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NON-CONTRACTIBLE ACYCLIC NORMAL SPINES

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1. Introduction

In [3], we have defined fake surfaces to study 3-manifolds with boundary from their spines. We use the notations in [3] and [4], for example, $\mathcal{F}(s, t)$ denotes the set of all the acyclic closed fake surfaces P with $\#\mathfrak{S}_2(P)=s$ and $\#\mathfrak{S}_3(P)=t$, where $\mathfrak{S}_i(P)$ means the i-th singularity of P and $\#$ denotes the number of the connected components. And, $\mathcal{E}(s, t)$ is the subset of $\mathcal{F}(s, t)$ each of whose elements is a noraml spine, that is, for any element P of $\mathcal{E}(s, t)$, there exists a 3-manifold in which P can be embedded as a spine. The following theorems are proved in [3] and [4].

Theorem. *$\mathcal{F}(s, t)=\phi$, if and only if $t=0$.*

Theorem. *$\mathcal{E}(s, t)=\phi$, if and only if $s\geqq 2$t.*

Then, when $t\geqq 1$, it is known that the difference $\mathcal{F}(s, t)-\mathcal{E}(s, t)$ is non-empty.

Let $C(s, t)$ denote the subset of $\mathcal{E}(s, t)$ each of whose elements is contractible and $\mathcal{B}(s, t)$ the subset of $C(s, t)$ each of whose elements is a normal spine of a 3-ball. Define the two difference sets $\mathcal{D}(s, t)$ and $\mathcal{A}(s, t)$ by

$$\mathcal{D}(s, t) = \mathcal{E}(s, t)-C(s, t)\,,$$
$$\mathcal{A}(s, t) = C(s, t)-\mathcal{B}(s, t)\,.$$

Then, Poincaré conjecture asks "Is the set $\bigcup_{s,t} \mathcal{A}(s, t)$ empty?". On the other hand, the following theorem is well-known.

Theorem. $\bigcup_{s,t} \mathcal{D}(s, t)\neq\phi$.

And, in [3] and [4], we proved the following.

Theorem. *$\mathcal{D}(s, t)=\phi=\mathcal{A}(s, t)$ for the cases $s=2t-1$ and $s=2t-2$, and $\mathcal{D}(1, 2)=\phi=\mathcal{A}(1, 2)$.*

In this paper, we show the following.

Theorem 1. *For the case* $1 \leqq s \leqq 2t-11$ *and* $t \geqq 6$, *the set* $\mathcal{D}(s, t)$ *is non-empty.*

In § 2, we construct a non-contractible acyclic mormal spine P_k with $\#\mathfrak{S}_2(P_k)=1$ and $\#\mathfrak{S}_3(P_k)=8k-1$ for any integer $k \geqq 1$. And, in § 3, we can prove that a 3-manifold W_1 has a normal spine P' with $\#\mathfrak{S}_2(P')=1$ and $\#\mathfrak{S}_3(P')=6$, where W_k is the 3-manifold containing P_k as its normal spine. And, the proof of Theorem 1 is obtained. It is known, by the uniqueness theorem of [1], that W_k is uniquely determined. In § 4, we define the *Dehn space of type k* and show, in Theorem 2, that W_k is the Dehn space of type k.

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2. The construction of non-contractible acyclic normal spines P_k

It has been proved in Theorem 4 [3] that $\mathcal{E}(1, 1)$ contains a unique element $F^1_{1,1}$, called an abalone. Let the set $\{M_1, M_2, f\}$ be the polygonal representation of the abalone, that is, M_i is a 2-ball for $i=1, 2$, and f means the identification map from $M_1 \cup M_2$ to $F^1_{1,1}$ (for M_1, M_2 and the identification by f, see Theorem 2 [3]).

Through out this paper, the subpolyhedron $f(M_2)$ of the abalone is denoted by F, which is written in Fig. 1. Then, F is a closed fake surface with $\#\mathfrak{S}_2(F)=1$ and $\#\mathfrak{S}_3(F)=0$, more precisely, $U(F)=S\times_\sigma T$. And, by a little geometrical consideration, it is seen that F is a normal spine of the exterior of the clover-leaf knot in 3-sphere. The fundamental group of F is as follows.

$$\pi_1(F) = (S_1, S_2;\ S_1 S_2^{-1} S_1 S_2^2 = 1)$$

(for the generators S_1 and S_2, see Fig. 1).

Lemma 1. *For any integer* $k \geqq 0$, *there exists an embedding* h_k *from* 1-*sphere* S *into* F *which represents the homotopy class* $S_2^{3k} S_1^{6k-1}$ *and the intersection* $h_k(S) \cap S_2$ *consists of* $|8k-1|$ *points.*

Proof. When $k=0$, we can take h_0 to be the homeomorphism from S onto S_1 which reverses the orientation. Then, clearly, h_0 represents the homotopy class S_1^{-1} and we have $\#(h_0(S) \cap S_2)=1$. Let us construct the required embedding h_k for the cases $k \geqq 1$.

Step 1. Suppose $k=1$. For the point a, a', b, b', c, c', d and x_i, $i=1, 2, 3$, see Fig. 2. Now, starting from the point a, go to b along the orientation of S_2. From b', go to c along S_2. Intersecting with S_2 at the point x_1, go to c' as shown in Fig.2. From c', go to d along S_2. And, intersecting with S_2 at x_2, go to x_3.

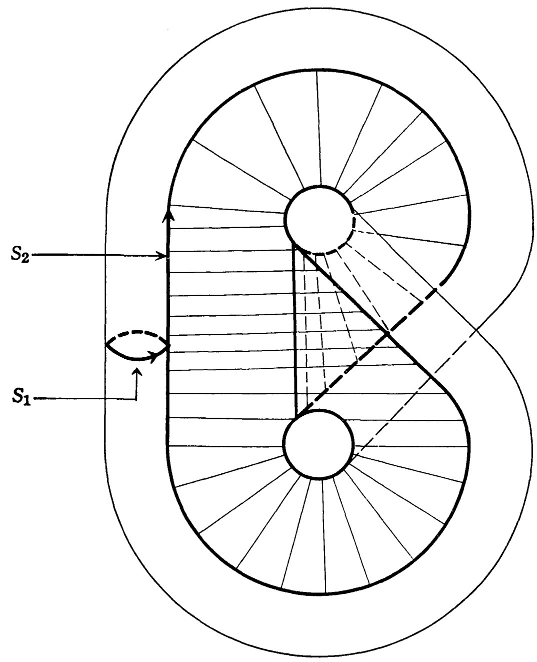

Fig. 1

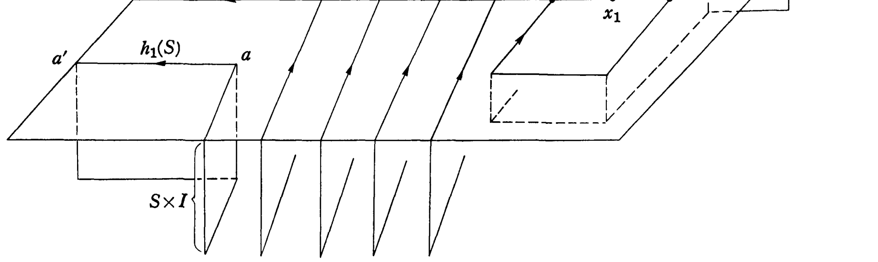

Fig.2

Finally, going along S_1 five times from x_3, we come back to the starting point a. Thus, we obtain an embedding h_1 representing the homotopy class $S_2^3 S_1^5$ and $\#(h_1(S)\cap S_2)=7$.

Step 2. Because S_2^3 lies in the center of $\pi_1(F)$, we obtain the following.

$$S_2^{3k} S_1^{6k-1} = (S_2^3 S_1^5) \prod_{p=2}^{k} (S_2^3 S_1^6)_p, \quad k \geqq 2 .$$

So, we try to construct the required embedding h_k to represent the homotopy class $(S_2^3 S_1^5) \prod_{p=2}^{k} (S_2^3 S_1^6)_p$, as follows. Let $a_2, \cdots, a_k$ be the points between a and S_1 as shown in Fig. 3. And formally, set $a_1 = a$. Then, by the same way as in Step 1, we obtain an embedding h_p' from S into F which represents the homotopy class $(S_2^3 S_1^6)_p$ and whose initial point and end point is a_p. And set $h_1' = h_1$. More strictly, we can choose h_p' to satisfy the following conditions.

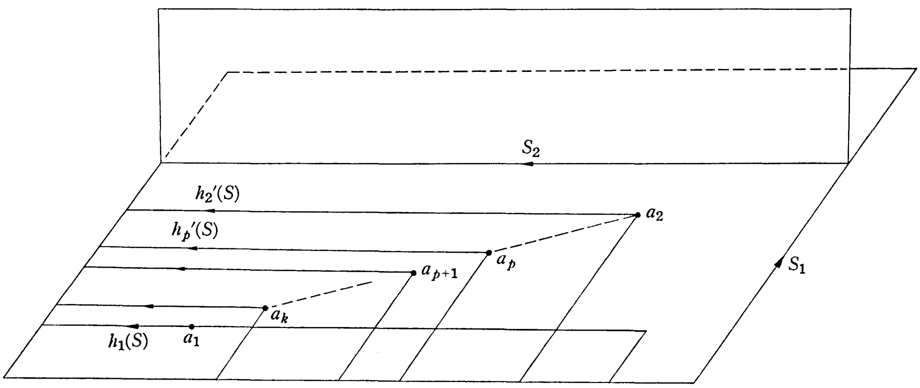

Fig. 3

(1) $h_p'(S) \cap h_q'(S) = \phi$, if $p \neq q$ and $p, q \geqq 2$.
(2) $h_p'(S) \cap h_1(S)$ is one point in the small neighborhood of a_1 (see Fig. 3).
(3) $\#(h_p'(S) \cap S_2) = 8$.

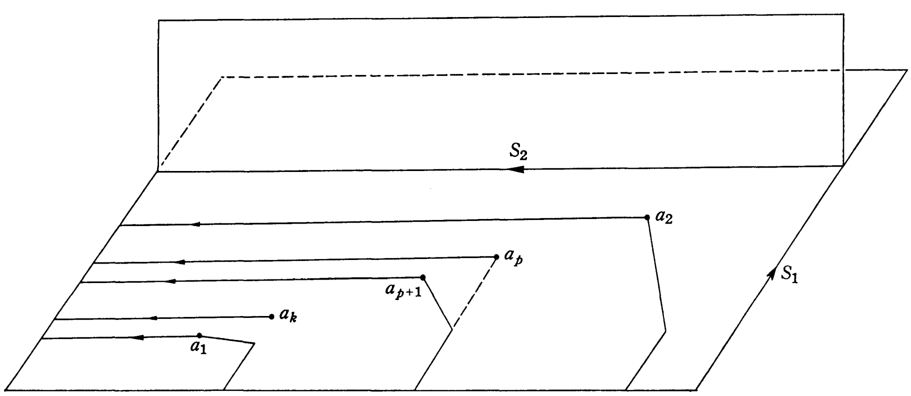

Fig. 4

Now, changing the end point a_p of h_p' to the initial point a_{p+1} of h'_{p+1}, for $1 \leqq p \leqq k-1$, and the end point a_k to a_1 (see Fig. 4), we obtain the required embedding h_k from S into F.

DEFINITION 1. Let P_k be the closed fake surface obtained from F by attaching a 2-ball B by the homeomorphism h_k from B to F.

REMARK. From the construction, it is clear that P_0 is homeomorphic to an abalone $F^1_{1,1}$.

Lemma 2. *If $k \geqq 1$, then P_k is a non-contractible element of $\mathcal{E}(1, 8k-1)$.*

Proof. We can prove that P_k is acyclic, because

$$\begin{aligned} H_1(P_k) &= (S_1, S_2;\, 2S_1+S_2 = 0 \quad (6k-1)S_1+3kS_2 = 0) \\ &= 0 \end{aligned}$$

and $H_2(P_k)$ is trivially trivial. And the fact that $\pi_1(P_k)$ is non-trivial follows from the calculation in [2]. Hence, P_k is a non-contractible acyclic closed fake surface. It follows from the construction of P_k that $U(P_k)$ can be embedded in the euclidean 3-space R^3. Then, by Lemma 2 [4], P_k is a normal spine. And, again from the construction, we see $\#\mathfrak{S}_2(P_k)=1$ and $\#\mathfrak{S}_3(P_k)=8k-1$, more precisely, $\mathfrak{S}_2(P_k)=S_2 \cup h_k(S)$ and $\mathfrak{S}_3(P_k)=S_2 \cap h_k(S)$ is the union $\bigcup_{p=1}^{k} (S_2 \cap h_p'(S))$, and we obtain $\#\mathfrak{S}_3(P_k)=8k-1$.

3. The element P' of $\mathcal{D}(1, 6)$ and the proof of Theorem 1

Let W_k denote the 3-manifold containing P_k as its normal spine, $k=1, 2, \dots$. In this section, we consider P_1 in W_1 and consturct another normal spine P' of W_1 from P_1 in $\mathcal{D}(1, 6)$. For the polygonal representation of P_1, see Fig. 5.

Proposition 1. *W_1 has a normal spine P' in $\mathcal{D}(1, 6)$.*

Proof. Let us consider M_1 of the polygonal representation of P_1, and let N be the regular neighborhood of M_1 mod $\dot{M}_1$ in W_1 chosen to satisfy

$$N \cap (P_1 - \mathring{M}_1) = \dot{N} \cap P_1 = \dot{M}_1 \times I\,,$$

as shown in Fig. 6, where I is the closed unit interval [0, 1] and $M_1 = M_1 \times 1/2$. Put $A = \dot{N} \cap P_1$. Then, $A = (A \cap \mathfrak{S}_2(P_1))$ has three connected components each of whose closures is a 2-ball. Take such a 2-ball B. Regarding B as a free face of $P_1 \cup N$, we can collapse $P_1 \cup N$ to $(P_1 - (N \cap P_1)) \cup (\dot{N} - \mathring{B})$ (see Fig. 7). Put $P' = (P_1 - (N \cap P_1)) \cup (\dot{N} - \mathring{B})$. Then, it is clear that P' is a closed fake surface embedded in the 3-manifold W_1. Since P_1 expands to $P_1 \cup N$ and $P_1 \cup N$ col-

Polygonal representation of P_1

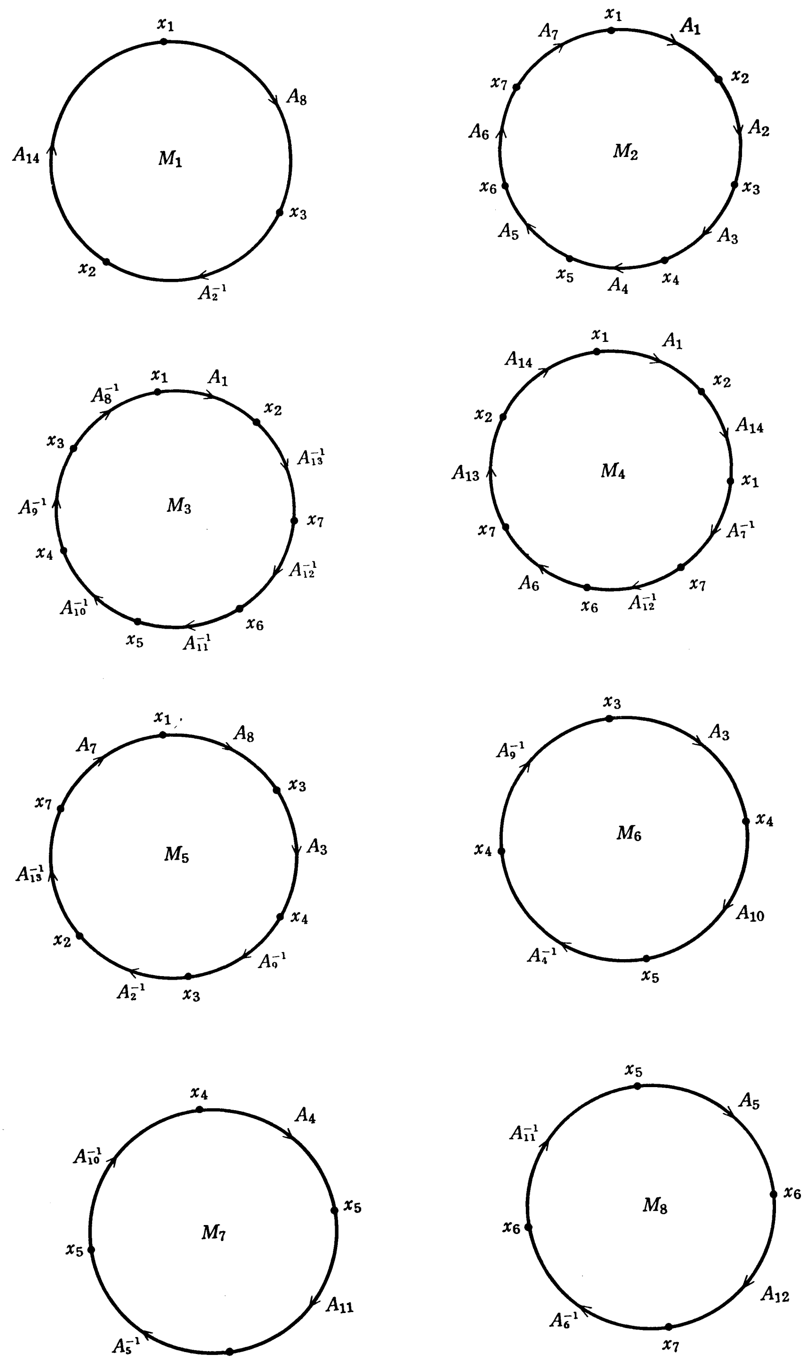

Fig. 5

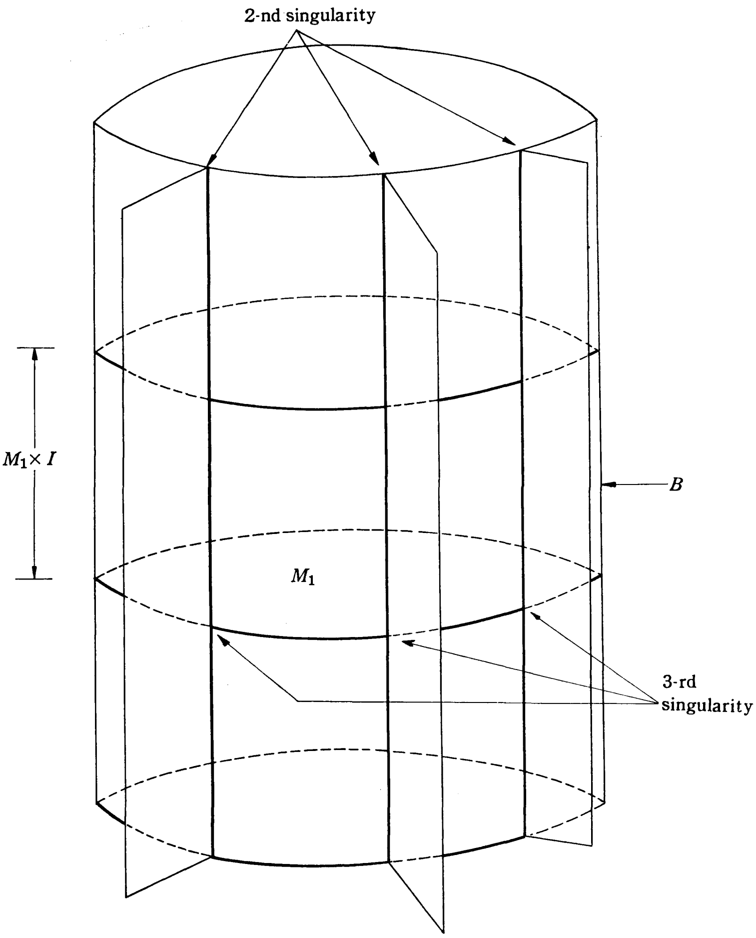

Fig. 6

lapses to P' in W_1, P_1 and P' belong to the same simple homotopy type in W_1, that is P' is also a spine of W_1. By the above construction, the conditions $\#\mathfrak{S}_2(P')=1$ and $\#\mathfrak{S}_3(P')=6$ are easily seen. Thus, W_1 has a normal spine P' in $\mathcal{D}(1. 6)$.

REMARK. The polygonal representation of P' is shown in Fig. 8.

Now, we can prove Theorem 1.

Theorem 1. *For the case* $1 \leqq s \leqq 2t-11$ *and* $t \geqq 6$. *the set* $\mathcal{D}(s, t)$ *is non-empty.*

Proof. First, it is shown that $\mathcal{D}(1, t)$ is non-empty for $t \geqq 6$ by the same argument as that of the proof of Lemma 12 [4], because P' and P_1 belong to

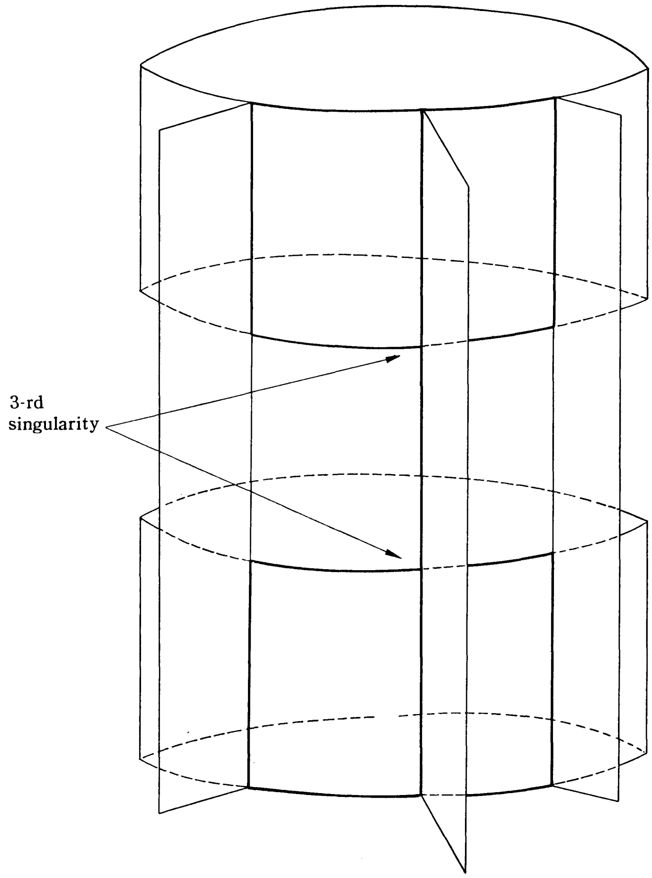

Fig. 7

$\mathcal{D}(1, 6)$ and $\mathcal{D}1, 7)$, respectively. And, we obtain an element of $\mathcal{D}(s, t)$ with $1 \leqq s \leqq 2t-11$ as in the proof of Theorem 6 [4].

4. The Dehn spaces

Let E denote the exterior of a clover knot k in a 3-sphere Σ, that is, $E=\Sigma-\mathring{N}(k, \Sigma)$ where $\mathring{N}(k, \Sigma)$ means the interior of a regular neighborhood $N(k, \Sigma)$ of k in Σ. Then, there exists a subpolyhedron F_0 in E which is homeomorphic to F. Of course, F_0 is a spine of E. Regarding the generators S_1 and S_2 of $\pi_1(F)$ as those of $\pi_1(F_0)$, we can write

$$\pi_1(E) = (S_1, S_2 \colon S_1 S_2^{-1} S_1 S_2^2 = 1)\,.$$

Take S_1 and $S_1^{-1} S_2$ as the generators of $\pi_1(E)$, and let i_* denote the homomorphism from $\pi_1(\dot{E})$ to $\pi_1(E)$ induced by the inclusion map. Since E is an exterior

Polygonal representation of P'

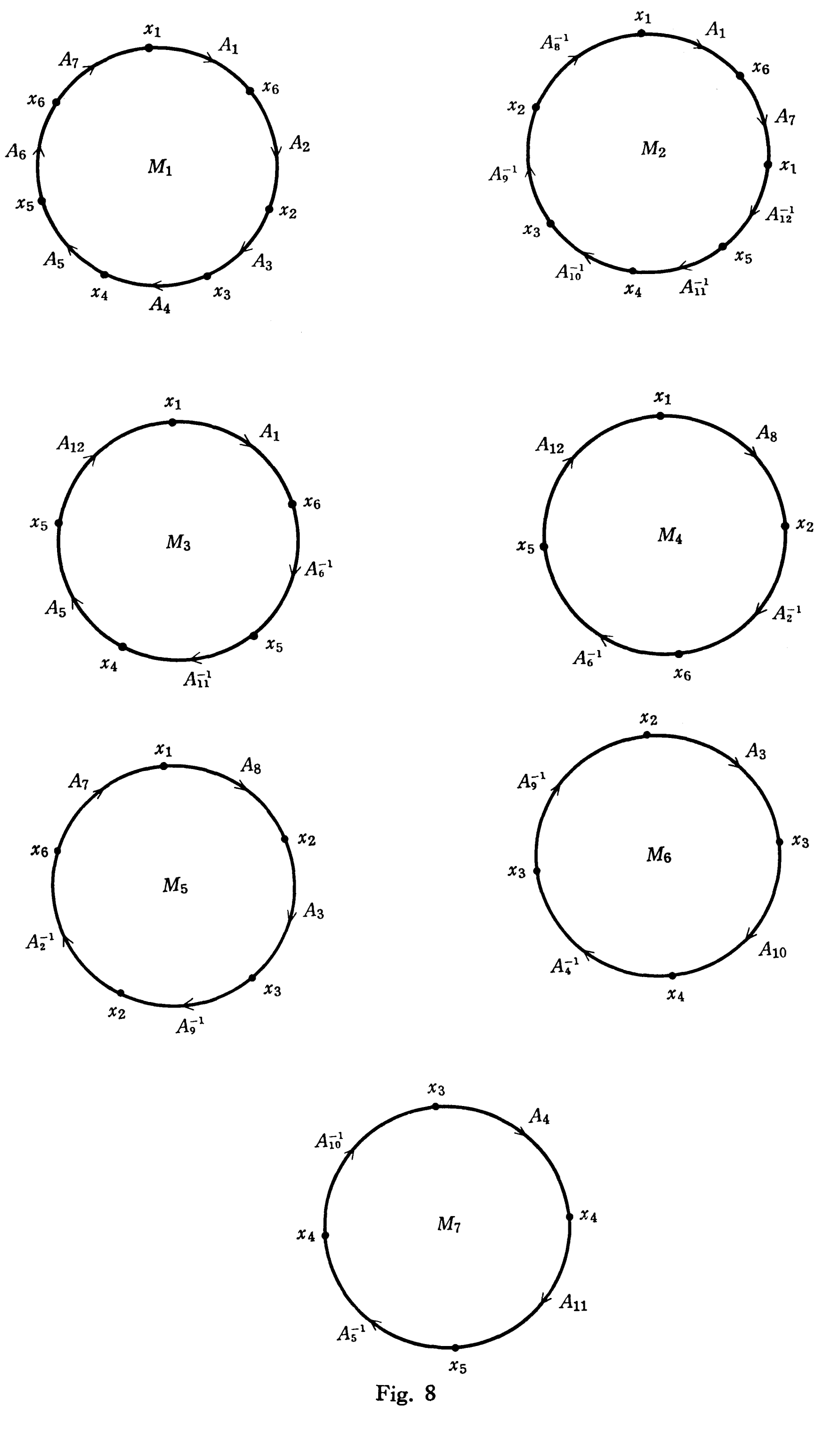

Fig. 8

of a knot, i_* is a monomorphism and we have $i_*^{-1}(S_2^{3k}S_1^{6k-1})=(S_1^{-1}S_2)^{2k}S_1^{6k-1}$. Let C_k denote the 1-sphere in $\dot{E}$ representing the homotopy class $(S_1^{-1}S_2)^{2k}S_1^{2k-1}$. Note that C_k exists because $2k$ and $6k-1$ are relatively prime.

DEFINITION 2. Define the *Dehn space* V_k *of type* k to be the 3-manifold obtained from E by attaching a 2-handle along C_k. (Cf. [2])

Theorem 2. *Let* W_k *be the* 3*-manifold containing* P_k *as its spine. Then,* W_k *is the Dehn space of type* k.

Proof. By the uniqueness theorem of [1], it is sufficient to prove that the Dehn space V_k contains P_k as its spine, because P_k clearly satisfies the conditions of *standard spine* of [1]. Let N_0 be the 3-rd derived neighborhood of $U(F_0)$ in E mod $\dot{U}(F_0)$. We can embed a cylinder $S\times I$ in N_0 in order to satisfy $(S\times I)\cap U(F_0)=S\times 0=h_k(S)$ and $(S\times I)\cap\dot{N}_0=S\times 1$ as shown in Fig. 2. Now, let $F_1=F\cup(S\times I)$ and N_1 the regular neighborhood of F_1 in E mod $\dot{F}_1=S\times 1$. Then, N_1 is homeomorphic to E keeping F_0 fixed, because F_1 collapses to F_0 by collapsing $S\times I$ to $S\times 0$ from $S\times 1$. And hence $S\times 1$ represents the homotopy class $(S_1^{-1}S_2)^{2k}S_2^{6k-1}$ in $\pi_1(N_1)$. Thus, V_k may be regarded as the 3-manifold obtained from N_1 by attaching a 2-handle along $S\times 1$. Then, the 2-handle $B^2\times I$ collapses to $(\dot{B}^2\times I)\cup(B^2\times 1/2)$, where B^2 is a 2-ball and $\dot{B}^2\times 1/2=S\times 1$. Thus, V_k collapses to $N_1\cup(B^2\times 1/2)$. Since N_1 is a regular neighborhood of F_1, $N_1\cup(B^2\times 1/2)$ collapses to $F_1\cup(B^2\times 1/2)$ which is clearly homeomorphic to P_k. Thus, V_k has a spine homeomorphic to P_k. This completes the proof of Theorem 2.

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References

[1] B.G. Casler: *An embedding theorem for a connected* 3*-manifold with boundary,* Proc. Amer. Math. Soc. **16** (1965), 559–566.

[2] M. Dehn: *Über die Topologie des dreidimensionalen Raumes,* Math. Ann. **69** (1910), 137–168.

[3] H. Ikeda: *Acyclic fake surfaces,* Topology **10** (1971), 9–36.

[4] ———: *Acyclic fake surfaces which are spines of* 3*-manifolds,* Osaka. J Math. **9** (1972), 391–408.