c^* denotes its complex conjugate.

1. Scattering theory for one dimensional Schrödinger equation

Consider the Schrödinger equation

$$-\phi_{xx}+u(x)\phi = \zeta^2\phi \qquad \zeta = \xi+i\eta \tag{2}$$

over $(-\infty, \infty)$ with potential $u(x)$ satisfying

$$\int(1+|x|)|u(x)|dx<\infty . \tag{3}$$

We summarize results from scattering theory for (2). We refer to [1] for detail.

If $\mathrm{Im}\ \zeta\geq 0$, there exist unique solutions $f_\pm(x, \zeta)$ which behave like $\exp(\pm i\zeta x)$ as $x\to\pm\infty$. They are called Jost solutions of (2). Jost solutions are analytic in ζ, $\mathrm{Im}\ \zeta>0$. If $\zeta=\xi$ is real, Wronskian $[f_+, f_+^*]$ is equal to $2i\xi$. As f_+ and f_+^* are independent solutions of (2) for $\xi\neq 0$, one can express f_- as

$$f_- = a(\xi)f_+^*+b(\xi)f_+ . \tag{4}$$

We have

$$a(\xi) = (2i\xi)^{-1}[f_+, f_-] , \tag{5}$$

$$b(\xi) = (2i\xi)^{-1}[f_-, f_+^*] \tag{6}$$

and

$$|a(\xi)|^2 = 1+|b(\xi)|^2 . \tag{7}$$

By the above expression for $a(\xi)$, it is the boundary value of the function $a(\zeta)$ analytic in $\mathrm{Im}\ \zeta>0$. The functions

$$r_\pm(\xi) = \pm b(\pm\xi)a(\xi)^{-1}$$

are called right and left reflection coefficients. They are defined for $\xi\neq 0$ and bounded by 1.

Put

$$h_\pm(x, \zeta) = \exp(\mp i\zeta x)f_\pm(x, \zeta) . \tag{8}$$

Then $h_\pm$ are expressed as

$$h_\pm(x, \zeta) = 1\pm\int_0^{\pm\infty} B_\pm(x, y)\exp(\pm 2i\zeta y)dy . \tag{9}$$

The kernels $B_\pm$ satisfy the integral equations

$$B_\pm(x,y)=\pm\int_{x+y}^{\pm\infty}u(z)\,dz+\int_0^y dz\int_{x+y-z}^{\pm\infty}u(s)B(s,z)\,ds\,. \tag{10}$$

By successive approximation, one has the estimate

$$|B_\pm(x,y)|\le\pm C_\pm(x)\int_{x+y}^{\pm\infty}|u(z)|\,dz\,,$$

where $C_\pm(\pm x)$ are non-increasing functions. Moreover, first derivatives of $B_\pm$ exist. Potential u is reconstructed by formula

$$u(x)=\mp\partial/\partial x B_\pm(x,\pm 0)\,. \tag{11}$$

Lemma 1. *Suppose that $u(x)$ is n-times differentiable and*

$$\sigma_{\pm j}(x)=\pm\int_x^{\pm\infty}|u^{(j)}(y)|\,dy,\qquad j\le n,$$

are finite for any x. Then $B_\pm^{(j,k)}=\partial^{j+k}B_\pm/\partial x^j\partial y^k$ exist for $j+k\le n+1$ and are estimated as

$$|B_\pm^{(j,k)}(x,y)\pm u^{(j+k-1)}(x+y)|\le C_\pm(x)\sum_{l=0}^{j+k-1}\sigma_{\pm l}(x+y)\,.$$

Proof. First consider the case $k=0$. By (10), we have

$$B_{\pm x}(x,y)\pm u(x+y)=-\int_0^y u(x+y-z)B_\pm(x+y-z,z)\,dz \tag{12}$$

and the estimates for $B_{\pm x}$ follow. Existence and estimates of $B_\pm^{(j,1)}$ are obtained by repeated differentiation of (12) with respect to x.

Suppose that, in general, the statement has been proved for $j+k\le m$ and for $j+k=m+1$, $k\le k'$. Then existence of $B_\pm^{(m-k',-k'+1)}$ and its estimate follow from

$$B_{\pm x}(x,y)-B_{\pm y}(x,y)=-\int_x^{\pm\infty}u(z)B_\pm(z,y)\,dz \tag{13}$$

and its repeated differentiations.

Q.E.D.

If second derivatives exist, (13) gives

$$B_{\pm xx}(x,y)-B_{\pm xy}(x,y)=u(x)B_\pm(x,y)\,. \tag{14}$$

Put (9) into (6). Then we have

$$2i\xi b(\xi)=\exp(-2ix\xi)\int\Pi_1(y)\exp(-2i\xi y)\,dy$$

where

$$\begin{aligned}\Pi_1(y)=\Pi_1(x,y)&=-B_{+x}(x,y)+B_{-x}(x,y)\\&\quad+\int B_{-x}(x,y)B_+(x,y-z)\,dz-\int B_{+x}(x,z)B_-(x,y-z)\,dz\,.\end{aligned}$$

Here we have extended $B_\pm(x,y)$ as 0 for $\pm y<0$.

If u is sufficiently differentiable with rapidly decreasing derivatives, by

Lemma 1 $\Pi_1(y)$ is differentiable except at $y=0$. For $\pm y>0$ we have

$$\Pi_1^{(n)}(y) = \mp B_{\pm}^{(1,n)}(x,y)+\sum_{j=0}^{n-1} B_{\pm}^{(1,j)}(x,y)B_{\mp}^{(0,n-j-1)}(x,\mp 0)$$
$$+\int B_{-x}(x,z)B_{+}^{(0,n)}(x,y-z)\,dz-\int B_{+x}(x,z)B_{-}^{(0,n)}(x,y-z)\,dz\,.$$

Putting (9) into (5), we have

$$2i\zeta a(\zeta) = 2i\zeta-\int u(y)\,dy+\int_0^\infty \Pi_2(y)\exp(2i\zeta y)\,dy$$

where

$$\Pi_2(y) = B_{+x}(x,y)-B_{-x}(x,-y)-B_{+y}(x,y)$$
$$+B_{-y}(x,-y)-B_{+}(x,0)B_{-}(x,-y)+\int[B_{+x}(x,z)-B_{+y}(x,z)]B_{-}(x,z-y)\,dz$$
$$-\int B_{+}(x,z)B_{-x}(x,z-y)\,dz\,.$$

If $a(\zeta_0)=0$, $\operatorname{Im}\zeta_0>0$, then $f_{\pm}(x,\zeta_0)$ are linearly dependent by (5). They are square-integrable because of the asymptotic behavior. Because L_u is symmetric, ζ_0^2 is real, *i.e.* ζ_0 is purely imaginary. One has

$$a'(\zeta_0) = -i\int f_{+}(x,\zeta_0)f_{-}(x,\zeta_0)\,dx\,. \tag{15}$$

The functions $f_{\pm}(x,\zeta)$ being real valued for $\zeta=i\eta$, zeros of $a(\zeta)$ are all simple. Number of zeros is finite because $a(\zeta)\to 1$ as $|\zeta|\to\infty$. We denote them by $i\eta_1,\cdots,i\eta_N$. Put

$$c_{\pm j}^{-1} = \int f_{\pm}(x,i\eta_j)^2\,dx\,.$$

Then $c_{\pm j}$ are related by

$$c_j c_{-j} = -a'(i\eta_j)^{-2}\,. \tag{16}$$

We call the triplets $s_{\pm}=\{r_{\pm}(\xi),\eta_j,c_{\pm j}\}$ right and left scattering data of the potential u. Put

$$F_{\pm}(y) = \pi^{-1}\int r_{\pm}(\xi)\exp(\pm 2i\xi y)\,d\xi \tag{17}$$

$$\Omega_{\pm}(y) = 2\sum_{j=1}^{N} c_{\pm j}\exp(\mp 2\eta_j y)+F_{\pm}(y)\,. \tag{18}$$

Then $B_{\pm}$ satisfy the Marchenko equations

$$B_{\pm}(x,y)\pm\int_0^{\pm\infty}\Omega_{\pm}(x+y+z)B_{\pm}(x,z)\,dz+\Omega_{\pm}(x+y) = 0\,. \tag{19}$$

We now turn to the inverse problem. Let N be a non-negative integer and $\eta_1,\cdots,\eta_N$ be prositive numbers different from each other. Let $c_1,\cdots,c_N$ be positive umbers. Let $r(\xi)$ be a function which satisfies $r(-\xi)=r(\xi)^*$, $|r(\xi)|<1$ for $\xi\neq 0$ and $r(\xi)=O(\xi^{-1})$ as $\xi\to\pm\infty$.

Determine $a(\zeta)$ by following conditions: it is analytic in $\zeta(\operatorname{Im}\zeta>0)$,

$|a(\xi)|^{-2}=1-|r(\xi)|^2$ on the real axis and $i\eta_j(j=1, \cdots, N)$ are its simle zeros (see [1, p. 151] for an explicit formula). Put $r_+(\xi)=r(\xi)$ and

$$r_-(\xi) = -r(-\xi)a(-\xi)a(\xi)^{-1}.$$

Define c_{-j} be the relations (16). Define functions $F_\pm$ and $\Omega_\pm$ by the formulas (17) and (18).

Assume that $F_\pm$ are absolutely continuous and $(1+|x|)F_\pm'(\pm x)$ belong to $L^1(a, \infty)$ for any a. Then Marchenko equations (19) are uniquely solvable for each x. Let $B_\pm(x, y)$ be their solutions. Then the estimates

$$|B_\pm(x, y)| \le C_\pm(x)\alpha_\pm(x+y),$$

where

$$\alpha_\pm(x) = \pm\int_x^{\pm\infty} |\Omega_+'(y)|\,\mathrm{dy},$$

hold. $B_\pm$ are differentiable in x and estimates

$$|B_{\pm x}(x, y)+\Omega_\pm'(x+y)| \le C_\pm(x)\alpha_\pm(x)\alpha_\pm(x+y)$$

hold. See [1, pp. 159–160].

Define $h_\pm$ by (9) and then $f_\pm$ by (8). Put

$$u_\pm(x) = \mp B_{\pm x}(x, \pm 0).$$

Then $(1+|x|)u_\pm(\pm x)$ are in $L^1(a, \infty)$ for any a and

$$-f_\pm''+u_\pm f_\pm = \zeta^2 f_\pm$$

hold.

To show that $u_\pm(x)$ conincide, $r(\xi)$ should satisfy an additional condition [1, Lemma 3.1]: if $\xi a(\xi)\neq 0$ at $\xi=0$, then $r(0)=-1$. In this case $\{r(\xi), \eta_j, c_j\}$ is right reflection coefficient of $u(x)=u_\pm(x)$.

Lemma 2. *Suppose that $F_\pm(x)$ are n-times continuously differentiable and $F_\pm^{(j)}(\pm x)$ $(j\le n)$ belong to $L^1(a, \infty)$ for any a. Then continuous derivatives $B_\pm^{(j,k)}$ exist for $j+k\le n$ and are estimated as*

$$\pm\int_0^{\pm\infty} |B_\pm^{(j,k)}(x, y)|\,dy \le C_\pm(x)$$

and for $(j, k)\neq(0, 0)$

$$|B_\pm^{(j,k)}(x, y)+\Omega_\pm^{(j,k)}(x+y)| \le C_\pm(x)\sum_{m=k}^{j+k}\alpha_{\pm m}(x+y)$$

where

$$\alpha_{\pm j}(x) = \pm\int_x^{\pm\infty} |\Omega_\pm^{(j)}(y)|\,dy.$$

Proof. Derivatives $B_\pm^{(j,0)}$ satisfy

$$B_\pm^{(j,0)}(x,y)\pm\int_0^{\pm\infty}\Omega_\pm(x+y+z)B_\pm^{(j,0)}(x,z)\,dz \tag{20}$$
$$=-\Omega_\pm^{(j)}(x+y)\mp\textstyle\sum_{m=1}^{j}{}_jC_m\displaystyle\int_0^{\pm\infty}\Omega_\pm^{(m)}(x+y+z)B_\pm^{(j-m,0)}(x,z)dz\,.$$

The existence and estimates of $B_\pm^{(j,0)}$ are quite analogous to the case of $B_\pm^{(1,0)}$ and are shown by induction based on (20). Then differentiating (20) with respect to y, we obtain the results for $B_\pm^{(j,k)}$.

Q.E.D.

2. Time dependence of scattering data

Put

$$L_u=-D^2+u \qquad D=d/dx$$

and

$$B_u=-4D^3+3uD+3Du\,.$$

Then the operator $[B_u, L_u]=B_uL_u-B_uL_u$ is the multiplication by the function $6uu_x-u_{xxx}$. Thus the equation

$$dL_u/dt=[B_u, L_u] \tag{21}$$

for $u=u(t)=u(x,t)$ is equivalent to (1). This observtion is due to Lax (See [4] and [5]).

We now describe a formal procedure to determine time dependence of Jost functions and scattering data of $L_{u(t)}$ for smooth real-valued solution $u=u(t)=u(x,t)$ which is rapidly decreasing in x for each t. Relations found here will be used later to construct solutions.

We differentiate the relation $L_uf_\pm=\zeta^2f_\pm$ with respect to t. Making use of (21), we see that $df_\pm/dt-B_uf_\pm$ again satisfies (2). Because they behave like $4(\pm i\zeta)^3\exp(\pm i\zeta x)$ as $x\to\pm\infty$, by the uniqueness of Jost functions, we have

$$df_\pm/dt-B_uf_\pm=4(\pm i\zeta)^3f_\pm\,. \tag{22}$$

Differentiating the relation (4) with respect to t and eliminating t-derivative of Jost functions by (22), we obtain an identity

$$(da/dt)f_+^*+(db/dt-8i\xi^3b)f_+=0\,.$$

Thus we have

$$a(\xi,t)=a(\xi,0)$$
$$b(\xi,t)=b(\xi,0)\exp(8i\xi^3t)\,.$$

$a(\zeta,t)$ is independent of t and so are its zeros $i\eta_1,\cdots,i\eta_N$. Reflection coefficients vary as

$$r_\pm(\xi,t)=r_\pm(\xi)\exp(\pm 8i\xi^3t)\,. \tag{23}$$

Differentiating

$$f_-(x, i\eta_j) = d_j f_+(x, i\eta_j)$$

with respect to t and eliminating df_+/dt by (22), we have

$$d_j'(t) = 8\eta_j^3 d_j(t)\,.$$

Because the relation $d_j = ia'(i\eta_j)c_j$ hold by (15), c_j satisfy the same differential equation as d_j and consequently we have

$$c_j(t) = c_j(0) \exp(8\eta_j^3 t) \qquad j = 1, \cdots, N\,. \tag{24}$$

Formulas (23) and (24) have been discovered by GGKM [2]. See also Lax [5]. Present derivation is somewhat different from those of [2] and [5].

3. Properties of the reflection coefficients

In this section, we study the analytical properties of the reflection coefficients in detail.

Assume that $u(x)$ is a real differentiable function such that $u^{(j)}(x)$ $(j \leq 6)$ are rapidly decreasing. Then $B_\pm^{(j,k)}(x, y)$ $(j+k \leq 7)$ exist and estimated as in Lemma 1.

$f_\pm(x, \xi)$ and $f_\pm'(x, \xi)$ are C^∞ in ξ. By (5) and (6), $\xi a(\xi)$ and $\xi b(\xi)$ are C^∞. If $\xi \neq 0$, $\xi a(\xi)$ is not zero by (7). So $r(\xi)$ is C^∞ for $\xi \neq 0$ and even at $\xi = 0$ if $\xi a(\xi)|_{\xi=0} \neq 0$. Suppose that $\xi a(\xi)|_{\xi=0} = 0$. Then by (5) and (6), $a(\xi)$ and $b(\xi)$ are smooth at $\xi = 0$. By (7), $a(0) \neq 0$ and $r_\pm$ are smooth at $\xi = 0$.

We next study the behavior of a, b and $r_\pm$ as $|\xi| \to \infty$. By integral representation each derivatives of $\xi(a(\xi)-1)$ are bounded. $\Pi_1^{(n)}(y)$ $(n \leq 6)$ exist and are continuous except at $y = 0$. They have right and left limits at $y = 0$ and are rapidly decreasing as $|y| \to \infty$.

Lemma 3. $\Pi_1^{(n)}(y)$ $(n \leq 5)$ *are continuous even at* $y = 0$.

Proof. By the expression for the derivatives of $\Pi_1(y)$, it is sufficient to show that

$$\mp B_\pm^{(1,n)}(x, \pm 0) + \textstyle\sum_{j=0}^{n-1} B_\pm^{(1,j)}(x, \pm 0) B_\mp^{(0,n-j-1)}(x, \mp 0) \tag{25}$$

do not depend on signs. Using (14), we replace $B_\mp^{(1,j)}(x, \pm 0)$ by derivatives of $u(x)$ and $B_\pm^{(0,k)}(x, \pm 0)$. We get thus rather complicated expressions for (25). We describe here only two of them: (25) is equal to

$$\mp[\mp u'(x) - u(x) B_\pm(x, \pm 0)] \mp u(x) B_\mp(x, \mp 0)$$

for $n = 1$ and to

$$\mp[-u(x)B_{\pm y}(x, \pm 0)-u'(x)B_{\pm}(x, \pm 0)\pm u(x)^2]$$
$$\mp u(x)B_{\mp y}(x, \mp 0)+[\mp u'(x)-u(x)B_{\pm}(x, \pm 0)]B_{\mp}(x, \mp 0)$$

for $n=2$.

Q.E.D.

So we have

$$(2i\xi)^7 b(\xi) = \exp(-2ix\xi)\int \Pi_1^{(6)}(y) \exp(-2i\xi y) dy$$

after integration by parts. Therefore $\xi^7 b(\xi)$ is bounded and belongs to L^2. Also $\xi^6(\xi b(\xi))'$ and $\xi^6(\xi b(\xi))''$ are bounded and belong to L^2. It follows that $\xi^7 r_{\pm}(\xi)$, $\xi^7 r_{\pm}'(\xi)$ and $\xi^7 r_{\pm}''(\xi)$ also are bounded and belong to L^2.

Put

$$F_{\pm}(x, t) = \pi^{-1}\int r_{\pm}(\xi) \exp(\pm 8i\xi^3 t \pm 2i\xi x) d\xi .$$

Recall that if a function and its first derivative are square integrable, then its Fourier transform is integrable. By the properties of $r_{\pm}$ just described, we have

Lemma 4. *Continuous derivatives $F_{\pm}^{(j,k)}(x, t)$ $(j+3k\leq 5)$ exist and are integrable. $xF_{\pm x}(x, t)$ and $xF_{\pm xx}(x, t)$ are also integrable.*

Define $c_{\pm j}(t)$ by the formula (24). Put

$$\Omega_{\pm}(x, t) = 2\sum_j c_{\pm j}(t) \exp(\mp 2\eta_j x)+F_{\pm}(x, t) .$$

Then by Lemma 4, Marchenko equation (19) is solvable for each t. Denote by $B_{\pm}(x, y; t)$ the solutions. Put

$$u_{\pm}(x, t) = \mp B_{\pm x}(x, \pm 0; t) .$$

As $r_{\pm}(0, t)=r_{\pm}(0)$, the additional condition required to show $u_+(x, t)=u_-(x, t)$ is clearly satisfied. We have thus proved

Theorem 1. *For each t there exists the unique potential $u(x, t)$ which satisfies* (3) *such that $\{r_{\pm}(\xi, t), \eta_j, c_{\pm j}(t)\}$ are its scattering data.*

By Lemma 2 and 4, $B_{\pm}$ are C^5 in x. Its x-derivatives are integrable with respect to y. So the corresponding Jost functions $f_{\pm}(x, \zeta; t)$ are C^5 in x.

Next we study the differentiability in t. Note that $\Omega_{\pm}$ satisfy the differential equations

$$(26) \qquad (\Omega_{\pm})_t+(\Omega_{\pm})_{xxx} = 0 .$$

Differentiation of (19) with respect to t and (26) lead to the integral equation for the t-derivative:

$$(27) \qquad B_{\pm t}(x, y)\pm\int_0^{\pm\infty}\Omega_{\pm}(x+y+z)B_{\pm t}(x, z)dz+D_{\pm}(x, y) = 0 ,$$

where

$$D_{\pm}(x, y) = \mp\int_0^{\pm\infty} \Omega_{\pm xxx}(x+y+z) B_{\pm}(x, z)\, dz - \Omega_{\pm xxx}(x+y)$$

(we omit the variable t when confusion does not occur).

Comparing (27) with (20) and arguing analogously, we conclude that continuous drivatives $B_{\pm}^{(j,k,l)}(x, y; t)\ (j+k+3l \leq 5)$ exist and integrable with respect to y. So j-th x and k-th $t(j+3k \leq 5)$ derviatives of $f_{\pm}(x, \zeta; t)$ exist. Also $u^{(j,k)}(x, t)\ (j+3k \leq 4)$ exist and are continuous.

Lemma 5. *$u^{(j,k)}(x, t)\ (j+3k \leq 3)$ are integrable with respect to x and tend to zero as $|x| \to \infty$.*

Proof. By putting results of Lemma 2 again in (20), we obtain finer estimates for x-derivatives of u:

$$\begin{aligned} &|\mp u'(x) + \Omega_{\pm}''(x)| \leq C_{\pm}(x)\alpha_{\pm 1}(x) \\ &|\mp u''(x) + \Omega_{\pm}^{(3)}(x)|,\ |\mp u^{(3)}(x) + \Omega_{\pm}^{(4)}(x)| \\ &\quad \leq C_{\pm}(x)\, [\alpha_{\pm 1}(x) + \alpha_{\pm 2}(x)]\,. \end{aligned}$$

Then the assertion follows from Lemma 4. The argument for the t-derivative is quite similar.

Q.E.D.

4. Solution of the initial value problem

Having studied the analytical properties, we are now in a position to prove

Theorem 2. *The function $u(x, t)$ in Theorem 1 satisfies the KdV equation.*

Proof. It is sufficient to show that the relation (22) hold. In fact, differentiating $L_u f_{\pm} = \zeta^2 f_{\pm}$ with respect to t and eliminating $df_{\pm}/dt$ by (22), we obtain

$$(dL_u/dt - [B_u, L_u]) f_{\pm} = 0$$

i.e.

$$(u_t - 6uu_x + u_{xxx}) f_{\pm} = 0\,.$$

We prove the equation (22) for f_+. We omit the plus sign from the quantiites involved.

(22) is equivalent to $dh/dt = g$ where $g = g(x, \zeta; t)$ is defined by

$$\begin{aligned} g = {} & 12\zeta^2 h_x - 12i\zeta h_{xx} - 4h_{xxx} \\ & + 6u(i\zeta h + h_x) + 3u_x h\,. \end{aligned}$$

After repeated applications of (14) and integration by parts we get

$$g(x, \zeta; t) = \int_0^{\infty} C(x, y; t) \exp(2i\zeta y)\, dy\,,$$

where

$$C = -B_{xxx} + 3uB_x .$$

The kernel C satisfies the same integral equation as B_t:

$$C(x, y) + \int_0^\infty \Omega(x+y+z) C(x, z) = -D(x, y) . \tag{28}$$

In fact, denote the left hand side of (28) by $E(x, y)$. We differentiate Marchenko equation (19) with respect to x, multiply $3u$ and then substract three times x-differentiations of (19). We obtain thus

$$E(x, y) + 3\int_0^\infty \Omega'(x+y+z)\, [u(x) B(x, z) - B_{xx}(x, z)]\, dz$$
$$-3\int_0^\infty \Omega''(x+y+z) B_x(x, z)\, dz + 3\Omega'(x+y) u(x) + D(x, y) = 0 .$$

The identity (14) and integration by parts lead to

$$E(x, y) + D(x, y) = 0 .$$

The homogenous equation associated with the Marchenko equation has only trivial solution. Thus by (27) and (28), we have $B_t = C$ and consequently $h_t = g$ has been established.

Q.E.D.

REMARK. For the special case of $r_\pm(\xi) \equiv 0$, results analogous to Theorem 2 have been proved by Hirota [3] and the present author [6] by different methods.

By Lemma 5, the solution constructed in this paper satisfies the conditions which is needed for the uniqueness of solutions of the initial value problem. See Lax [4 pp. 467–468].

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