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# CHARACTERISTIC CLASSES WITH VALUES IN COMPLEX COBORDISM

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## Introduction

This paper is concerned with the characteristic classes for complex bundles with values in the complex cobordism $U^*(\cdot)$. These are the dual Chern classes $\bar{c}^R$, the Wu classes $u^R$ and the classes $q$ corresponding to the power operations $P$. On these classes with values in the classical cohomology, Haefliger and Wu have proved some interesting theorems in [11], [19], [20]. The aim of this paper is to show the complex cobordism version of their theorems.

Quillen [17] has given a formula relating the power operation $P$ to the Landweber-Novikov operations $s^R$, and a formula relating the class $q$ to the Chern classes $c^R$. These formulae play a central role in this paper.

The layout of this paper is as follows.

§1 contains a recall of the Landweber-Novikov operations and the conjugation in Hopf algebras. In §2 we consider the dual Chern classes $\bar{c}^R(\xi)$ and the Wu classes $u^R(\xi)$ of a complex bundle $\xi$ in connection with the Landweber-Novikov operations $s^R$ and their conjugations $\bar{s}^R$. §3 is devoted to the dual Chern classes $\bar{c}^R(M)$ and the Wu classes $u^R(M)$ of a weakly complex manifold $M$, with which a Riemann-Roch type theorem is proved along the line of Atiyah-Hirzebruch [4]. We have in particular the following formula which may be regarded as a complex cobordism version of the formulae in Wu [19], [20]:

$$\langle s^R\alpha, [M]\rangle = \sum_{I+J=R} s^I\langle\alpha\cdot u^J(M), [M]\rangle,$$

where $\alpha\in U^*(M)$ and $[M]\in U_*(M)$ is the fundamental class of $M$.

In §4 and §5, we consider the power operations $P$ and the corresponding characteristic classes $q$, and give a proof of the formulae due to Quillen.

In §6 an element $\Delta\in U^*(E_G\underset{G}{\times} M^p)$ is defined after Haefliger [11] for a closed almost complex manifold $M$, where $E_G$ is the universal $G$-bundle for a cyclic group $G$ of order $p$ (prime). We prove a formula connecting $\Delta$ to $u^R(M)$ in terms of $P$, which may be regarded as a complex cobordism version of Theorem 3.2 in Haefliger [11].

§7–§9 are concerned with immersions and imbeddings of closed almost complex manifolds. In §8 we prove a complex cobordism version of the theorem of Haefliger [11] on immersions and imbeddings. This is given in a form of integrality condition in localization. In §9, this is converted to a theorem given in terms of $K$-theory, and is employed to give another proof of the results due to Atiyah-Hirzebruch [5] and Sanderson-Schwarzenberger [18] on non-imbeddability and non-immersibility of complex projective spaces in Euclidean spaces. This fact make us expect that the theorems of §8 would yield better results on immbedding and immersion problem if we could manage well the complex cobordism theory, but I am not successful.

## 1. Landweber-Novikov operations

We shall consider the complex cobordism theory, that is, the generalized cohomology theory with values in the Milnor spectrum $MU$(see [2]). We denote by $U^*(X, A)$ the complex cobordism of a $CW$ pair $(X, A)$.

We shall first recall some facts on characteristic classes and cohomology operations in the complex cobordism theory from Landweber [12] and Novikov [16] (see also [1]).

Let $\mathcal{S}^*$ denote the $\boldsymbol{Z}$-algebra (under composition) of stable cohomology operations of complex cobordism, and $\mathcal{C}^*$ the $\boldsymbol{Z}$-algebra of stable characteristic classes of complex bundles with values in complex cobordism. Each of these contains $U^*(pt)$ naturally as a subalgebra. An isomorphism $\psi\colon \mathcal{S}^*\cong\mathcal{C}^*$ of graded modules can be defined by

$$\psi(\tau)(\xi)=\phi_\xi^{-1}\tau\phi_\xi(1)\in U^*(X),$$

where $\tau\in\mathcal{S}^*$, $\xi$ is a complex bundle over $X$, and $\phi_\xi$ is the Thom isomorphism of $\xi$ in complex cobordism. Later on $\psi(\tau)(\xi)$ will be denoted by $\psi(\tau, \xi)$.

Let $R=(r_1, r_2, \cdots)$ be a sequence of non-negative integers which are almost all zero, and $\mathcal{R}$ be the set of such sequences. We put

$$|R|=\sum_i r_i, \qquad ||R||=\sum_i i r_i.$$

For $I=(i_1, i_2, \cdots)$, $J=(j_1, j_2, \cdots)\in\mathcal{R}$, we define

$$I+J=(i_1+j_1, i_2+j_2, \cdots)\in\mathcal{R}.$$

We write $O=(0, 0, \cdots, 0, \cdots)$.

Consider the elementary symmetric functions $\sigma_1, \sigma_2, \cdots$ in a sufficiency of variables $t_1, t_2, \cdots, t_n$, and define for each $R\in\mathcal{R}$ a polynomial $f_R$ by

$$f_R(\sigma_1, \sigma_2, \cdots)=\sum t_1^{m_1}t_2^{m_2}\cdots t_n^{m_n},$$

where the sum runs over $n$-tuples $(m_1, m_2, \cdots, m_n)$ such that $r_1$ of the $m$'s are 1, $r_2$

of the $m$'s are 2, and so on, while the rest of the $m$'s are 0.

Given a complex bundle $\xi$ over $X$, we define the *Chern class* $c^R(\xi)\in U^{2\|R\|}(X)$ by

$$c^R(\xi)=f_R(c_1(\xi),\ c_2(\xi),\ \cdots)\,,$$

where $c_i(\xi)\in U^{2i}(X)$ are the characteristic classes of Conner-Floyd [7]. Since $c^R\in\mathcal{C}^*$, the cohomology operation $s^R\in\mathcal{S}^*$ of degree $2\|R\|$ is defined by

$$(1.1)\qquad \psi(s^R)=c^R \qquad\text{or}\qquad c^R(\xi)=\psi(s^R,\,\xi)\,.$$

$s^R$ is called the *Landweber-Novikov operation.* It holds that

$$(1.2)\qquad c^R(\xi\oplus\eta)=\sum_{I+J=R}c^I(\xi)c^J(\eta)\,,$$

$$(1.3)\qquad s^R(\alpha\beta)=\sum_{I+J=R}s^I\alpha\cdot s^J\beta\,.$$

Let $S^*\subset\mathcal{S}^*$ denote the submodule generated by all $s^R$. Then $\{s^R\}_{R\in\mathcal{R}}$ is a basis of the module $S^*$, and $S^*$ is a subalgebra of $\mathcal{S}^*$. Furthermore $S^*$ is a connected Hopf algebra with a commutative coproduct $\psi$: $S^*\to S^*\otimes S^*$ defined by

$$\psi(s^R)=\sum_{I+J=R}s^I\otimes s^J\,.$$

The Hopf algebra $S^*$ is called the *Landweber-Novikov algebra.*

Next we shall recall the following result due to Milnor-Moore [15]. Let $A$ be a connected Hopf algebra with commutative coproduct. Then there is associated to each $a\in A$ an element $\bar a\in A$ so as to satisfy the following properties:

i) $\deg\bar a=\deg a$, ii) $\bar 1=1$,

iii) $\bar{\bar a}=a$, iv) $\overline{a+b}=\bar a+\bar b$,

v) $\overline{a\,b}=(-1)^{\deg a\deg b}\,\bar b\,\bar a$,

vi) if $\psi(a)=\sum_i a_i'\otimes a_i''$ for the coproduct $\psi$, then

$$\sum_i a_i'\bar a_i''=\begin{cases}0 & (\deg a>0)\,,\\ a & (\deg a=0)\,.\end{cases}$$

The element $\bar a$ is called the *conjugation* of $a$.

We shall denote by $\bar s^R$ the conjugation of $s^R$ in the Landweber-Novikov algebra $S^*$. It follows that

$$(1.4)\qquad \sum_{I+J=R}s^I\bar s^J=\sum_{I+J=R}\bar s^I s^J=\begin{cases}0 & (R\neq 0)\,,\\ id & (R=0)\,.\end{cases}$$

We have also

$$(1.5)\qquad \bar s^R(\alpha\beta)=\sum_{I+J=R}\bar s^I\alpha\cdot\bar s^J\beta\,.$$

This is proved as follows by induction. To do this we introduce an order in $\mathcal{R}$ such that $R<R'$ if $|R|<|R'|$. Since (1.5) is obvious if $R=0$, we assume $R\neq 0$. Then (1.3) and (1.4) imply

$$\bar{s}^R(\alpha\beta) = -\sum_{\substack{I+K+L=R\\ I\neq R}} \bar{s}^I(s^K\alpha\cdot s^L\beta)\,.$$

Since $I<R$, we have inductively

$$\begin{aligned}\bar{s}^R(\alpha\beta) &= -\sum_{\substack{I+K+L=R\\ I\neq R}}\ \sum_{P+Q=I} \bar{s}^P s^K\alpha\cdot\bar{s}^Q s^L\beta\\ &= -\sum_{I+J=R}\Big(\sum_{P+K=I}\bar{s}^P s^K\alpha\Big)\Big(\sum_{Q+L=J}\bar{s}^Q s^L\beta\Big)+\sum_{P+Q=R}\bar{s}^P\alpha\cdot\bar{s}^Q\beta\\ &= \sum_{P+Q=R}\bar{s}^P\alpha\cdot\bar{s}^Q\beta.\end{aligned}$$

## 2. Wu classes and the dual Chern classes

Corresponding to (1.1) we put

$$\bar{u}^R(\xi) = \psi(\bar{s}^R, \xi)\in U^{2\|R\|}(X)$$

for a complex bundle $\xi$ over $X$. We have

$$\bar{u}^R(\xi\oplus\eta) = \sum_{I+J=R}\bar{u}^I(\xi)\bar{u}^J(\eta)\,, \tag{2.1}$$

which is shown as follows (compare [10], Appendix 2).

Let $D(\xi)$, $S(\xi)$ denote respectively the disc bundle, the sphere bundle associated to $\xi$, and $\pi_\xi$: $D(\xi)\to X$ the projection. Then, for the Thom isomorphism $\phi_\xi$: $U^*(X)\cong U^*(D(\xi), S(\xi))$, we have $\phi_\xi(\alpha)=\pi_\xi^*(\alpha)\cdot\phi_\xi(1)$. Therefore it follows

$$\begin{aligned}&\phi_{\xi\times\eta}(\psi(\bar{s}^I, \xi)\times\psi(\bar{s}^J, \eta))\\ &= \pi_{\xi\times\eta}^*(\psi(\bar{s}^I, \xi)\times\psi(\bar{s}^J, \eta))\cdot\phi_{\xi\times\eta}(1)\\ &= (\pi_\xi^*\psi(\bar{s}^I, \xi)\times\pi_\eta^*\psi(\bar{s}^J, \eta))\cdot(\phi_\xi(1)\times\phi_\eta(1))\\ &= \pi_\xi^*\psi(\bar{s}^I, \xi)\cdot\phi_\xi(1)\times\pi_\eta^*\psi(\bar{s}^J, \eta)\cdot\phi_\eta(1)\\ &= \bar{s}^I\phi_\xi(1)\times\bar{s}^J\varphi_\eta(1)\,.\end{aligned}$$

Consequently we have

$$\begin{aligned}&\phi_{\xi\oplus\eta}\Big(\sum_{I+J=R}\psi(\bar{s}^I, \xi)\cdot\psi(\bar{s}^J, \eta)\Big)\\ &= \phi_{\xi\oplus\eta}d^*\Big(\sum_{I+J=R}\psi(\bar{s}^I, \xi)\times\psi(\bar{s}^J, \eta)\Big)\\ &= d^*\phi_{\xi\times\eta}\Big(\sum_{I+J=R}\psi(\bar{s}^I, \xi)\times\psi(\bar{s}^J, \eta)\Big)\\ &= d^*\sum_{I+J=R}\bar{s}^I\phi_\xi(1)\times\bar{s}^J\phi_\eta(1)\\ &= \sum_{I+J=R}\bar{s}^I\phi_\xi(1)\cdot\bar{s}^J\phi_\eta(1)\end{aligned}$$

$$\begin{aligned}&= \bar{s}^R(\phi_\xi(1)\cdot\phi_\xi(1)) \qquad \text{by (1.5)}\\ &= \bar{s}^R\phi_{\xi\oplus\eta}(1)\\ &= \phi_{\xi\oplus\eta}\psi(\bar{s}^R, \xi\oplus\eta)\,,\end{aligned}$$

where $d^*$ is induced by the diagonal map. Since $\phi_{\xi\oplus\eta}$ is an isomorphism, we get (2.1).

For a complex bundle $\xi$ over $X$ and $R\in\mathcal{R}$, we define the *Wu class* $u^R(\xi)\in U^{2\|R\|}(X)$ and the *dual Chern class* $\bar{c}^R(\xi)\in U^{2\|R\|}(X)$ by

$$\begin{aligned}u^R(\xi) &= \sum_{I+J=R} \bar{s}^I c^J(\xi)\,,\\ \bar{c}^R(\xi) &= \sum_{I+J=R} s^I \bar{u}^J(\xi)\,.\end{aligned} \tag{2.2}$$

Obviously $u^R$, $\bar{c}^R\in\mathcal{C}^*$, and it follows from (1.4) that

$$\begin{aligned}c^R(\xi) &= \sum_{I+J=R} s^I u^J(\xi)\,,\\ \bar{u}^R(\xi) &= \sum_{I+J=R} \bar{s}^I\bar{c}^J(\xi)\,.\end{aligned} \tag{2.3}$$

Moreover it follows from (1.2), (1.5) that

$$u^R(\xi\oplus\eta) = \sum_{I+J=R} u^I(\xi)u^J(\eta)\,, \tag{2.4}$$

and from (1.3), (2.1) that

$$\bar{c}^R(\xi\oplus\eta) = \sum_{I+J=R} \bar{c}^I(\xi)\bar{c}^J(\eta)\,. \tag{2.5}$$

We have also

$$\sum_{I+J=R} u^I(\xi)\bar{u}^J(\xi) = \begin{cases}0 & (R\neq O)\,,\\ 1 & (R=O)\,;\end{cases} \tag{2.6}$$

$$\sum_{I+J=R} c^I(\xi)\bar{c}^J(\xi) = \begin{cases}0 & (R\neq O)\,,\\ 1 & (R=O)\,.\end{cases} \tag{2.7}$$

In fact, it follows from (1.5), (1.4) that

$$\begin{aligned}&\phi_\xi\Big(\sum_{I+J=R} u^I(\xi)\bar{u}^J(\xi)\Big)\\ &= \phi_\xi\Big(\sum_{K+L+J=R} \bar{s}^K(\psi(s^L, \xi))\cdot\psi(\bar{s}^J, \xi)\Big)\\ &= \sum_{K+L+J=R} \pi_\xi^*\bar{s}^K\phi_\xi^{-1}s^L\phi_\xi(1)\cdot\pi_\xi^*\phi_\xi^{-1}\bar{s}^J\phi_\xi(1)\cdot\phi_\xi(1)\\ &= \sum_{K+L+J=R} \bar{s}^K\pi_\xi^*\phi_\xi^{-1}s^L\phi_\xi(1)\cdot\bar{s}^J\phi_\xi(1)\\ &= \sum_{M+L=R} \bar{s}^M(\pi_\xi^*\phi_\xi^{-1}s^L\phi_\xi(1)\cdot\phi_\xi(1))\\ &= \sum_{M+L=R} \bar{s}^M s^L\phi_\xi(1)\end{aligned}$$

$$= \begin{cases} 0 & (R \neq O), \\ \phi_\xi(1) & (R = O). \end{cases}$$

This proves (2.6). Similar for (2.7).

By (2.4) and (2.6) and by (1.2) and (2.7), we have

**Lemma 1.** *If $\xi$ and $\eta$ are complex bundles such that $\xi \oplus \eta$ is trivial, then*

$$\bar{c}^R(\xi) = c^R(\eta), \qquad u^R(\xi) = \bar{u}^R(\eta).$$

The following relations can be proved by the argument similar to the proof of (2.1).

$$\text{(2.8)} \qquad \begin{aligned} \phi_\xi^{-1} s^R \phi_\xi(\alpha) &= \sum_{I+J=R} s^I \alpha \cdot c^J(\xi), \\ \phi_\xi^{-1} \bar{s}^R \phi_\xi(\alpha) &= \sum_{I+J=R} \bar{s}^I \alpha \cdot \bar{u}^J(\xi). \end{aligned}$$

REMARK 1. Let $C^* \subset \mathcal{C}^*$ be the subalgebra generated by $c_i$ ($i=0, 1, 2, \cdots$). Then $\psi$ gives rise to an isomorphism $S^* \cong C^*$ of modules. We see $c^R$, $\bar{u}^R \in C^*$, and hence $\bar{c}^R$, $u^R \in C^*$ by (2.6) and (2.7).

REMARK 2. For a prime $p$, let $\mu_p$: $U^*(\cdot) \to H^*(\cdot\,; \boldsymbol{Z}_p)$ be the natural transformation. Let $i\Delta(j) \in \mathcal{R}$ be a sequence with $i$ in the $j$-th place and zero elsewhere. Then $s^{i\Delta(p-1)}$ corresponds to $\mathcal{P}^i$ or $Sq^{2i}$ according as $p > 2$ or $p=2$ under $\mu_p$ (see [12], p. 107). Therefore $\mu_2$ sends $u^{i\Delta(1)}$ to the classical Wu class $U^i_{(2)}$, and $\bar{c}^{i\Delta(1)}$ to the dual Stiefel-Whitney class $\bar{W}^{2i}$. Similarly $\mu_p$ ($p>2$) sends $u^{i\Delta(p-1)}$ to $U^i_{(p)}$, and $\bar{c}^{i\Delta(p-1)}$ to $\bar{Q}^i$ (see [11] for the notations).

## 3. Riemann-Roch type theorem

Let $M$ be a weakly complex manifold. Then the stable tangent bundle $\tau$ is endowed with the complex structure. We write $u^R(M)$ for $u^R(\tau)$, and call it the Wu class of $M$. Similar for $c^R(\tau)$ and $\bar{c}^R(\tau)$.

The following Riemann-Roch type theorem holds.

**Theorem 1.** *Let $M$ and $N$ be closed weakly complex manifolds, and $f$: $M \to N$ be a continuous map. Then, for the Gysin homomorphism $f_!$: $U^i(M) \to U^{i+n-m}(N)$ ($m=\dim M$, $n=\dim N$), we have*

$$\begin{aligned} \sum_{I+J=R} s^I f_! \alpha \cdot \bar{c}^J(N) &= \sum_{I+J=R} f_!(s^I \alpha \cdot \bar{c}^J(M)), \\ \sum_{I+J=R} \bar{s}^I f_! \alpha \cdot u^J(N) &= \sum_{I+J=R} f_!(\bar{s}^I \alpha \cdot u^J(M)) \end{aligned}$$

*for* $\alpha \in U^i(M)$.

Proof (compare [10], Theorem 10). Take a differentiable imbedding $i$ of $M$ into the interior of the $k$-dimensional disc $D^k$ such that the imbedding $(f, i)$:

$M \to N \times D^k$ is homotopic to a differentiable imbedding $\tilde{f}\colon M \to N \times D^k$, where $k$ is a sufficiently large interger such that $n+k-m$ is even. The normal bundle $\nu(\tilde{f})$ of the imbedding $\tilde{f}$ is endowed with the complex structure. Consider the collapsing map $c$ of the Thom complex $T(\boldsymbol{k}) = N \times D^k / N \times S^{k-1}$ to the Thom complex $T(\nu(\tilde{f}))$, where $\boldsymbol{k}$ denotes the real $k$-dimensional trivial bundle over $N$. By definition $f_!$ is the composite

$$U^i(M) \xrightarrow{\phi_{\nu(\tilde{f})}} \tilde{U}^{i+n+k-m}(T(\nu(\tilde{f}))) \\ \xrightarrow{c^*} \tilde{U}^{i+n+k-m}(T(\boldsymbol{k})) \xrightarrow{\phi_k^{-1}} U^{i+n-m}(N)\,.$$

Take a differentiable imbedding $j$ of $N$ into the interior of $D^l$, where $l$ is a sufficiently large integer such $l-n$ is even. Let $\nu(M)$ be the normal bundle of the imbedding

$$M \xrightarrow{\tilde{f}} N \times D^k \xrightarrow{j \times id} D^l \times D^k\,,$$

and $\nu(N)$ the normal bundle of the imbedding $j$. Then it follows that

$$\nu(M) \cong \nu(\tilde{f}) \oplus \nu(N)$$

as complex bundles. Therefore we have the following commutative diagram:

$$\begin{array}{ccccccc} & & \tilde{U}^*(T(\nu(\tilde{f})\oplus\nu(N))) & \xrightarrow{c^*} & \tilde{U}^*(T(\boldsymbol{k}\oplus\nu(N))) & \xleftarrow{\phi_k} & \tilde{U}^*(T(\nu(N))) \\ & \nearrow^{\phi_{\nu(M)}} & \uparrow{\scriptstyle \phi_{\nu(N)}} & & \uparrow{\scriptstyle \phi_{\nu(N)}} & & \uparrow{\scriptstyle \phi_{\nu(N)}} \\ U^*(M) & & & & & & \\ & \searrow_{\phi_{\nu(\tilde{f})}} & \tilde{U}^*(T(\nu(\tilde{f}))) & \xrightarrow{c^*} & \tilde{U}^*(T(\boldsymbol{k})) & \xleftarrow{\phi_k} & U^*(N) \end{array}$$

(see [6], p. 97). Thus we have

$$f_! = \phi_{\nu(N)}^{-1} \circ \phi_k^{-1} \circ c^* \circ \phi_{\nu(M)}\,. \tag{3.1}$$

Since $\phi_k$ is the iterated suspension, it commutes with $\bar{s}^R$. Therefore it follows from Lemma 1, (2.8) and (3.1) that

$$\begin{aligned} &\sum_{I+J=R} f_!(\bar{s}^I(\alpha)\cdot u^J(M)) = \sum_{I+J=R} f_!(\bar{s}^I(\alpha)\cdot \bar{u}^J(\nu(M))) \\ &= f_!(\phi_{\nu(M)}^{-1}\bar{s}^R\phi_{\nu(M)}(\alpha)) = \phi_{\nu(N)}^{-1}\phi_k^{-1}c^*\bar{s}^R\phi_{\nu(M)}(\alpha) \\ &= \phi_{\nu(N)}^{-1}\bar{s}^R\phi_k^{-1}c^*\phi_{\nu(M)}(\alpha) = \phi_{\nu(N)}^{-1}\bar{s}^R\phi_{\nu(N)}f_!(\alpha) \\ &= \sum_{I+J=R} \bar{s}^I f_!(\alpha)\cdot \bar{u}^J(\nu(N)) = \sum_{I+J=R} \bar{s}^I f_!(\alpha)\cdot u^J(N)\,, \end{aligned}$$

and the second equality has been proved. Similar for the first equality.

Let $U_i(X)$ denote the complex bordism group of a $CW$ complex $X$, and let

$$\langle\ ,\ \rangle\colon U^i(X) \otimes U_j(X) \to U_{j-i}(pt) = U^{i-j}(pt)$$

be the Kronecker product.

**Theorem 2.** *If $M$ is a closed weakly complex manifold, we have*

$$\langle s^R\alpha, [M]\rangle = \sum_{I+J=R} s^I\langle\alpha\cdot u^J(M), [M]\rangle,$$
$$\langle \bar{s}^R\alpha, [M]\rangle = \sum_{I+J=R} \bar{s}^I\langle\alpha\cdot \bar{c}^J(M), [M]\rangle$$

*for $\alpha\in U^*(M)$, where $[M]\in U_*(M)$ is the fundamental class of $M$.*

Proof. Let $c\colon M\to pt$ be the collapsing map. Then it is easily seen that $c_!(\alpha)=\langle\alpha, [M]\rangle$. Therefore the first equality is equivalent to

$$c_! s^R(\alpha) = \sum_{I+J=R} s^I c_!(\alpha\cdot u^J(M)).$$

It follows from Theorem 1 that

$$\bar{s}^R c_!(\alpha) = \sum_{I+J=R} c_!(\bar{s}^I\alpha\cdot u^J(M)).$$

Hence in virtue of (1.4) we have

$$\begin{aligned} c_! s^R(\alpha) &= \sum_{I+P+Q=R} s^I\bar{s}^P c_! s^Q(\alpha) \\ &= \sum_{I+P+Q=R} s^I \sum_{J+K=P} c_!(\bar{s}^K s^Q\alpha\cdot u^J(M)) \\ &= \sum_{I+J+U=R} s^I c_!(\sum_{K+Q=U} \bar{s}^K s^Q\alpha\cdot u^J(M)) \\ &= \sum_{I+J=R} s^I c_!(\alpha\cdot u^J(M)). \end{aligned}$$

This proves the first equality. Similarly we can prove the second equality.

REMARK 1. If $V$ is a closed weakly complex manifold of dimension $i$ and $\nu$ is its stable normal bundle, it is known by Novikov [16] that $s^R$ sends the element of $U^{-i}(pt)=U_i(pt)$ represented by $V$ to $c_*D^{-1}c^R(\nu)$, where $D\colon U_*(V)\cong U^*(V)$ is the Atiyah-Poincaré duality and $c_*\colon U_*(V)\to U_*(pt)$ is induced by the collapsing map (see also [1]).

REMARK 2. With the classical (co)homology, Wu proves

$$\langle Sq^i\alpha, [M]\rangle = \langle\alpha\cdot U^i_{(2)}, [M]\rangle, \qquad (p=2),$$
$$\langle \mathcal{P}^i\alpha, [M]\rangle = \langle\alpha\cdot U^i_{(p)}, [M]\rangle, \qquad (p>2)$$

for $\alpha\in H^*(M; \boldsymbol{Z}_p)$, where $M$ is a closed manifold and is assumed to be oriented if $p>2$. The first formula in Theorem 2 may be regarded as a complex cobordism version of these formulae (see Remark 2 of §2). The classical form of the second formula in Theorem 2 is seen in Massey-Peterson [14].

## 4. The classes $q$

Throughout the remainder of this paper, we denote by $G$ a cyclic group of order $k$, where $k$ is a fixed integer.

Denote by $L$ the complex 1-dimensional $G$-module where the generator multiplies by $\exp(2\pi\sqrt{-1}/k)$, and define a complex $(k-1)$-dimensional $G$-module $\Lambda$ as a linear subspace

$$\{(z_1, z_2, \cdots, z_k)\in \boldsymbol{C}^k;\, z_1+z_2+\cdots+z_k=0\}$$

of $\boldsymbol{C}^k$ on which $G$ acts by the cyclic permutation of coordinates. Let $\rho$ resp. $\lambda$ denote the bundle associated to the universal $G$-bundle $E_G\to B_G$ with fibre $L$ resp. $\Lambda$. Since there is an isomorphism $\Lambda\cong L\oplus L^2\oplus\cdots\oplus L^{k-1}$ of complex $G$-modules, we have an isomorphism

$$\lambda\cong\rho\oplus\rho^2\oplus\cdots\oplus\rho^{k-1} \tag{4.1}$$

of complex bundles.

We shall put

$$v=e(\rho)\in U^2(B_G), \qquad w=e(\lambda)\in U^{2(k-1)}(B_G),$$

where $e$ stands for the Euler class, *i.e.* the top dimensional Chern class.

For a complex $m$-dimensional bundle $\xi$ over a $CW$ complex $X$, we put

$$q(\xi)=e(\lambda\hat{\otimes}\xi)\in U^{2m(k-1)}(B_G\times X),$$

where $\hat{\otimes}$ denotes the external tensor product. It follows that $q$ is natural and multiplicative:

$$q(f^*\xi)=(1\times f)^*q(\xi), \qquad q(\xi\oplus\eta)=q(\xi)q(\eta).$$

Let

$$F(x, y)=x+y+\sum_{i,j\geqq 1}a_{ij}x^iy^j\in U^*(pt)[[x, y]]$$

be the formal group law for the complex cobordism theory, that is, a formal power series on $x$ and $y$ with coefficients in $U^*(pt)$ such that

$$e(\eta_1\otimes\eta_2)=F(e(\eta_1), e(\eta_2))$$

for complex line bundles $\eta_1$ and $\eta_2$ (see [9], [16]). Define $[i](x)\in U^*(pt)[[x]]$ $(i=1, 2, \cdots)$ by

$$[1](x)=x, \qquad [i](x)=F([i-1](x), x),$$

and define $a_j(x)\in U^*(pt)[[x]]$ $(j=0, 1, 2, \cdots)$ by

$$\prod_{i=1}^{k-1} F([i](x), y) = \sum_{j\geqq 0} a_j(x)y^j .$$

Since $e(\rho^i)=[i](v)$, it follows from (4.1) that

$$a_0(v) = \prod_{i=1}^{k-1} [i](v) = w .$$

It is easily seen that $a_j(v)\in U^{2(k-1)-2j}(B_G)$. We shall write

$$a(v)^R = a_1(v)^{r_1} a_2(v)^{r_2}\cdots a_j(v)^{r_j}\cdots$$

for $R=(r_1, r_2, \cdots, r_j, \cdots)\in \mathfrak{R}$ .

**Theorem 3** (Quillen [17]). *For a eomplex m-dimensional bundle* $\xi$, *we have*

$$q(\xi) = \sum_{|R|\leqq m} w^{m-|R|}a(v)^R \times c^R(\xi) .$$

Proof. For a complex line bundle $\eta$ over $X$, we have

$$\begin{aligned} q(\eta) &= e(\sum_{i=1}^{k-1} \rho^i \hat{\otimes} \eta) = \prod_{i=1}^{k-1} e(p_1^*\rho^i \otimes p_2^*\eta) \\ &= \prod_{i=1}^{k-1} F([i](p_1^* e(\rho)), p_2^* e(\eta)) \\ &= \sum_{j\geqq 0} p_1^* a_j(e(\rho))\cdot p_2^* e(\eta)^j \\ &= w\times 1 + \sum_{j\geqq 1} a_j(v)\times e(\eta)^j , \end{aligned}$$

where $p_1$: $B_G\times X\to B_G$, $p_2$: $B_G\times X\to X$ are the projections. Therefore, if $\xi=\eta_1\oplus \cdots\oplus\eta_m$ is a sum of line bundles, it follows that

$$\begin{aligned} q(\xi) &= \prod_{i=1}^{m} (w\times 1+a_1(v)\times e(\eta_i)+a_2(v)\times e(\eta_i)^2+\cdots) \\ &= \sum_{|R|\leqq m} w^{m-|R|}a(v)^R\times f_R(c_1(\xi), c_2(\xi), \cdots, c_m(\xi)) \\ &= \sum_{|R|\leqq m} w^{m-|R|}a(v)^R\times c^R(\xi) . \end{aligned}$$

To prove the result for $\xi$ which is general, we apply the splitting principle. Let $f$: $Y\to X$ be a splitting map. Since $f^*\xi$ is a sum of line bundles, we have

$$\begin{aligned} &(1\times f)^* q(\xi) = q(f^*\xi) \\ &= \sum_{|R|\leqq m} w^{m-|R|}a(v)^R\times c^R(f^*\xi) \\ &= (1\times f)^*(\sum_{|R|\leqq m} w^{m-|R|}a(v)^R\times c^R(\xi)) . \end{aligned}$$

Since $(1\times f)^*$ is monic, we have the desired result.

We shall regard $U^*(B_G\times X)$ as a $U^*(B_G)$-module via the homomorphism

$U^*(B_G)\to U^*(B_G\times X)$ induced by the projection, and consider the localization $U^*(B_G\times X)[w^{-1}]$ of $U^*(B_G\times X)$ with respect to the multiplicative set generated by $w$.

We put

$$q_0(\xi)=w^{-m}q(\xi)\in U^*(B_G\times X)\,[w^{-1}]$$

for a complex $m$-dimensional bundle $\xi$ over $X$. Then it follows that $q_0$ is natural, multiplicative and stable.

**Corollary**. *For a complex bundle $\xi$ over a finite dimensional complex $X$, we have*

$$q_0(\xi)=\sum_R w^{-|R|}a(v)^R\times c^R(\xi)\,.$$

Proof. Since $q(i)=w^i$ for a trivial complex bundle of dimension $i$, Theorem 3 implies

$$w^i q(\xi)=\sum_{|R|\leq m+i} w^{m+i-|R|}a(v)^R\times c^R(\xi)\,.$$

Since $c^R(\xi)$ is in $U^{2||R||}(X)$ which is zero if $2||R||>\dim X$, we have for a sufficiently large $i$

$$w^{m+i}q_0(\xi)=w^{m+i}\sum_R w^{-|R|}a(v)^R\times c^R(\xi)\,,$$

which proves the corollary.

REMARK. Suppose $k$ is a prime $p$, and let $e\in H^*(B_G;\boldsymbol{Z}_p)$ denote the usual Euler class of $\rho$. Then it is easily seen that

$$\mu_p(w)=-e^{p-1},\qquad \mu_p(a_{p-1}(v))=1,\qquad \mu_p(a_j(v))=0\;(j\neq 0,p-1)\,,$$

and hence

$$\mu_p(q(\xi))=\begin{cases}\sum_{i=0}^{m}(-1)^{m-i}e^{(m-i)(p-1)}\times Q^i(\xi) & (p>2)\,,\\ \sum_{i=0}^{m}e^{m-i}\times W^{2i}(\xi) & (p=2)\end{cases}$$

(see Remark 2 of §2).

## 5. Power operations

Let $Y$ be a pointed $CW$ complex, and consider the smash product $B_G^+\wedge Y$, where $B_G^+$ is the disjoint union of $B_G$ and a point. In [8] tom-Dieck defines the $k$-th power operation

$$P\colon \tilde{U}^{2i}(Y)\to\tilde{U}^{2ik}(B_G^+\wedge Y)\,,$$

where $\tilde{U}^*(\cdot)$ is the reduced complex cobordism theory. For a $CW$ complex $X$, taking $Y=X^+$ he defines the power operation

$$P\colon U^{2i}(X) \to U^{2ik}(B_G\times X)\,.$$

He shows that $P$ is natural, multiplicative, and

$$P(\sigma^2\alpha) = \sigma^2(wP(\alpha))$$

holds for $\alpha\in \tilde{U}^{2i}(Y)$, where $\sigma^2\colon \tilde{U}^{2i}(Y)\to\tilde{U}^{2(i+1)}(Y\wedge S^2)$ is the double suspension, and $\tilde{U}^*(B_G^+\wedge Y)$ is regarded as a $U^*(B_G)$-module as usual. He shows also that $q$ is the characteristic class corresponding to $P$ in the following sense:

$$q(\xi) = \phi^{-1}_{id\times\xi}P\phi_\xi(1)\,,$$

where $\xi$ is a complex bundle over $X$, and $\phi_{id\times\xi}\colon U^*(B_G\times X)\to\tilde{U}^*(T(id\times\xi))= \tilde{U}^*(B_G^+\wedge T(\xi))$ is the Thom isomorphism.

We shall define

$$P_0\colon U^{2i}(X) \to (U^*(B_G\times X)[w^{-1}])^{2i}$$

by $P_0(\alpha)=w^{-i}P(\alpha)$. It follows that $P_0$ is natural, additive, multiplicative and stable.

**Theorem 4** (Quillen [17]). *For a finite complex $X$ we have*

$$P_0(\alpha) = \sum_R w^{-|R|}a(v)^R\times s^R\alpha, \qquad (\alpha\in U^{2i}(X))\,.$$

Proof. Let $\alpha$ be represented by $f\colon X^+\wedge S^{2n-2i}\to MU(n)$, where $MU(n)$ is the Thom complex of the universal complex bundle $\zeta=\zeta_n$ of dimension $n$. Then we have

$$f^*(\phi_\zeta(1)) = \sigma^{2n-2i}(\alpha)\,.$$

Therefore it follows from the properties of $P$ mentioned above and Theorem 3 that

$$\begin{aligned}
&\sigma^{2n-2i}w^{n-i}P(\alpha) = P\sigma^{2n-2i}(\alpha)\\
&= Pf^*(\phi_\zeta(1)) = (1\times f)^*P\phi_\zeta(1)\\
&= (1\times f)^*\phi_{id\times\zeta}q(\zeta)\\
&= (1\times f)^*\phi_{id\times\zeta}\sum_{|R|\leqq n} w^{n-|R|}a(v)^R\times c^R(\zeta)\\
&= \sum_{|R|\leqq n} w^{n-|R|}a(v)^R\times f^*\phi_\zeta c^R(\zeta)\,.
\end{aligned}$$

Since

$$\begin{aligned}
&f^*\phi_\zeta c^R(\zeta) = f^*s^R\phi_\zeta(1) = s^R f^*\phi_\zeta(1)\\
&= s^R\sigma^{2n-2i}(\alpha) = \sigma^{2n-2i}s^R(\alpha)\,,
\end{aligned}$$

we have

$$\sigma^{2n-2i}w^{n-i}P(\alpha) = \sigma^{2n-2i}\sum_{|R|\leq n} w^{n-|R|}a(v)^R \times s^R\alpha\ .$$

Since $\sigma^{2n-2i}$ is monic, this proves the desired result.

**Corollary.** *For a complex bundle $\xi$ over a finite complex, we have*

$$q_0(\xi) = \sum_R w^{-|R|}a(v)^R P_0(u^R(\xi))\ .$$

Proof. From the corollary of Theorem 3, (2.3) and Theorem 4 it follows that

$$\begin{aligned} q_0(\xi) &= \sum_R w^{-|R|}a(v)^R \times c^R(\xi) \\ &= \sum_R w^{-|R|}a(v)^R \times \sum_{I+J=R} s^I u^J(\xi) \\ &= \sum_J w^{-|J|}a(v)^J \sum_I w^{-|I|}a(v)^I \times s^I u^J(\xi) \\ &= \sum_J w^{-|J|}a(v)^J P_0(u^J(\xi))\ . \end{aligned}$$

REMARK. The power operations $P$ for $k=p$ (a prime) correspond to the usual Steenrod reduced power under the transformation $\mu_p$. Therefore the formula in Theorem 4 may be regarded as a complex cobordism version of the Steenrod formula given in 2.5 of [11] (see Remark of §4).

## 6. The class $\Delta$

Let $M$ be a closed almost complex manifold, and $\tau(M)$ be the tangent bundle of $M$ endowed with the complex structure. Consider the $k$-fold product $M^k$ on which $G$ acts by the cyclic permutation of coordinates. Let $\nu$: $W \to M$ be the normal bundle of the diagonal imbedding $d$: $M \to M^k$. Then $\nu$ is endowed with a $G$-equivariant complex structure which is isomorphic with $\tau(M)\hat{\otimes}\Lambda$. This is seen from an exact sequence

$$0 \to \tau(M) \to \tau(M^k)|M \to \tau(M)\hat{\otimes}\Lambda \to 0$$

of complex $G$-bundles over $M$, which comes from the exact sequence $0 \to \boldsymbol{C} \xrightarrow{d} \boldsymbol{C}^k \to \Lambda \to 0$ of complex $G$-modules.

Consider the complex bundle

$$\nu_1 = id \underset{G}{\times} \nu \colon E_G \underset{G}{\times} W \to B_G \times M\ .$$

Then we have isomorphisms

$$\nu_1 \cong id \underset{G}{\times} (\tau(M)\hat{\otimes}\Lambda) \cong \lambda \hat{\otimes} \tau(M)$$

of complex bundles, and hence

(6.1) $$e(\nu_1)=q(\tau(M)).$$

If we regard $W$ as an equivariant tubular neighborhood of $d(M)$ in $M^k$, we have the Thom class

$$t(\nu_1)\in U^{2m(k-1)}(E_G\underset{G}{\times}M^k,\ E_G\underset{G}{\times}(M^k-W))$$

(dim $M=2m$). We define

(6.2) $$\Delta=j^*(t(\nu_1))\in U^{2m(k-1)}(E_G\underset{G}{\times}M^k),$$

where $j^*$ is induced by the inclusion.

We have obviously

(6.3) $$e(\nu_1)=(id\underset{G}{\times}d)^*\Delta$$

for the homomorphism $(id\underset{G}{\times}d)^*\colon U^*(E_G\underset{G}{\times}M^k)\to U^*(B_G\times M)$.

Remark. If we consider the standard $G$-action on the sphere $S^{2n+1}$ and define $\Delta_n\in U^{2m(k-1)}(S^{2n+1}\underset{G}{\times}M^k)$ to be the Atiyah-Poincaré dual of the element $[S_G^{2n+1}\times M,\ id\underset{G}{\times}d]\in U_{2(n+m)+1}(S^{2n+1}\underset{G}{\times}M^k)$, then it is seen that $\Delta_n$ is the image of $\Delta$ under the homomorphism $U^*(E_G\underset{G}{\times}M^k)\to U^*(S^{2n+1}\underset{G}{\times}M^k)$ induced by the inclusion.

Let

$$P^{ext}\colon U^{2i}(X)\to U^{2ik}(E_G\underset{G}{\times}X^k)$$

denote the external power operation. By definition we have

(6.4) $$P=(id\underset{G}{\times}d)^*\circ P^{ext}.$$

We shall regard $U^*(E_G\underset{G}{\times}X^k)$ as a $U^*(B_G)$-module as usual and consider the localization $U^*(E_G\underset{G}{\times}X^k)[w^{-1}]$. Define now

$$P_0^{ext}\colon U^{2i}(X)\to (U^*(E_G\underset{G}{\times}X^k)[w^{-1}])^{2i}$$

by $P_0^{ext}(\alpha)=w^{-i}P^{ext}(\alpha)$, $\alpha\in U^{2i}(X)$.

**Theorem 5.** *If $k$ is a prime, for a closed almost complex manifold $M$ of dimension $2m$ we have*

$$\Delta=\sum_R w^{m-|R|}a(v)^R P_0^{ext}(u^R(M))$$

*in $U^*(E_G\underset{G}{\times}M^k)[w^{-1}]$.*

Proof. By (6.1), (6.2), (6.4) and Corollary of Theorem 4, we have

$$\begin{aligned}(id\underset{G}{\times} d)^*\Delta &= q(\tau(M))\\ &= \sum_R w^{m-|R|}a(v)^R P_0(u^R(M))\\ &= (id\underset{G}{\times} d)^* \sum_R w^{m-|R|}a(v)^R P_0^{ext}(u^R(M))\end{aligned}$$

in $U^*(B_G\times M)[w^{-1}]$. Since $k$ is a prime, $d(M)$ is the fixed point set of the $G$-space $M^k$. Therefore, by the localization theorem for the equivariant cohomology theory $U_G^*(\cdot)=U^*(E_G\underset{G}{\times}\cdot)$ (see [9]), we see that $(id\underset{G}{\times} d)^*$ induces an isomorphism $U^*(E_G\underset{G}{\times} M^k)[w^{-1}]\cong U^*(B_G\times M)[w^{-1}]$. Thus we have the desired result.

**Corollary.** *For a continuous map $f$: $S^{2n+1}\to M$ to a closed almost complex manifold $M$ of dimension $2m$, we have*

$$(id\underset{G}{\times} f^k)^*\Delta = w^m$$

in $U^*(E_G\underset{G}{\times}(S^{2n+1})^k)[w^{-1}]$.

Proof. Since both $U^{2i}(S^{2n+1})$ and $U^{2i}(pt)$ are zero if $i>0$, we have $f^*U^R(M)=0$ $(R\neq 0)$. Therefore Theorem 5 implies

$$\begin{aligned}(id\underset{G}{\times} f^k)^*\Delta &= \sum_R w^{m-|R|}a(v)^R P_0^{ext}(f^*u^R(M))\\ &= w^m\,.\end{aligned}$$

REMARK. Theorem 5 may be regarded as a complex cobordism version of Theorem 3.2 in [11].

## 7. The imbedding class and the immersion class

In next section we shall prove theorems on immersions and imbeddings of closed almost complex manifolds. To do this, given a continuous map $f$: $M\to M'$ between closed almost complex manifolds, we shall define for each prime $k$ the imbedding class $\phi_f$ and the immersion class $\psi_f$ after Haefliger [11] and Wu [21].

Consider the $G$-space $M^k$ as in the preceeding section, and identify $M$ with the diagonal $d(M)$. Since $k$ is a prime, we have a principal $G$-bundle $M^k-M\to(M^k-M)/G$. Let $h$: $M^k-M\to E_G$ be a bundle map classifying this bundle.

The bundle $(M^k-M)\underset{G}{\times} M'^k\to(M^k-M)/G$ associated to $M^k-M\to(M^k-M)/G$ with fibre $M'^k$ has a cross section $s$: $(M^k-M)/G\to(M^k-M)\underset{G}{\times} M'^k$ determined by $f^k$: $M^k\to M'^k$.

We shall now write $\Delta'$ for the element $\Delta$ of (6.2) for $M'$, and define $\phi_f$ to be the image of $\Delta'$ under the composite

$$U^*(E_G\underset{G}{\times} M'^k)\xrightarrow{(h\underset{G}{\times} id)^*} U^*((M^k-M)\underset{G}{\times} M'^k)\xrightarrow{s^*} U^*((M^k-M)/G)\,.$$

Obviously $\varphi_f$ depends on the homotopy class of $f$. If $f$ is a topological imbedding, then $(h \underset{G}{\times} id) \circ s$ takes $(M^k - M)/G$ into $E_G \underset{G}{\times} (M'^k - M')$. Therefore it follows from the definition of $\Delta'$ that $\varphi_f = 0$ if $f$ is a topological imbedding. Thus we have

**Lemma 2.** *If $f$ is homotopic to a topological imbedding, then $\varphi_f = 0$.*

Consider the following diagram:

$$\begin{array}{ccc} U^*(E_G \underset{G}{\times} M'^k) & \xrightarrow{(h \underset{G}{\times} id)^*} & U^*((M^k - M) \underset{G}{\times} M'^k) \\ \Big\downarrow (id \underset{G}{\times} f^k)^* & & \Big\downarrow s^* \\ & & U^*((M^k - M)/G) \\ & & \Big\downarrow p^* \\ U^*(E_G \underset{G}{\times} M^k) & \xrightarrow{i^*} & U^*(E_G \underset{G}{\times} (M^k - M)), \end{array}$$

where $p$ is the projection and $i$ is the inclusion. It follows that $p^*$ is an isomorphism and the map sending $(x_1, \cdots, x_k) \in M^k - M$ to $(h(x_1, \cdots, x_k), x_1, \cdots, x_k)$ induces the inverse of $p^*$. Therefore the above diagram is commutative, and we have

$$(7.1) \qquad p^*(\varphi_f) = i^*(id \underset{G}{\times} f^k)^* \Delta' .$$

Consider the direct limit $\varinjlim U^*((W - M)/G)$, where $W$ runs over all equivariant neighborhoods of $M$ in $M^k$. We have the canonical homomorphism

$$\kappa \colon U^*((M^k - M)/G) \to \varinjlim U^*((W - M)/G) .$$

We shall define $\psi_f = \kappa(\varphi_f)$.

If $f$ is a topological immersion, $(h \underset{G}{\times} id) \circ s$ takes $(W - M)/G$ into $E_G \underset{G}{\times} (M'^k - M')$ for sufficiently small $W$. Therefore, as in Lemma 2, we have

**Lemma 3.** *If $f$ is homotopic to a topological immersion, then $\psi_f = 0$.*

Consider the homomorphisms

$$\begin{aligned} & U^*(B_G \times M') \xrightarrow{(id \times f^*)} U^*(B_G \times M) \xleftarrow{\iota} \varinjlim U^*(E_G \underset{G}{\times} W) \\ & \xrightarrow{i^*} \varinjlim U^*(E_G \underset{G}{\times} (W - M)) \xleftarrow{p^*} \varinjlim U^*((W - M)/G) , \end{aligned}$$

where $\iota$ and $i^*$ are induced by the inclusion maps and $p^*$ is induced by the projection. It follows that $\iota$ and $p^*$ are isomorphisms. Lemma 3 and (6.3) prove the following equality by diagram-chasing:

(7.2) $$p^*(\psi_f) = i^*\iota^{-1}(id\times f)^*e(\nu_1'),$$

where $\nu_1'$ is the bundle $\nu_1$ for $M'$.

## 8. Theorems on immersion and imbedding

In this section we shall prove a complex cobordism version of the immersion and imbedding theorems due to Haefliger and Wu (see §5 in [11]).

Consider the localization homomorphism $U^*(B_G\times M)\to U^*(B_G\times M)[w^{-1}]$. An element in the image of this homomorphism is said to be *integal.*

**Theorem 6.** *Let $M$ and $M'$ be closed almost complex manifolds with* dim $M=2m$, dim $M'=2m'$. *Let $f$: $M\to M'$ be a continuous map homotopic to a topological immersion. Then, for any prime $k$, the element*

$$\sum_R w^{m'-m-|R|}a(v)^R\times(\sum_{I+J=R} f^*c^I(M')\cdot\bar{c}^J(M))$$

*of $U^*(B_G\times M)[w^{-1}]$ is integral.*

Proof. Consider the bundle $\nu_1$: $E_G\times_G W\to B_G\times M$. Then we have the Thom isomorphism

$$U^i(B_G\times M)\cong U^{i+2m(k-1)}(E_G\times_G W,\ E_G\times_G(W-M)).$$

Therefore the exact sequence for $(E_G\times_G W,\ E_G\times_G(W-M))$ yields an exact sequence

$$\cdots\to U^{i-2m(k-1)}(B_G\times M)\to U^i(E_G\times_G W)\to U^i(E_G\times_G(W-M))\to\cdots.$$

Passing to the limit we have an exact sequence

$$\cdots\to U^{i-2m(k-1)}(B_G\times M)\xrightarrow{\cdot e(\nu_1)} U^i(B_G\times M) \xrightarrow{i^*\circ\iota^{-1}} \varinjlim_G U^i(E_G\times(W-M))\to\cdots$$

with the notations of (7.2). Therefore, in virtue of Lemma 3 and (7.2), there exists $\alpha\in U^*(B_G\times M)$ such that

$$(id\times f)^*e(\nu_1') = \alpha\cdot e(\nu_1),$$

i.e.

$$(id\times f)^*q(\tau(M')) = \alpha\cdot q(\tau(M))$$

(see (6.1)). This shows that

$$w^{m'-m}\cdot(id\times f)^*q_0(\tau(M'))\cdot q_0(\nu(M))\in U^*(B_G\times M)[w^{-1}]$$

is integral, where $\nu(M)$ is the stable normal bundle of $M$. It follows from

Corollary of Theorem 3 and Lemma 1 that

$$\begin{aligned}&(id\times f)^*q_0(\tau(M'))\cdot q_0(\nu(M))\\&=(\sum_I w^{-|I|}a(v)^I\times f^*c^I(M'))\cdot(\sum_J w^{-|J|}a(v)^J\times\bar{c}^J(M))\\&=\sum_R w^{-|R|}a(v)^R\times(\sum_{I+J=R}f^*c^I(M')\cdot\bar{c}^J(M)).\end{aligned}$$

This completes the proof.

**Theorem 7.** *Let $M$ and $M'$ be closed almost complex manifolds with* dim $M=2m$, dim $M'=2m'$. *Let $f\colon M\to M'$ be a continuous map which is null-homotopic. Then, if $f$ is also homotopic to a topological imbedding, for any prime $k$ the element*

$$\sum_R w^{m'-m-|R|}v^{-1}a(v)^R\times\bar{c}^R(M)$$

*of $U^*(B_G\times M)[w^{-1}]$ is integral.*

Proof. It follows from Lemma 2 and (7.1) that $i^*(id\underset{G}{\times}f^k)^*\Delta'=0$ for $i^*$: $U^*(E_G\underset{G}{\times}M^k)\to U^*(E_G\underset{G}{\times}(M^k-M))$ induced by the inclusion. Therefore there exists $\beta\in U^*(B_G\times M)$ such that

$$(id\underset{G}{\times}f^k)^*\Delta'=j^*\phi_{\nu_1}(\beta)$$

with the notations in the following diagram:

$$\begin{array}{ccc}U^*(B_G\times M)&\xrightarrow{r_0^*}&U^*(M)\\ \downarrow{\phi_{\nu_1}}&&\downarrow{\phi_\nu}\\ U^*(E_G\underset{G}{\times}(M^k,M^k-M))&\xrightarrow{r^*}&U^*(M^k,M^k-M))\\ \downarrow{j^*}&&\downarrow{j^*}\\ U(E_G\underset{G}{\times}M^k)&\xrightarrow{r^*}&U^*(M^k)\\ \uparrow{(id\underset{G}{\times}f^k)^*}&&\uparrow{(f^k)^*}\\ U^*(E_G\underset{G}{\times}M'^k)&\xrightarrow{r'^*}&U^*(M'^k)\end{array}$$

where $r_0$, $r$, $r'$ and $j$ are the inclusion maps. The diagram is commutative, and $(f^k)^*=0$ since $f$ is null-homotopic. Therefore we have $j^*\phi_\nu r_0^*(\beta)=0$.

Consider the commutative diagram

$$\begin{array}{ccccc}U^*(E_G\underset{G}{\times}(M^k-M))&\xrightarrow{\delta}&U_G(E_G\underset{G}{\times}(M^k,M^k-M))&\xrightarrow{j^*}&U^*(E_G\underset{G}{\times}M^k)\\ \downarrow{r^*}&&\downarrow{r^*}&&\downarrow{r^*}\\ U^*(M^k-M)&\xrightarrow{\delta}&U^*(M^k,M^k-M)&\xrightarrow{j^*}&U^*(M^k)\end{array}$$

in which the horizontal lines are the exact sequences of pairs. Since $r^*$ in the left is an isomorphism, it follows that there exists $\beta_1 \in U^*(E_G \underset{G}{\times} (M^k - M))$ such that $\phi_\nu r_0^*(\beta) = r^*\delta(\beta_1)$. Take $\beta_2 \in U^*(B_G \times M)$ such that $\delta(\beta_1) = \phi_{\nu_1}(\beta_2)$, and put $\alpha = \beta - \beta_2$. Then it follows that

$$\begin{aligned} j^*\phi_{\nu_1}(\alpha) &= j^*\phi_{\nu_1}(\beta) - j^*\phi_{\nu_1}(\beta_2) \\ &= j^*\phi_{\nu_1}(\beta) - j^*\delta(\beta_1) = j^*\phi_{\nu_1}(\beta) \end{aligned}$$

and

$$\begin{aligned} \phi_\nu r_0^*(\alpha) &= \phi_\nu r_0^*(\beta) - \phi_\nu r_0^*(\beta_2) \\ &= r^*\delta(\beta_1) - r^*\phi_{\nu_1}(\beta_2) = r^*\phi_{\nu_1}(\beta_2) - r^*\phi_{\nu_1}(\beta_2) = 0\,. \end{aligned}$$

Consequently we have

$$j^*\phi_{\nu_1}(\alpha) = (id \underset{G}{\times} f^k)^*\Delta', \tag{8.1}$$

$$r_0^*(\alpha) = 0\,. \tag{8.2}$$

Since

$$\alpha e(\nu_1) = (id \underset{G}{\times} d)^* j^*\phi_{\nu_1}(\alpha)\,,$$

it follows from (6.1), (6,3) and (8.1) that

$$\begin{aligned} \alpha q(\tau(M)) &= (id \underset{G}{\times} d)^*(id \underset{G}{\times} f^k)^*\Delta' \\ &= (id \times f)^*(id \underset{G}{\times} d')^*\Delta' = (id \times f)^* q(\tau(M'))\,. \end{aligned}$$

Since $f$ is null-homotopic, we have

$$\alpha q(\tau(M)) = w^{m'}\,.$$

We know that

$$\begin{aligned} &\times : U^*(B_G) \underset{U^*(pt)}{\otimes} U^*(M) \cong U^*(B_G \times M)\,, \\ &U^*(B_G) \cong U^*(pt)[[v]]/([k](v)) \end{aligned}$$

(see [13]). Therefore it follows from (8.2) that there exists $\alpha_1 \in U^*(B_G \times M)$ such that $\alpha = v\alpha_1$. Thus we have

$$w^m v\ \alpha_1 q_0(\tau(M)) = w^{m'}\,,$$

which shows that

$$w^{m'-m} v^{-1} q_0(\nu(M)) = \sum_R w^{m'-m-|R|} v^{-1} a(v)^R \times \bar{c}^R(M)$$

is integral. This completes the proof.

**Corollary**. *If a closed almost complex manifold $M$ of dimension $2m$ can be* *immersed* (*resp. imbedded*) *in* $R^{2n}$, *for any prime k the element*

$$\sum_R w^{n-m-|R|} a(v)^R \times \bar{c}^R(M)$$
$$(\text{resp. } \sum_R w^{n-m-|R|} v^{-1} a(v)^R \times \bar{c}^R(M))$$

*of* $U^*(B_G \times M)[w^{-1}]$ *is integral.*

REMARK. Applying $\mu_p$ converts the conclusion for $k=p$ of the above corollary to the following (see Remark of §4): if $p=2$ then $\bar{W}^{2i}(M)=0$ for $i>n-m$ (resp. $\bar{W}^{2i}(M)=0$ for $i\geqq n-m$); *if* $p>2$ then $\bar{Q}^i(M)=0$ for $i>n-m$ (resp. $\bar{Q}^i(M))=0$ for $i\geqq n-m$).

## 9. Imbeddings and immersions of $CP^m$

In this section, we shall give a $K$-theory version of Corollary of §8 for $k=2$, and apply it to prove non-existence of imbedding and immersion of complex projective spaces in Euclidean spaces.

For a complex bundle $\xi$ over $X$, let $\gamma_i(\xi)\in K(X)$ denote the Atiyah class of $\xi$ (see [3].) There exists a natural transformation $\mu_c: U^*(\cdot)\to K^*(\cdot)$ such that $\mu_c(c_i(\xi))=\gamma_i(\xi)$ (see [7]). We define the *dual Atiyah class* $\bar{\gamma}_i(\xi)\in K(X)$ $(i=0, 1, 2, \cdots)$ by

$$\sum_{i+j=k} \gamma_i(\xi)\bar{\gamma}_j(\xi) = 0 \ (k>0), \qquad \bar{\gamma}_0(\xi) = 1 .$$

It follows that $\mu_c(\bar{c}_i(\xi))=\bar{\gamma}_i(\xi)$. If $M$ is an almost complex manifold and $\tau$ is its tangent bundle, we write $\bar{\gamma}_i(M)$ for $\bar{\gamma}_i(\tau)$. It follows that $\bar{\gamma}_i(M)=0$ $(i>m)$ if $\dim M=2m$.

**Theorem 8.** *Let M be a closed almost complex manifold such that* $K(M)$ *has no elements of finite order. Then, if M can be imbedded* (*resp. immersed*) *in* $R^{2n}$, *the element*

$$\sum_{i=0}^{m} 2^{m-i}\bar{\gamma}_i(M)\in K(M)$$

*is divisible by* $2^{2m-n+1}$ (*resp.* $2^{2m-n}$).

Proof. Since $\gamma_1(\eta)=\eta-1$ for a complex line bundle $\eta$, we have $\gamma_1(\eta\otimes\eta')=\gamma_1(\eta)+\gamma_1(\eta')+\gamma_1(\eta)\gamma_1(\eta')$. Therefore if $k=2$ it holds

$$\mu_c(a_1(v)) = 1+\gamma, \qquad \mu_c(a_i(v)) = 0 \ (i\geqq 2)$$

with $\gamma=\mu_c(v)=\mu_c(w)\in K(B_G)$.

It is known that $K(B_G)\cong Z[\gamma]/(\gamma^2+2\gamma)$ if $k=2$ (see [3]). Therefore we have $(1+\gamma)^2=1$ and $\gamma^i=(-2)^{i-1}\gamma$ $(i\geqq 1)$. From these we see

$$(1+\gamma)^i\gamma^{-j} = (-1)^{i+j}2^{-j} \quad (i\geqq 0, j\geqq 1)$$

in the localization $K(B_G)[\gamma^{-1}]$.

It follows now from Corollary in §8 for $k=2$ that if $M$ can be imbedded in $\boldsymbol{R}^{2n}$ ,the element

$$\sum_{i=n-m}^{m}\gamma^{n-m-i-1}(1+\gamma)^i\times\bar{\gamma}_i(M)$$
$$=(-1)^{n-m}2^{n-2m-1}\sum_{i=n-m}^{m}2^{m-i}\bar{\gamma}_i(M)$$

of the localization $K^*(B_G\times M)[\gamma^{-1}]$ is integral. Since $K(B_G)$ and $K(M)$ have no element of finite order, it is easily seen that the above integrality condition implies that

$$\sum_{i=n-m}^{m}2^{m-i}\bar{\gamma}_i(M)$$

is divisible by $2^{2m-n+1}$ in $K(M)$. This proves the desired result for imbeddings. Similarly we have the result for immersions.

REMARK If $k$ is an odd prime $p$, we see that

$$\mu_c(a_i(v))=\frac{(p-1)!}{(i+1)!(p-i-1)!}N \quad (0\leqq i<p-1),$$
$$\mu_c(a_{p-1}(v))=1, \qquad \mu_c(a_i(v))=0\ (i\geqq p),$$

where $N=\mu_c(w)=\sum_{i=1}^{p-1}(1-\rho^i)$.

As an application of the above theorem, we shall prove the following result due to Atiyah-Hirzebruch [5] and Sanderson-Schwarzenberger [18].

**Theorem 9.** *The complex m-dimensional projective space $CP^m$ can not be imbedded (resp. immersed) in $\boldsymbol{R}^{4m-2\alpha(m)}$ (resp. $\boldsymbol{R}^{4m-2\alpha(m)-1}$), where $\alpha(m)$ is the number of 1's in the dyadic expansion of m.*

Proof. Put $\theta=\eta-1\in K(CP^m)$, where $\eta$ is the canonical line budle over $CP^m$. Then it is easily seen that

$$\bar{\gamma}^i(CP^m)=(-1)^i\binom{m+i}{i}\theta^i.$$

Since $K(CP^m)\cong\boldsymbol{Z}[\theta]/(\theta^{m+1})$ has no elements of finite order, it follows from Theorem 8 that if $CP^m$ is imbedded in $\boldsymbol{R}^{2n}$ then

$$\sum_{i=0}^{m}(-1)^i2^{m-i}\binom{m+i}{i}\theta^i\in K(CP^m)$$

is divisible by $2^{2m-n+1}$, and hence $\binom{2m}{m}$ is divisible by $2^{2m-n+1}$. This means

$\alpha(m) \geqq 2m-n+1$. Thus $CP^m$ can not be imbedded in $R^{4m-2\alpha(m)}$.

To prove the result for non-immersion we borrow the device of [18]. Suppose that $CP^m$ is immersed in $R^{2n-1}$. Take an integer $s$ which is a power of 2 and is greater than $m$. Since $CP^s$ can be imbedded in $R^{4s-1}$, $CP^m \times CP^s$ can be imbedded in $R^{2n+4s-2}$ (see [18]). Apply Theorem 8 to this imbedding. Since

$$K(CP^m \times CP^s) \cong K(CP^m) \otimes K(CP^s)\,,$$
$$\bar{\gamma}_k(CP^m \times CP^s) = \sum_{i+j=k} \bar{\gamma}_i(CP^m) \times \bar{\gamma}_j(CP^s)\,,$$

it follows then that

$$\binom{2m}{m}\binom{2s}{s}$$

is divisible by $2^{2m-n+2}$, and hence $\alpha(m) \geqq 2m-n+1$. Thus $CP^m$ can not be immersed in $R^{4m-2\alpha(m)-1}$. This completes the proof.

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