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MODULES OVER DEDEKIND PRIME RINGS I

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The purpose of this paper is the investigation of modules over Dedekind prime rings. In Section 1, we shall prove that the double centralizer of a P-primary module over a Dedekind prime ring R is isomorphic to $\hat{R}_P$ or $\hat{R}_P/\hat{P}^n$, where P is a nonzero prime ideal of R and $\hat{R}_P$ is the P-adic completion of R with unique maximal ideal $\hat{P}$. Using this result we shall determine the structure of the double centralizer of primary modules over bounded Dedekind prime rings. In Section 2, we shall give a characterization of quasi-injective modules over bounded Dedekind prime rings. This paper is a continuation of [7] and [8]. A number of concepts and results are needed from [7] and [8].

1. The double centralizer of torsion modules

Throughout this paper, R will denote a Dedekind prime ring with the two-sided quotient ring Q, we denote the completion of R with respect to P by $\hat{R}_P$ and its maximal ideal by $\hat{P}$. By Theorem 1. 1 of [6], $\hat{R}_P$ is a complete, g-discrete valuation ring in the sense of [8] and $\hat{R}_P=(\hat{L})_k$, where $\hat{L}$ is a complete, discrete valuation ring with unique maximal ideal $\hat{P}_0$. Further, $\hat{P}=p_0\hat{R}_P=\hat{R}_Pp_0$, where $p_0\in\hat{L}$ with $\hat{P}_0=p_0\hat{L}=\hat{L}p_0$. Since the proper ideals of $\hat{R}_P$ are only the powers of $\hat{P}$, we obtain $\hat{P}^n=\hat{R}_PP^n\hat{R}_P$ for $n=0, 1, 2, \cdots$ (cf. the proof of Theorem 4.5 of [4]). In this section we denote the complete set of the martix units of $\hat{R}_P=(\hat{L})_k$ by e_{ij} $(i, j=1, 2, \cdots, k)$.

Let M be a P-primary module. Then, by the same way as in Lemma 3.14 of [7], M is an $\hat{R}_P$-module by a natural way. It is evident that $\mathrm{Hom}_R(M, M)=\mathrm{Hom}_{\hat{R}_P}(M, M)$ and that M is torsion as an $\hat{R}_P$-module. If M is indecomposable, P-primary and divisible, then M is isomorphic to $\varinjlim e_{11}\hat{R}_P/e_{11}\hat{P}^n$, and we denote it by $R(P^\infty)$. If M is indecomposable, P-primary with $O(M)=P^n$, then M is isomorphic to $e_{11}\hat{R}_P/e_{11}\hat{P}^n$, and we denote it by $R(P^n)$.

Lemma 1.1. *Let R be a Dedekind prime ring. Then the double centralizer D_n of the module $R(P^n)$ is isomorphic to $\hat{R}_P/\hat{P}^n$.*

Proof. By Lemma 3.20 of [7], $L_n=\mathrm{Hom}_R(R(P^n), R(P^n))$, where $L_n=\hat{L}/\hat{P}_0^n$. Hence we have

$$R(P^n) = L_n(e_{11}+e_{11}\hat{P}^n)+\cdots+L_n(e_{1k}+e_{11}\hat{P}^n)\,.$$

From this the assertion is evident.

Lemma 1.2. *Let R be a Dedekind prime ring. Then the double centralizer D of the module $R(P^\infty)$ is isomorphic to $\hat{R}_P$.*

Proof. It is clear that $R(P^\infty)$ is faithful as an $\hat{R}_P$-module. Hence $D\supseteqq\hat{R}_P$. Let d be any nonzero element of D. Then $p_0^n[(e_{11}\hat{R}_P/e_{11}\hat{P}^n)d]=0$, because $\mathrm{Hom}_R(R(P^\infty), R(P^\infty))=e_{11}\hat{R}_Pe_{11}$ (cf. Theorem 3.21 of [7]). Therefore we may assume that $d_n=d\,|\,e_{11}\hat{R}_P/e_{11}\hat{P}^n=r_n$ ($r_n\in\hat{R}_P$) by Lemma 1.1, where $|$ means the restriction and r_n is unique up to mod $\hat{P}^n$. Since $R(P^\infty)$ is injective, the natural homomorphism $e_{11}\hat{R}_P/e_{11}\hat{P}^{n+1}\rightarrow e_{11}\hat{R}_P/e_{11}\hat{P}^n$ can be extended to a map $\varphi_n\colon R(P^\infty)\rightarrow R(P^\infty)$. Because

$$\begin{aligned}(e_{11}\hat{R}_P/e_{11}\hat{P}^n)r_n &= [\varphi_n(e_{11}\hat{R}_P/e_{11}\hat{P}^{n+1})]\,d = \varphi_n[(e_{11}\hat{R}_P/e_{11}\hat{P}^{n+1})\,d]\\ &= (e_{11}\hat{R}_P/e_{11}\hat{P}^n)\,r_{n+1}\,,\end{aligned}$$

we have $r_n-r_{n+1}\in\hat{P}^n$. Therefore $\hat{r}=(\cdots, r_n+\hat{P}^n, \cdots)\in\hat{R}_P$ and it is easily seen that $d=\hat{r}$.

Lemma 1.3. *Let S be a g-discrete valuation ring with unique maximal ideal P (cf. [8]). Assume that B is a submodule of the torsion S-module M and that $B=\sum_n\oplus B_n$, where B_n is a direct sum of cocyclic modules of order P^n. Then B is a basic submodule of M if and only if*

$$M = B_1\oplus\cdots\oplus B_n\oplus(B_n^*+MP^n) \qquad \textit{for every } n\,,$$

where $B_n^*=B_{n+1}\oplus B_{n+2}\oplus\cdots$ (cf. Theorem 32.4 of [2]).

In the case of indecomposable, injective and P-primary modules the following theorem was proved by Kuzmanovich [6].

Theorem 1.4. *Let R be a Dedekind prime ring, let M be a P-primary module and let D be the double centralizer of M. Then*

(a) *If $O(M)=P^n$, then $D\cong\hat{R}_P/\hat{P}^n$.*

(b) *If M is faithful, then $D\cong\hat{R}_P$.*

Proof. We may assume without loss of generality that R is a complete, g-discrete valuation ring with unique maximal ideal P. Let $H=\mathrm{Hom}_R(M, M)$ and $D=\mathrm{Hom}_H(M, M)$.

(a) It is evident that $D\supseteqq R/P^n$. By Theorems 3.7 and 3.38 of [7], $M=\sum\oplus e_iM$, where $e_iM\cong R(P^{n_i})$ and e_i is an idempotent in $\mathrm{Hom}_R(M, M)$. Since $O(M)=P^n$, there is $e_{i_0}\in H$ such that $O(e_{i_0}M)=P^n$. Let d be any element of D. Then $(e_{i_0}M)d=e_{i_0}(Md)\subseteqq e_{i_0}M$. Thus, by Lemma 1.1, $d_{i_0}=d\,|\,e_{i_0}M=r$, where $r\in R$ and it is unique up to mod P^n. Now, for any direct summand

e_iM, there exists $\varphi_i \in H$ such that $\varphi_i(e_{i_0}M)=e_iM$. Let u be any element of e_iM. Then $ud=\varphi_i(v)d=\varphi_i(vd)=\varphi_i(vr)=ur$, and thus we obtain $d=r$, as desired.

(*b*) It is evident that $D \supseteqq R$. To prove the converse inclusion, let d be any nonzero element of D.

Case I. If M is divisible, then $M=\sum \oplus M_i$, where $M_i=R(P^\infty)$. Let π_i be the projection map from M to M_i. Then $M_id=(\pi_iM)d=\pi_i(Md)\subseteqq M_i$. Therefore, by Lemma 1.2, $d_i=d\,|\,M_i=r_i$, where $r_i \in R$. For any i, j, there is an element $\varphi_{ij}\in H$ such that $\varphi_{ij}(M_i)=M_j$. Let y be any element of M_j and let $\varphi_{ij}(x)=y(x\in M_i)$. Then $yr_j=yd=[\varphi_{ij}(x)]d=\varphi_{ij}(xd)=yr_i$. Thus we have $r_i=r_j$. and so $d=r$ for some $r\in R$.

Case. II. If M is reduced, then, it is evident that $B_n^* \neq 0$ for every natural integer n, where B_n^* is defined in Lemma 1.3. Hence we have submodules $\{M_i\}$ with the following properties:

(1) $M_i=R(P^{n_i})$, where $n_1<n_2<\cdots$,

(2) $M_i=e_iM$, where e_i is an idempotent element of H. Then $(e_iM)d=e_i(Md)\subseteqq e_iM$ and $H\supseteqq \mathrm{Hom}(e_iM, e_iM)$. Hence $d_i=d\,|\,M_i=r_i$ by Lemma 1.2, where $r_i\in R$ and r_i is unique up to mod P^{n_i}. For any i, j $(j>i)$, there is an element $e_{ji}\in H$ such that $e_{ji}(M_j)=M_i$. Now let x be any element of e_jM. Then we have

$$(e_{ji}x)r_i=(e_{ji}x)d=e_{ji}(xd)=e_{ji}(xr_j)=(e_{ji}x)r_j\,.$$

Hence $r_i-r_j\in P^{n_i}$, and so $\hat{r}=(\cdots, r_l+P^l, \cdots)\in R$, where $r_l=r_i$ $(n_{i-1}<l\leqq n_i)$. It is evident that $d_i=\hat{r}$ for every i. Let u be any uniform element of M. Then $uR\cong R(P^l)$ for some l by Lemma 3.37 of [7]. So there is $\theta_i\in H$ such that θ_i maps e_iM onto uR. Let $\theta_i(e_iy)=u$, where $y\in M$. Then we obtain

$$ud=[\theta_i(e_iy)]d=\theta_i[(e_iy)d]=\theta_i[(e_iy)\hat{r}]=u\hat{r}\,.$$

Let m be any element of M. Then, by Theorem 3.38 of [7], mR is a direct sum of a finite number of reduced cocyclic modules, and so $md=m\hat{r}$, as desired.

Case III. If M is not reduced, then there are idempotent elements $e_1, e_2\in H$ such that $M=e_1M\oplus e_2M$, where e_1M is divisible and e_2M is reduced. First we assume that e_2M is not bounded, then, by Cases I, II, there exist $r_1, r_2\in R$ such that $d_i=r_i$, where $d_i=d\,|\,e_iM$ $(i=1, 2)$. Let u be any uniform element in e_1M. Then there is $\varphi\in H$ such that $\varphi(e_2M)=uR$, because e_2M contains a reduced, cocyclic direct summand U such that $O(U)\subseteqq O(uR)$. Let $\varphi(x)=u$, where $x\in e_2M$. Then we have

$$ur_1=ud=[\varphi(x)]d=\varphi(xd)=\varphi(xr_2)=ur_2\,.$$

Therefore $r_1=r_2$. Second assume that e_2M is of bounded order. By Case I, there is $r_1\in R$ such that $d_1=d\,|\,e_1M=r_1$ and $e_2M=\sum \oplus N_i$ by Theorem 3.7 of [7], where $N_i=R(P^{n_i})$. For each i, there is $\theta_i\in H$ such that it induces a monomorphism from N_i to e_1M. Let u be any element of N_i and let $\theta_i(u)=x\in e_1M$. Then we obtain

$$\theta_i(ur_1) = xd = [\theta_i(u)]d = \theta_i(ud) .$$

Hence $ur_1=ud$, and thus we have $r_1=d$. This completes the proof of Theorem 1.4.

Corollary 1.5. *Let R be a bounded Dedekind prime ring, let M be a torsion module and let $M=\sum \oplus M_P$ be the primary decomposition of M* (cf. Theorem 3.2 of [7]). *Then the double centralizer D of M is isomorphic to $\Pi\, \hat{R}_P/\hat{P}^{n_p}$, where $O(M_P)=P^{n_p}$, n_p is a natural integer or ∞ and $\hat{P}^{\infty}=0$.*

Proof. Let $\alpha=(r_p+\hat{P}^{n_p})$ be any element of $\Pi\, \hat{R}_P/\hat{R}^{n_p}$, where $r_p\in\hat{R}_P$ and let $m=\sum m_{pi}$ be any element of M, where $m_{pi}\in M_{P_i}$. Define $m\alpha=\sum m_{pi}r_{pi}$. By Theorem 1.4, it is easily seen that $\alpha\in D$. Conversely let d be any element of D. Since $M_Pd\subseteq M_P$, we have $d_p=r_p+\hat{P}^{n_p}$, where $d_p=d|M_P$. Then it is evident that $d=(r_p+\hat{P}^{n_p})$.

2. Quasi-injective modules

Let R be a bounded Dedekind prime ring and let Q be the quotient ring of R. In [7], the author proved that any injective module is a direct sum of minimal right ideals of Q and modules of type P^{∞} for various prime ideals P.

In this section, we shall characterize quasi-injective modules. By virtue of Goldie's theorem, $Q=(F)_k$, where F is a division ring. Throughout this section we denote a complete martix units of $Q=(F)_k$ by e_{ij}.

Lemma 2.1. *If a module $M=\sum \oplus M_\alpha$ and if N is a fully invariant submodule of M, then $N=\sum \oplus(M_\alpha\cap N)$* (cf. Lemma 9.3 of [3]).

Theorem 2.2 *Let R be a bounded Dedekind prime ring and let M be a module. Then M is quasi-injective if and only if it is;*

(i) *injective, or*

(ii) *a torsion module such that every P-primary component M_P is a direct sum of isomorphic cocyclic modules.*

Proof. The sufficiency easily follows from Theorem 1.1 of [5] and Proposition 1.1 of [8].

Conversely assume that M is quasi-injective. Then the injective envelope $E(M)$ of M is isomorphic to $\sum \oplus\bar{M}_\alpha$, where $\bar{M}_\alpha$ is a minimal right ideal of Q or a module of type P^{∞}. By Lemma 2.1 and Theorem 1.1 of [5], we have $M=\sum \oplus M_\alpha$, where $M_\alpha=\bar{M}_\alpha\cap M$.

Case I. If M is torsion-free then we may assume that $\bar{M}_\alpha=e_{11}Q$ for all α. Assume that M is not injective, then there is M_α such that $M_\alpha\subsetneqq\bar{M}_\alpha=e_{11}Q$. By virtue of Faith-Utumi's Theorem (cf. Theorem 6 of [1], p. 91] there is an Ore domain D such that

$$S = \sum_{i,j=1}^{k} De_{ij} \subseteq R \subseteq Q = (F)_k ,$$

and F is the quotient division ring of D. Now let

$$U = \left\{ \begin{pmatrix} d_{11}, \cdots, d_{1k} \\ 0 \end{pmatrix} \middle| d_{1i} \in D \right\}.$$

Since U is a uniform right ideal of S and Q is a quotient ring of S, we have $0 \neq M_\alpha U$. Hence there exists an element $u_\alpha \in M_\alpha$ such that $0 \neq u_\alpha U \cong U$ as an S-module. Let q be any element of $\bar{M}_\alpha (= e_{11}Q)$. Then there is an element $d \in D$ such that $dq = v \in U$, because D is an Ore domain. It is clear that $O(v) = O(q)$. Since $u_\alpha U \cong U$, there exists an element $u \in U$ such that $O(u_\alpha u) = O(v)$. The map $\theta : u_\alpha uR \rightarrow qR$ defined by $u_\alpha ur \rightarrow qr$, for $r \in R$, can be extended to the map $\bar{\theta} : \bar{M}_\alpha \rightarrow \bar{M}_\alpha$. Since $\bar{\theta}(M) \subseteq M$ and $\bar{\theta}(u_\alpha u) = q \in M$, we have $\bar{M}_\alpha = M_\alpha$, which is a contradiction. Therefore M is injective.

Case II. If M is torsion, then $M = \sum \oplus M_P$, where M_P is the P-primary part of M and M_P is also quasi-injective. Hence we may assume that M is P-primary, quasi-injective and that $M = \sum \oplus M_\alpha$, where $M_\alpha = R(P^{n_\alpha})$ ($n_\alpha = 1, 2, \cdots$, or ∞). If $M_\alpha = R(P^n)$ and $M_\beta = R(P^m)$ for $\alpha \neq \beta$, where $\infty \geqq n > m$, then there exists a monomorphism $\varphi : M_\alpha \rightarrow \bar{M}_\beta$ $(= R(P^\infty))$, and it can be extended to an isomorphism $\bar{\varphi} : \bar{M}_\alpha \rightarrow \bar{M}_\beta$. It is clear that $\bar{\varphi}(M_\alpha) \subseteq \bar{M}_\beta \cap M = M_\beta$. This is a contradiction, and thus $m = n$.

Case III. If M is mixed, then since $E(M) = \bar{C} \oplus \bar{T}$, where $\bar{C}$ is torsion-free and $\bar{T}$ is the torsion part of $E(M)$, we obtain $M = C \oplus T$, where $C = \bar{C} \cap M$ and $T = \bar{T} \cap M$. By Case I, $C = \sum \oplus e_{11}Q$ and, by Case II, $T = \sum \oplus T_P$, $T_P = \sum \oplus R(P^{n_p})$ for fixed n_p, where T_P is the P-primary part of T and n_p is a natural integer or ∞. Now assume that M is not injective, then there exists a prime ideal P such that T_P is not injective, i.e., n_p is a natural integer. Consider the module $e_{11}R/e_{11}P^m$ for a fixed m $(> n_p)$. By Theorem 3.7 of [7], $e_{11}R/e_{11}P^m$ contains $R(P^m)$ as a direct summand. Hence there exists a map η such that $e_{11}R \xrightarrow{\eta} R(P^m) \rightarrow 0$ is exact. It can be extended to a map $\bar{\eta} : e_{11}Q \rightarrow R(P^\infty)$. Thus we have $R(P^m) \subseteq \bar{\eta}(e_{11}Q) \subseteq M$, which is a contradiction.

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