Miyamoto, I.
Osaka J. Math.
11 (1974), 9-13

MULTIPLY TRANSITIVE PERMUTATION GROUPS AND ODD PRIMES

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(Received June 11, 1973)

In [4] M. Hall determined all 4-fold transtitive permutation groups whose stabilizer of 4 points is of odd order. In this note we give some analogous version of M. Hall's theorem for any odd prime p on $3p$-fold transtive permutation groups. We note that such a version is also already obtained by E. Bannai [1] on (p^2+p)-fold transitive permutation groups.

Theorem. *Let p be an odd prime. Let G be a $3p$-fold transitive permutation group on $\Omega=\{1, 2, \cdots, n\}$. If the order of a stabilizer of $3p$ points in G is prime to p, then $G=S_n$ $(3p\leqslant n<4p)$ or $G=A_n$ $(3p+2\leqslant n<4p)$.*

Our notation follows Nagao [6]. Let us recall some of them: For a set S of permutations on Ω the set of the points left fixed by S will be denoted by $I(S)$. For a permutation x let $\alpha_i(x)$ denote the number of i-cycles. Also let $I^c(S)=\Omega-I(S)$ and $\alpha(x)=\alpha_1(x)$. The order of a permutation x will be denoted by $o(x)$. $p\mid o(x)$ will mean that $o(x)$ is divisible by p and $p\nmid o(x)$ will mean that $o(x)$ is not divisible by p.

1. On $2p$-fold transitive groups

The next lemma which is indebted to Nagao [6] is essential in the present work.

Lemma 1.1. *Let X be a p-fold transitive permutation group on a finite set Ω. Let P be a Sylow p-subgroup of X. If P is semiregular on Ω-$I(P)$, then*

(i) *X has only one conjugacy class of the elements of order p, and*

(ii) *for an element u of order p, $C_X(u)$ is transitive on $I^c(u)$.*

Proof. Since X is p-fold transitive,

$$\frac{|X|}{p}=\sum_{x\in X}\alpha_p(x)\,, \tag{1}$$

by a result of Frobenius [1][2]. On the other hand, since P is semiregular, any element x with p-cycle is uniquely expressed as a product of an element

u of order p and an element y of order prime to p which commute with each other. Then we can see easily that $\alpha_p(x)=\frac{1}{p}\alpha^*(y)$, where $\alpha^*(y)$ denotes the number of the fixed points of y on $I^c(u)$. Hence we have by (1)

$$\frac{|X|}{p}=\sum_i \frac{|X|}{|C_X(u_i)|}\frac{1}{p}\sum_y{}'\alpha^*(y), \tag{2}$$

where $\{u_1\}, \cdots, \{u_k\}$ are the conjugacy classes of X consisting of elements of order p and the second summuation $\sum\limits_y{}'$ ranges over all the elements of $C_X(u_i)$ of order prime to p. Now let t_i be the number of the orbits of $C_X(u_i)$ on $I^c(u_i)$, then by [5], Theorem 16.6.13,

$$\sum_{y\in C_X(u_i)}\alpha^*(y)=t_i\cdot|C_X(u_i)|\,.$$

Since P is semiregular, $\alpha^*(y)$ vanishes for an element y such that $p\,|\,o(y)$. Hence

$$\sum_{y\in C_X(u_i)}{}'\alpha^*(y)=t_i|C_X(u_i)|\,.$$

Then by (2),

$$\begin{aligned}\frac{|X|}{p}&=\sum_i\frac{|X|}{|C_X(u_i)|}\frac{1}{p}\cdot t_i\cdot|C_X(u_i)|\\&=\frac{|X|}{p}\sum_i t_i\,.\end{aligned}$$

Therefore we have that $k=1$ and $t_1=1$. Thus we have the assertion.

REMARK. The following inequality is valid whenever X is p-fold transitive.

$$\frac{|X|}{p}\geqslant\sum_i\frac{|X|}{|C_X(u_i)|}\frac{1}{p}\sum_{y\in C_X(u_i)}{}'\alpha^*(y).$$

In this section we always assume that p is an odd prime and that G is a $2p$-fold transitive permutation group on $\Omega=\{1,\cdots,n\}$, excluding S_n and A_n, where the stabilizer H of the points $1,\cdots,2p$ in G is of order prime to p. Then $I(H)=\{1,\cdots,2p\}$ by Theorem of Nagao [6]. Let $\Delta=\{1,\cdots,2p\}$ and let $N=N_G(H)$, then $N^\Delta=S_{2p}$ (cf. Wielandt [7], Theorem 9.4). Let P be a Sylow p-subgroup of N then P is an elementary abelian group of order p^2. We may assume that

$$a=(1)(2)\cdots(p)(p+1,\cdots,2p)\cdots\cdots$$
and
$$b=(1,2,\cdots,p)(p+1)\cdots(2p)\cdots\cdots$$

generate P; i.e., $\langle a,b\rangle=P$. Since $|H|$ is prime to p, P has at most $p+1$ orbits of length p. So we consider the following 3 cases separately.

Case (I) *P has exactly two orbits of length p; $\{1,\cdots,p\}$ and $\{p+1,\cdots,2p\}$,* *and P is semiregular on $\Omega-I(P)-\{1, \cdots, 2p\}$.*

Case (II) *P has i ordbits of length p $(2<i\leqslant p)$ and P is semiregular on $\Omega-I(P)-\{1, \cdots, ip\}$ and in this case we may assume that*

$$ab=(1, 2, \cdots, p)(p+1, \cdots, 2p)(2p+1)(2p+2)\cdots(3p)\cdots\cdots$$

Case (III) *P has $p+1$ orbits of length p and P is semiregular on $\Omega-I(P)-\{1, \cdots, (p+1)p\}$.*

Let $K=G_{1,\cdots,p}$ and $L=\langle b\rangle\cdot K$. Then we have the following corollary immediately from lemma 1.1.

Corollary 1.2.
$$|C_K(a)|=\sum_{y\in C_K(a)}\alpha^*(y)=\sum_{y\in C_K(a)}{}'\alpha^*(y),$$
$$|C_L(a)|=\sum_{y\in C_K(a)}\alpha^*(y)=\sum_{y\in C_L(a)-C_K(a)}\alpha^*(y),$$
where the summuation $\sum'$ ranges over all the elements of $C_K(a)$ of order prime to p and $\alpha^*(y)$ denotes the number of the fixed points of y on $I^c(a)$.

Lemma 1.3. *In cases* (I) *and* (II) *P is a Sylow p-subgroup of L. In case* (III) *P is not a Sylow p-subgroup of L.*

Proof. In case (I) and (II), $n=ip+rp^2+s$ for some integers $i\,(2\leqslant i\leqslant p)$, r and $s\,(0\leqslant s<p)$. $L_1=K$ and K is p-fold transitive on $\Omega-\{1, \cdots, p\}$. Hence the first assertion holds by the assumption that $|H|$ is prime to p. We have the second assertion similarly.

Lemma 1.4. *Case* (I) *does not hold.*

Proof. Let t denote the number of the orbits of $C_L(b)$ on $I^c(b)-\{1, \cdots, p\}$ and let $\alpha^*(y)$ denote the number of the fixed points of y on $I^c(b)-\{1, \cdots, p\}$. By Lemma 1.3 P is a Sylow p-subgroup of $C_L(b)$. In case (I) any element of P except the identity has no fixed points on $I^c(b)-\{1, \cdots, p\}$. Therefore $\alpha^*(y)=0$ for any element y of $C_L(b)$ such that $p\,|\,o(y)$ and

$$t|C_L(b)|=\sum_{y\in C_L(b)}{}'\alpha^*(y).$$

Hence by the remark after lemma 1.1,

$$\frac{|L|}{p}\geqslant\frac{|L|}{|C_L(b)|}\frac{1}{p}\sum_{y\in C_L(b)}{}'\alpha^*(y)+\frac{|L|}{|C_L(b^{-1})|}\frac{1}{p}\sum_{y\in C_L(b^{-1})}{}'\alpha^*(y)$$
$$=2\frac{|L|}{|C_L(b)|}\frac{1}{p}t|C_L(b)|,$$

because there exist two elements b and b^{-1} of L of order p which are not conjugate in L. Hence we have $t=0$, that is, b is a p-cycle. Then $G=S_n$ or A_n(cf. Wielandt [7] §13). This is not the case.

Lemma 1.5. *Case* (II) *does not hold.*

Proof. By corollary 1.2,

$$|C_L(a)| = \sum_{y \in C_L(a) - C_K(a)} \alpha^*(y) + |C_K(a)| .$$

Since $|C_L(a): C_K(a)| = p$, we have

$$(3) \qquad \frac{p-1}{p} |C_L(a)| = \sum_{y \in C_L(a) - C_K(a)} \alpha^*(y) .$$

$b, b^2, \cdots, b^{p-1}$ are not conjugate with one another in $C_L(a)$ since they are not conjugate in L. $b^i (i=2, 3, \cdots, p-1)$ and ab are not conjugate in $C_L(a)$. We shall show that b and ab are not conjugate in $C_L(a)$. If b and ab are conjugate in $C_L(a)$ by an element x, *i.e.*, $b^x = ab$. Then $x \in C_L(a) \cap N_L(P)$ and x^p centralizes b. Hence $p \mid o(x)$, but this is a contradiction since P is a Sylow p-subgroup of L. Thus we have p conjugacy classes in $C_L(a) - C_K(a)$ of order p represented by the elements $b, b^2, \cdots, b^{p-1}$ and ab, any of which has p fixed points on $I^c(a)$. Since the restriction of $C_L(P)$ on the orbits of P of length p is self-centralizing (cf. Wielandt [5] §4), we have

$$\begin{aligned} \sum_{y \in C_L(a) - C_K(a)} \alpha^*(y) &\geqslant p \cdot p |C_L(a): C_L(P)| \cdot |\{y \in C_L(P) \mid p \not| o(y)\}| \\ &= p^2 |C_L(a): C_L(P)| \cdot |C_L(P): P| . \end{aligned}$$

Hence

$$\sum_{y \in C_L(a) - C_K(a)} \alpha^*(y) \geqslant |C_L(a)| .$$

This contradicts the equality (3).

2. Proof of Theorem

Lemma 2.1. *Let p be an odd prime. Let G be a $2p$-fold transitive permutation group on $\Omega = \{1, 2, \cdots, n\}$. Let K be the stabilizer of the points* $1, 2, \cdots, 2p$ *in G and let P be a Sylow p-subgroup of K.*

If P is not identity and semiregular on $\Omega - \{1, 2, \cdots, 2p\}$, then P is of order p.

Proof. Let a be an element of order p which is conjugate with some element of P such that

$$a = (1)(2)\cdots(p)(p+1, p+2, \cdots, 2p)\cdots\cdots .$$

Then a normalizes K, hence also normalizes a Sylow p-subgroup P' of K. So we find an element b of P' of order p which commutes with a. Then a fixes exactly p points of a fixed block of b and $|I(a) \cap I(b)| = p$, *i.e.*, $|I(\langle a, b \rangle)| = p$. Conjugating a to a' and b to b', we may assume that

$$a' = (1)(2)\cdots(p)(p+1, \cdots, 2p)\cdots\cdots ,$$

$$b' = (1, 2, \cdots, p)(p+1) \cdots (2p) \cdots\cdots .$$

Let $Q = \langle a', b' \rangle$, then any element of Q has at least p fixed points on $\Omega - \{1, 2, \cdots, 2p\}$. Q normalizes K, hence also normalizes a Sylow p-subgroup P'' of K.

Assume $|P''| \geqslant p^2$. We shall find a subgroup S of P'' of order p^2 which is normalized by Q. Since Q normalizes $Z(P'')$, the center of P'', if $|Z(P'') \cap C_{P''}(Q)| \geqslant p^2$, we find such subgroup S immediately. Let $R = Z(P'') \cap C_{P''}(Q)$ and we assume $|R| = p$. We can find a Q-invariant subgroup $\bar{S}$ of order p in P''/R. Then the inverse image S in P'' is Q-invariant and of order p^2. S is a cyclic group of order p^2 or an elementary abelian group of order p^2. Anyhow the automorphism group of S does not contain an elementary abelian group of order p^2. Therefore some element $c(\neq 1)$ of Q centralizes S. Since c has fixed points on $\Omega - I(S)$, c has at least p^2 fixed points (cf. Wielandt [7] §4). Since p is odd, $p^2 > 2p$. This contradicts the semiregularity of P on $\Omega - \{1, 2, \cdots, 2p\}$. Thus we have the assertion.

Proof of Theorem. If G is $3p$-fold transitive on Ω, then by lemma 2.1 a Sylow p-subgroup of a stabilizer of $2p$ points in G is of order p. But this contradicts lemma 1.3. Thus we have the assertion of Theorem.

REMARK. A result corresponding to lemma 2.1 was also proved by E. Bannai in a little strong form. His result will be published elsewhere.

At the end the author thanks Dr. E. Bannai and Mr. H. Enomoto for their helpful advice and kind encouragement.

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